Local response in bulk-gapped interacting systems

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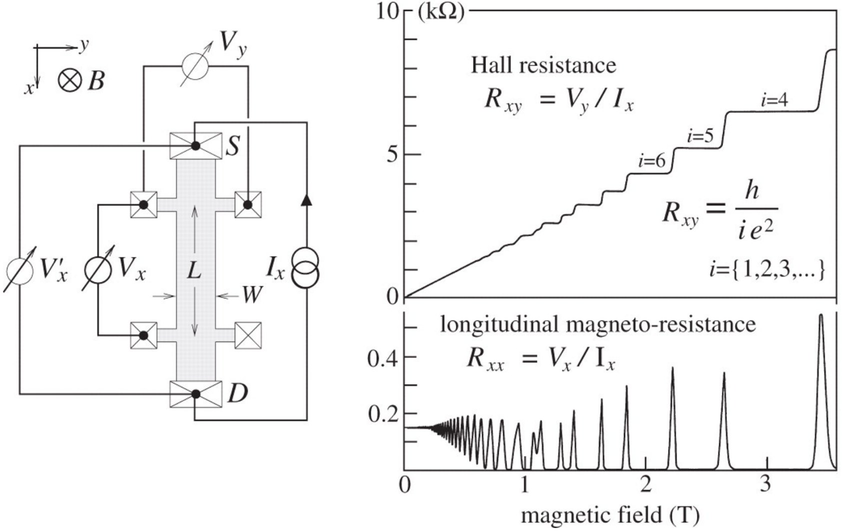
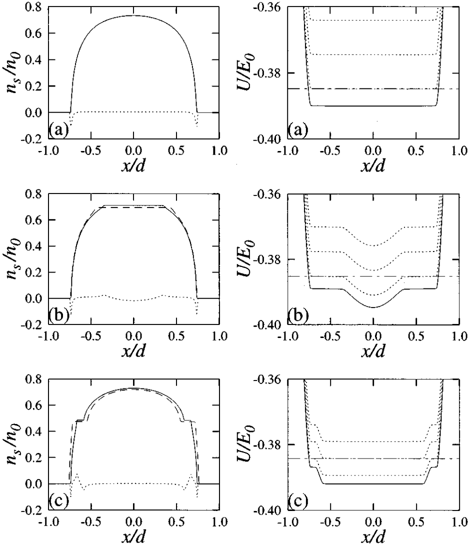
Schematic setup of a quantum hall experiment



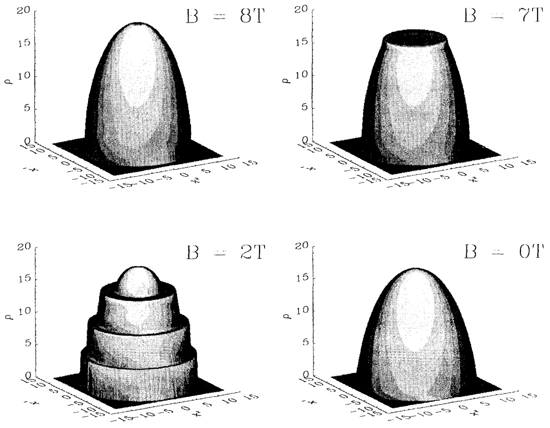
Weis, von Klitzing 2011

Electron density in equilibrium



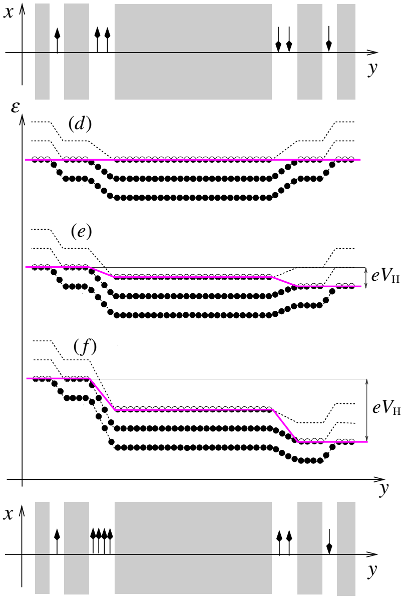
Oh, Gerhardts 1997

Magnetic Thomas-Fermi theory for 2d quantum dots



Lieb, Solovej, Yngvason 1995

Incompressible stripes



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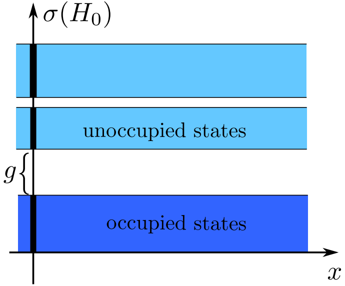

