Bosonization of Fermionic Many-Body Systems

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IPAM workshop
Model Reduction in Quantum Mechanics

April 11, 2022

Based on joint works with
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Bosonic systems

Statistics: we distinguish between

Bosons: described by symmetric wave functions, ie.

$$\psi_N(x_{\pi 1}, x_{\pi_2}, \ldots, x_{\pi N}) = \psi_N(x_1, x_2, \ldots, x_N) \qquad \text{for all } \pi \in S_N$$

Fermions: described by antisymmetric wave functions, ie.

$$\psi_N(x_{\pi 1}, x_{\pi_2}, \ldots, x_{\pi N}) = \sigma_\pi \psi_N(x_1, x_2, \ldots, x_N) \qquad \text{for all } \pi \in S_N$$

Bosonic mean-field regime: N particles in torus $\Lambda = [0; 2\pi]^3$, with Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j) \qquad \text{acting on } L^2_s(\Lambda^N)$$

Goals: ground state energy, excitations, time-evolution.

Bogoliubov theory

Fock space: describe bosons on

$$\mathcal{F}_s = \bigoplus_{n \geq 0} L^2_s(\Lambda^n)$$

For $p \in \mathbb{Z}^3$, we introduce a_p^*, a_p satisfying **CCR**

$$[a_p, a_q^*] = \delta_{pq}, \qquad [a_p, a_q] = [a_p^*, a_q^*] = 0$$

Then

$$a_p^* a_p = \text{ number of particles with momentum } p$$

In particular, $\sigma(a_p^* a_p) = \mathbb{N}$.

Hamilton operator: in second quantized form, we find

$$H_N = \sum_{p \in \mathbb{Z}^3} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \mathbb{Z}^3} \widehat{V}(r) a_{p+r}^* a_q^* a_{q+r} a_p$$

BEC: low-energy states exhibit Bose-Einstein condensation, ie.

$$a_0^*, a_0 \simeq \sqrt{N} \gg 1 = [a_0, a_0^*]$$

Hence,

$$H_N \simeq \frac{(N-1)\widehat{V}(0)}{2} + \sum_{p \neq 0} \left[p^2 + \widehat{V}(p)\right] a_p^* a_p + \frac{1}{2} \sum_{p \neq 0} \widehat{V}(p) \left[a_p^* a_{-p}^* + a_p a_{-p}\right]$$
$$+ \frac{1}{\sqrt{N}} \sum_{p,r \neq 0} \widehat{V}(r) \left[a_{p+r}^* a_{-r}^* a_p + \text{h.c.}\right]$$
$$+ \frac{1}{2N} \sum_{p,q,r \neq 0} \widehat{V}(r) a_{p+r}^* a_q^* a_{q+r} a_p$$

Neglect cubic and quartic terms: we find

$$H_N \simeq \frac{(N-1)\widehat{V}(0)}{2} + \sum_{p \neq 0} \left[p^2 + \widehat{V}(p)\right] a_p^* a_p + \frac{1}{2} \sum_{p \neq 0} \widehat{V}(p) \left[a_p^* a_{-p}^* + a_p a_{-p}\right]$$

Bogoliubov transformations: let

$$T = \exp \left[\frac{1}{2} \sum_{p \neq 0} \tau_p \Big(a_p^* a_{-p}^* - a_p a_{-p} \Big) \right]$$

Then

$$\begin{aligned} T^* \, a_p^* \, T &= (\cosh \tau_p) \, a_p^* + (\sinh \tau_p) \, a_{-p} \\ T^* \, a_p \, T &= (\cosh \tau_p) \, a_p + (\sinh \tau_p) \, a_{-p}^* \end{aligned}$$

Diagonalization: with **appropriate** choice of τ_p, we obtain

$$\begin{aligned} T^* H_N T \simeq \frac{(N-1)\widehat{V}(0)}{2} - \frac{1}{2} \sum_{p \neq 0} \left[p^2 + \widehat{V}(p) - \sqrt{|p|^4 + 2p^2 \widehat{V}(p)} \right] \\ + \sum_{p \neq 0} \sqrt{|p|^4 + 2p^2 \widehat{V}(p)} \cdot a_p^* a_p \end{aligned}$$

From this expression, we can read off ground state energy and excitations.