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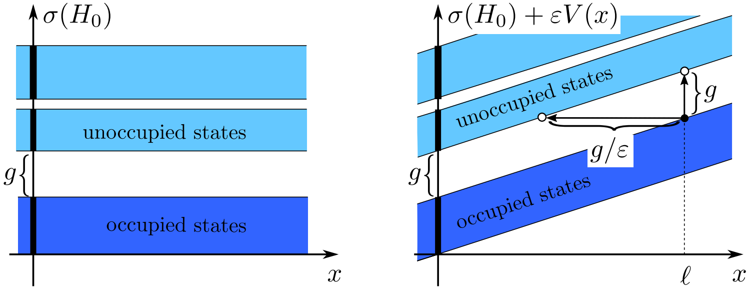

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The quasi-local algebra is the C^*-algebra $\mathcal{A} := \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$.

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In order to describe infinite systems of interacting fermions one takes the thermodynamic limit of a sequence of finite systems e.g. on cubes $\Lambda_k := \{-k, \ldots, k\}^d \subset \mathbb{Z}^d$, $k \in \mathbb{N}$.

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We consider also sequences of Hamiltonians that are sums of local terms ("SLT operator families'),

$$H_0^{\Lambda_k} = \sum_{X \subset \Lambda_k} \Phi(X)\,,$$

where the map

$$\Phi : \mathcal{P}_0(\mathbb{Z}^d) \to \mathcal{A}_{\mathrm{loc}}\,, \quad X \mapsto \Phi(X) \in \mathcal{A}_X$$

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A typical interaction for a physical Hamiltonian is of the form

$$\Phi(X) = \begin{cases} a_x^*\, T(x-y)\, a_y + h.c. + a_x^* a_x\, W(x-y)\, a_y^* a_y & \text{if } X = \{x, y\} \\ a_x^*(\phi(x) - \mu) a_x & \text{if } X = \{x\} \\ 0 & \text{otherwise} \end{cases}$$

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Then $\|H_0^{\Lambda_k}\| \sim |\Lambda_k| = (2k+1)^d$.

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To quantify locality of an interaction resp. of the corresponding SLT operators, one defines Banach spaces $\mathcal{B}_\zeta$ of SLT operators with norm

$$\|\Phi\|_\zeta \;:=\; \sup_{x\in\mathbb{Z}^d} \sum_{X\in\mathcal{P}_0(\mathbb{Z}^d),\, x\in X} \frac{\|\Phi(X)\|}{\zeta(\mathrm{diam}(X))}\,,$$

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In order to control also the localisation properties of elements of $\mathcal{A}$, one defines sub-algebras $\mathcal{D}_\zeta \subset \mathcal{A}$ with norm

$$\|B\|_\zeta := \|B\| + \sup_{k\in\mathbb{N}} \left(\frac{\|\,(1-\mathbb{E}_{\Lambda_k})\,(B)\|}{\zeta(k)} \right) < \infty\,,$$

where $\zeta : [0,\infty) \to (0,\infty)$ is again a rapidly decaying function and $\mathbb{E}_{\Lambda_k} : \mathcal{A} \to \mathcal{A}_{\Lambda_k}$ denotes the conditional expectation.

Interacting fermions on the lattice

Proposition: Thermodynamic limit for the dynamics

Let $H_0 \in \mathcal{B}_\zeta$. Then for any $B \in \mathcal{A}_{\mathrm{loc}}$ the limit

$$\mathfrak{U}_t(B) := \lim_{k\to\infty} \mathrm{e}^{\mathrm{i}H_0^{\Lambda_k} t}\, B\, \mathrm{e}^{-\mathrm{i}H_0^{\Lambda_k} t} \in \mathcal{A}$$

exists and defines a one-parameter family $t \mapsto \mathfrak{U}_t$ of automorphisms of the algebra $\mathcal{A}$ with densely defined generator $\mathcal{L}_{H_0} : D(\mathcal{L}_{H_0}) \to \mathcal{A}$.

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Moreover, for suitable pairs f_1, f_2 of rapidly decaying functions,

$$\mathfrak{U}_t : \mathcal{D}_{f_1} \to \mathcal{D}_{f_2}$$

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Lieb, Robinson '72; …; **Bru, Pedra** '16; **Nachtergaele et al.** '19; **Moon, Ogata** '20; **Henheik, T.** '21.

The standard gap condition

Standard gap assumption

Assume that smallest eigenvalue $E_0^{\Lambda_k}(t)$ (ground state) of $H_0^{\Lambda_k}(t)$ is separated from the rest of the spectrum uniformly in the volume $|\Lambda_k|$,

$$\inf_{\Lambda_k} \operatorname{dist}\left(E_0^{\Lambda_k}(t), \sigma(H_0^{\Lambda_k}(t)) \setminus \{E_0^{\Lambda_k}(t)\}\right) =: g > 0\,.$$

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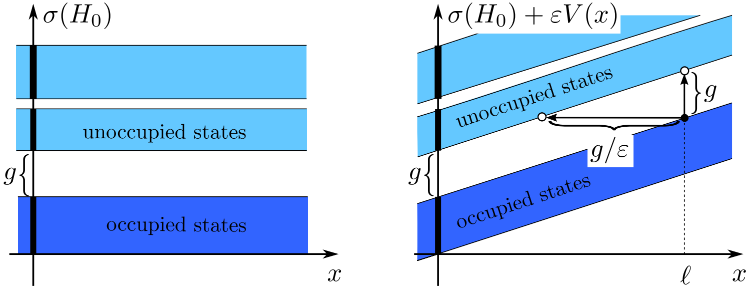
- The filled Dirac sea.

Why adiabatic response?



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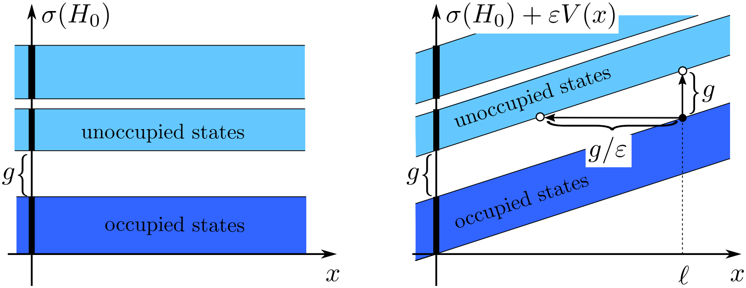

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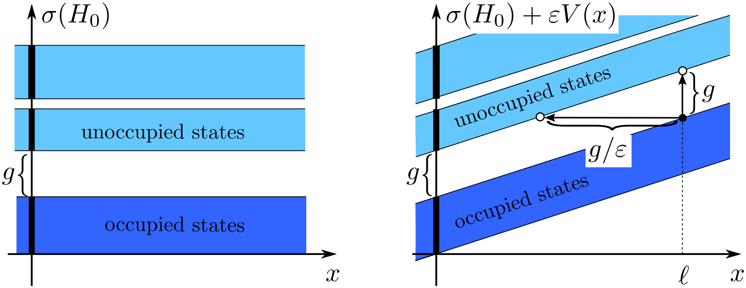

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In non-interacting systems: adiabatic theory for "almost invariant subspaces" was established in **Nenciu** '81, '02; **Nenciu, Sordoni** '03 (based on **Helffer, Sjöstrand** '89); **Panati, Spohn, T.** '03.

Adiabatic theorem

From now on we consider a time-dependent gapped Hamiltonian $H_0(t) \in \mathcal{B}_{\mathrm{e}^{-a\cdot}}$, $t \in I \subset \mathbb{R}$, possibly perturbed by a time-dependent operator $\varepsilon V(t)$, where $V(t) = V_v(t) + H_1(t)$ is the sum of an SLT operator $H_1(t) \in \mathcal{B}_{\mathrm{e}^{-a\cdot}}$ and a Lipschitz potential $V_v(t)$, i.e.

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Let

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and denote by $\mathfrak{U}_{t,t_0}^{\eta,\varepsilon}$ the corresponding adiabatic evolution family generated by the time-dependent Liouvillian $\frac{1}{\eta}\mathcal{L}_{H_\varepsilon(t)}$ with adiabatic parameter $\eta > 0$, i.e.

$$\mathfrak{U}_{t,t_0}^{\eta,\varepsilon}(B) := \lim_{k\to\infty} \mathfrak{U}_{t,t_0}^{\eta,\varepsilon,\Lambda_k}(B) \in \mathcal{A} \,.$$

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Note that for $\varepsilon = 0$ we are in the usual adiabatic situation of a time-dependent Hamiltonian with spectral gap.