Mean-field fermions

Mean-field regime: consider N fermions in torus $\Lambda = [0;2\pi]^3$. System described on

$$L^2_a(\Lambda^N) = \left\{\psi \in L^2(\Lambda^N), \text{ antisymmmetric w.r.t. permutations}\right\}$$

On $L^2_a(\Lambda^N)$, we have $\sum_{j=1}^N -\Delta_{x_j} \simeq N^{5/3}$.

Hamilton operator: given by

$$H_N = \sum_{j=1}^N -\varepsilon^2 \Delta_{x_j} + \frac{1}{N}\sum_{i<j}^N V(x_i - x_j)$$

with **semiclassical parameter** $\varepsilon = N^{-1/3}$. We assume $\widehat{V} \geq 0$.

We are interested in **ground state energy**

$$E_N = \min_{\psi \in L^2_a(\Lambda^N) : \|\psi\|=1} \langle \psi, H_N \psi \rangle$$

Hartree-Fock theory: consider **Slater determinants**

$$\psi_{\text{slater}}(x_1, \ldots, x_N) = C \det \left[f_i(x_j) \right]_{1 \le i,j \le N}$$

with $\{f_1, \ldots, f_N\}$ an orthonormal system in $L^2(\Lambda)$.

They are characterized by one-particle reduced density

$$\omega = N \operatorname{tr}_{2,\ldots,N} |\psi_{\text{slater}}\rangle\langle\psi_{\text{slater}}| = \sum_{j=1}^{N} |f_j\rangle\langle f_j|$$

Their energy is given by **Hartree-Fock functional**

$$\begin{aligned} \mathcal{E}_{\text{HF}}(\omega) = \operatorname{tr} \left[-\varepsilon^2 \Delta \right] \omega \\ + \frac{1}{2N} \int dx dy \, V(x-y) \left[\omega(x,x)\omega(y,y) - |\omega(x,y)|^2 \right] \end{aligned}$$

The **Hartree-Fock energy** is defined by

$$E_N^{\text{HF}} = \min_{0 \le \omega \le 1 : \operatorname{tr} \omega = N} \mathcal{E}_{\text{HF}}(\omega)$$

Fermi sea: If $V = 0$, Hartree-Fock energy minimized by

$$\omega_F = \frac{1}{(2\pi)^3} \sum_{p \in \mathbb{Z}^3 : |p| \le p_F} |e^{ip\cdot x}\rangle\langle e^{ip\cdot x}|$$

with **Fermi momentum**

$$p_F \simeq (3/4\pi)^{1/3} N^{1/3} =: \kappa_0 N^{1/3}$$

Here, we assume that Fermi ball is **completely filled**.

Then ω_F also minimizes Hartree-Fock functional if $V \neq 0$. Hence

$$E_N^{\text{HF}} = \mathcal{E}_{\text{HF}}(\omega_F) = \sum_{|p| \le p_F} \varepsilon^2 p^2 + \frac{N\widehat{V}(0)}{2} - \frac{1}{2N} \sum_{|p|,|p'| \le p_F} \widehat{V}(p - p')$$

Remark: in general, **[Gontier-Hainzl-Lewin, 19]** proved that $E_N^{\text{HF}} - \mathcal{E}_{\text{HF}}(\omega_F)$ is sub-exponentially small in N.

Remark: justification of Hartree-Fock theory by **[Bach, 92]**, **[Graf-Solovej, 94]** for Coulomb interaction.

Correlation energy

Theorem [Benedikter, Porta, S., Seiringer]: let $|k|\hat{V} \in \ell^1(\mathbb{Z}^3)$, $\hat{V} \geq 0$. Then

$$\lim_{N\to\infty} \Big[E_N - \mathcal{E}_{\text{HF}}(\omega_F)\Big]/\varepsilon$$
$$= \kappa_0 \sum_{k\in\mathbb{Z}^3} |k| \Big\{ \frac{1}{\pi} \int_0^\infty \log\Big[1 + 2\pi\kappa_0 \hat{V}(k)\Big(1 - \lambda \text{arctan}\Big(\frac{1}{\lambda}\Big)\Big)\Big]\, d\lambda - \frac{\pi}{2}\kappa_0 \hat{V}(k)\Big\}$$

Remark 1: for upper bound, only need $\sum_{k\in\mathbb{Z}^3} |k| \hat{V}^2(k) < \infty$.

Remark 2: Result agrees with formula first predicted by **Gell-Mann–Brueckner**, through random phase approximation.

Remark 3: for small V, theorem proven by **[Benedikter-Nam-Porta-S.-Seiringer, 20]**.

[Christiansen-Hainzl-Nam] proved similar result, with different approach.

Some ideas from proof

To estimate energy, it is convenient to factor out Fermi sea and focus on its **excitations**.

Fock space representation: on

$$\mathcal{F}_a = \bigoplus_{n \geq 0} L^2_a(\Lambda^n)$$

we introduce creation, annihilation operators satisfying canonical **anticommutation relations**

$$\left\{a_p, a_q^*\right\} = \delta_{p,q}, \qquad \left\{a_p, a_q\right\} = \left\{a_p^*, a_q^*\right\} = 0$$

We consider the Hamilton operator

$$\mathcal{H}_N = \sum_{p \in \mathbb{Z}^3} \varepsilon^2 p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \mathbb{Z}^3} \widehat{V}(r)\, a_{p+r}^* a_q^*\, a_{q+r} a_p$$

Particle-hole transformation: on $\mathcal{F}_a$, define **unitary** R with

$$R\Omega = \prod_{|p|\leq p_F} a_p^* \Omega = \text{ Fermi sea}$$

and

$$R a_p^* R^* = \begin{cases} a_p^*, & \text{if } |p| > p_F \\ a_p, & \text{if } |p| \leq p_F \end{cases}$$

After **conjugation**, a_p^* creates a hole in the Fermi sea, if $|p| \leq p_F$.

Observe: $R : \chi(\mathcal{N}_h = \mathcal{N}_p)\mathcal{F}_a \to L_a^2(\Lambda^N)$, where

$$\mathcal{N}_h = \sum_{|p|\leq p_F} a_p^* a_p, \qquad \mathcal{N}_p = \sum_{|p|>p_F} a_p^* a_p$$

count number of **holes** and of excited **particles**.

Excitation Hamiltonian: we define

$$\mathcal{L}_N = R^* \mathcal{H}_N R$$

Conjugation of kinetic energy: we compute

$$\begin{aligned}
\sum_{p\in\mathbb{Z}^3} \varepsilon^2 p^2 R^* a_p^* a_p R \\
&= \sum_{|p|\le p_F} \varepsilon^2 p^2 a_p a_p^* + \sum_{|p|>p_F} \varepsilon^2 p^2 a_p^* a_p \\
&= \sum_{|p|\le p_F} \varepsilon^2 p^2 - \sum_{|p|\le p_F} \varepsilon^2 p^2 a_p^* a_p + \sum_{|p|>p_F} \varepsilon^2 p^2 a_p^* a_p \\
&= \operatorname{tr}\left[-\varepsilon^2 \Delta\right] \omega_F + \mathbb{H}_0
\end{aligned}$$

with the **kinetic energy of excitations**:

$$\mathbb{H}_0 = \sum_{p\in\mathbb{Z}^3} |\varepsilon^2 p^2 - \varepsilon^2 p_F^2|\, a_p^* a_p$$