Adiabatic theorems for extended many-body systems

- Adiabatic theorems under the "standard gap assumption" in finite volumes and for $\varepsilon = 0$ with error estimates that are uniform in the volume were first shown by **Bachmann, De Roeck, Fraas** '18.

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- In **Henheik, T.** '20 we prove an adiabatic theorem under "standard gap assumption" in the infinite volume and for $\varepsilon > 0$, i.e. for $\mathfrak{U}_{t,t_0}$ on $\mathcal{A}$.

Adiabatic theorem with a gap in the bulk

Motivation: Response of Chern-nontrivial systems (e.g. quantum Hall systems), where the Hamiltonian has no spectral gap in the presence of edges.

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for all $B \in \mathcal{A}$ and $s \in \mathbb{R}$.

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for all $B \in \mathcal{A}$ and $s \in \mathbb{R}$.

H_ρ is called the bulk Hamiltonian associated with ρ.

Adiabatic theorem with a gap in the bulk

Gap assumption in the bulk (cf. Moon, Ogata, *JFA* '19)

There exists $g > 0$ such that for each $t \in I$ the Liouvillian $\mathcal{L}_{H_0(t)}$ has a unique ground state ρ_t and

$$\sigma(H_{\rho_t}) \setminus \{0\} \subset [g, \infty).$$

Adiabatic theorem with a gap in the bulk

Super-adiabatic theorem for NEASSs

Let the Hamiltonian $H_\varepsilon(t) = H_0(t) + \varepsilon V(t)$ satisfy the previous assumptions and denote by $\mathfrak{U}^{\varepsilon,\eta}_{t,t_0}$ the Heisenberg time-evolution it generates on $\mathcal{A}$.

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Then for any $\varepsilon, \eta \in (0,1]$ and $t \in I$ there exists a near-identity automorphism $\beta^{\varepsilon,\eta}(t)$ of $\mathcal{A}$ such that the super-adiabatic NEASS defined by

$$\Pi_t^{\varepsilon,\eta} := \rho_t \circ \beta_t^{\varepsilon,\eta}$$

has the following properties:

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has the following properties:

(1) It almost intertwines the time evolution: For any $n \in \mathbb{N}$ and any $f \in \mathcal{S}$, there exists a constant C_n such that for any $t \in I$ and $B \in \mathcal{D}_f$

$$\left|\left(\Pi^{\varepsilon,\eta}_{t_0} \circ \mathfrak{U}^{\varepsilon,\eta}_{t,t_0} - \Pi^{\varepsilon,\eta}_t\right)(B)\right| \leq C_n \, \frac{\varepsilon^{n+1} + \eta^{n+1}}{\eta^{d+1}} \, \left(1 + |t - t_0|^{d+1}\right) \|B\|_f .$$

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(2) It is local in time: $\beta^{\varepsilon,\eta}_t$ depends only on H_ε and its time derivatives at time t.

(3) It is stationary whenever the Hamiltonian is stationary: if H_ε is constant on an interval $J \subset I$ then $\Pi^{\varepsilon,\eta}_t = \Pi^{\varepsilon,0}_t$ is constant for $t \in J$.

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(4) $\Pi^{\varepsilon,0}_t$ has an explicit asymptotic expansion in powers of ε.

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has the following properties:

(5) It equals the ground state of H_0 whenever the perturbation vanishes and the Hamiltonian is stationary: if for some $t \in I$ all time-derivatives of H_ε vanish at time t and $V(t) = 0$, then $\Pi^{\varepsilon,\eta}_t = \Pi^{\varepsilon,0}_t = \rho_t$.

Concluding remarks

- Why is this useful at all? It provides an immediate justification of standard linear response (and higher order response) formulas for systems that initially in gapped ground states.
  See e.g. **Henheik, T.** '21.

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  $$|(\rho - \rho^{\Lambda})(B)| \leq C_n \|B\| \mathrm{dist}(X, \partial\Lambda)^{-n} \tag{1}$$

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- Assuming (1), in **Henheik, T.** '20 we also show an adiabatic theorem for finite systems with a gap in the bulk.

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Thanks for your attention!