Conjugation of potential energy: we find

$$\frac{1}{2N}\sum_{p,q,r}\widehat{V}(r)R^*a^*_{p+r}a^*_q a_{q+r}a_p R = \frac{N\widehat{V}(0)}{2} - \frac{1}{2N}\sum_{|p|,|p'|\leq p_F}\widehat{V}(p-p') + Q_B + \text{corrections}$$

where

$$\begin{aligned} Q_B &= \frac{1}{2N}\sum_{|p|,|q|\leq p_F,|p+r|,|q+r|>p_F}\widehat{V}(r)\, a^*_{p+r}a^*_p a_{q+r}a_q \\ &\quad + \frac{1}{2N}\sum_{|p|,|q+r|\leq p_F,|p+r|,|q|>p_F}\widehat{V}(r)\Big[a^*_{p+r}a^*_p a^*_{q+r}a^*_q + a_q a_{q+r}a_p a_{p+r}\Big] \\ &= \frac{1}{2N}\sum_{r\in\mathbb{Z}^3}\widehat{V}(r)\left[b^*_r b_r + \frac{1}{2}(b^*_r b^*_{-r} + b_r b_{-r})\right] \end{aligned}$$

in terms of **particle-hole pair** creation operators

$$b^*_r = \sum_{|p|\leq p_F,|p+r|>p_F} a^*_{p+r}a^*_p$$

satisfying (after normalization) **approximately bosonic** relations

$$[b^*_r, b^*_k] = [b_r, b_k] = 0, \qquad [b_r, b^*_k] = \text{const}\cdot\delta_{r,k} + \text{corrections}$$

Conclusion: excitation Hamiltonian is given by

$$\mathcal{L}_N = \mathcal{E}_{\mathsf{HF}}(\omega_F) + \mathbb{H}_0 + Q_B + \text{corrections}$$

where

$$\mathbb{H}_0 = \sum_{p\in\mathbb{Z}^3} |\varepsilon^2 p^2 - \varepsilon^2 p_F^2|\, a_p^* a_p$$

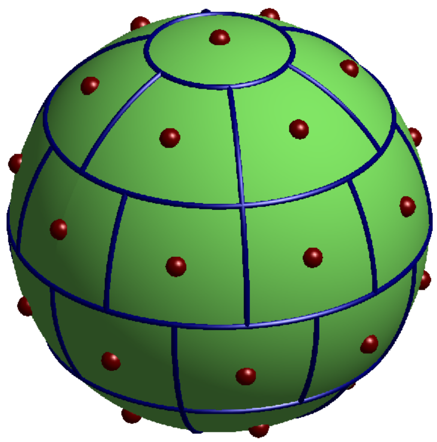
$$Q_B = \frac{1}{2N} \sum_{r\in\mathbb{Z}^3} \widehat{V}(r) \left[b_r^* b_r + \frac{1}{2}(b_r^* b_{-r}^* + b_r b_{-r}) \right]$$

and where corrections are **small** on states with few excitations.

Question: can $\mathbb{H}_0$ be expressed through operators b, b^*?

$$\begin{aligned} \mathbb{H}_0\, b_r^* \Omega &= \sum_{|q|\leq p_F, |q+r|>p_F} \mathbb{H}_0\, a_{q+r}^* a_q^* \Omega \\ &= \sum_{|q|\leq p_F, |q+r|>p_F} \left[\varepsilon^2 (q+r)^2 - \varepsilon^2 q^2 \right] a_{q+r}^* a_q^* \Omega \\ &\simeq \sum_{|q|\leq p_F, |q+r|>p_F} 2\varepsilon^2 q \cdot r a_{q+r}^* a_q^* \Omega \neq \epsilon(r)\, b_r^* \Omega \end{aligned}$$

Localized bosonic modes: we decompose Fermi sphere into $M = N^\delta$ **patches** $\{B_\alpha\}_{\alpha=1,\dots,M}$, and define **modes**



$$b^*_{k,\alpha} = \frac{1}{n_\alpha(k)} \sum_{\substack{p\in B_\alpha,\\ |p|\le p_F, |p+k|>p_F}} a^*_{p+k} a^*_p$$

with $n_\alpha(k)$ chosen so that

$$\left[b_{\alpha,k}, b^*_{\beta,q}\right] \simeq \delta_{\alpha,\beta}\delta_{k,q}$$

Now, we find, with w_α the **center** of B_α,

$$\mathbb{H}_0\, b^*_{r,\alpha}\Omega \simeq \sum_{\substack{q\in B_\alpha,\\ |q|\le p_F, |q+r|>p_F}} 2\varepsilon^2 q\cdot r a^*_{q+r} a^*_q \Omega \simeq 2\varepsilon^2 (w_\alpha \cdot r)\, b^*_{r,\alpha}\Omega$$

suggesting

$$\mathbb{H}_0 \simeq 2\kappa_0\varepsilon \sum_{r\in\mathbb{Z}^3} (\widehat{w}_\alpha \cdot r)\, b^*_{r,\alpha} b_{r,\alpha}$$

Quadratic Hamiltonian: combining modes with momenta k and $-k$, we arrive at

$$\mathcal{L}_N = \mathcal{E}_{\mathsf{HF}}(\omega_F) + \sum_{k \in \Gamma^{\mathsf{nor}}} 2\varepsilon\kappa_0 |k| \mathfrak{h}(k) + \text{corrections}$$

with

$$\mathfrak{h}(k) = \sum_{\alpha,\beta=1}^{M} (D(k) + W(k))_{\alpha,\beta}\, b^*_{k,\alpha} b_{k,\beta} + \frac{1}{2} \sum_{\alpha,\beta=1}^{M} \widetilde{W}(k)_{\alpha,\beta} \Big[b^*_{k,\alpha} b^*_{k,\beta} + b_{k,\beta} b_{k,\alpha} \Big]$$

where the $M \times M$ **matrices** $D(k), W(k), \widetilde{W}(k)$ have entries

$$D(k)_{\alpha,\beta} = \delta_{\alpha,\beta} |\widehat{k} \cdot \widehat{\omega}_\alpha|,$$

$$W(k)_{\alpha,\beta} = \frac{\hat{V}(k)}{2\hbar\kappa_0 N|k|} \times \begin{cases} n_\alpha(k) n_\beta(k) & \text{if } \alpha,\beta \in \mathcal{I}_k^+ \text{ or } \alpha,\beta \in \mathcal{I}_k^- \\ 0 & \text{otherwise,} \end{cases}$$

$$\widetilde{W}(k)_{\alpha,\beta} = \frac{\hat{V}(k)}{2\hbar\kappa_0 N|k|} \times \begin{cases} 0 & \text{if } \alpha,\beta \in \mathcal{I}_k^+ \text{ or } \alpha,\beta \in \mathcal{I}_k^- \\ n_\alpha(k) n_\beta(k) & \text{otherwise.} \end{cases}$$

Bogoliubov theory: let

$$T = \exp\Big[\frac{1}{2}\sum_{k\in\Gamma^{\text{nor}}}\sum_{\alpha,\beta} K(k)_{\alpha,\beta}\, b^*_{k,\alpha} b^*_{k,\beta} - \text{h.c.}\Big]$$

Up to small corrections,

$$T^* b_{k,\alpha} T \simeq \sum_\beta \cosh(K(k))_{\alpha,\beta}\, b_{k,\beta} + \sum_\beta \sinh(K(k))_{\alpha,\beta}\, b^*_{k,\beta}$$

With **appropriate choice** of K, we find

$$T^*\mathcal{L}_N T = \mathcal{E}_{\text{HF}}(\omega_F) + 2\varepsilon\kappa_0 \sum_{k\in\Gamma^{\text{nor}}} |k|\, \text{tr}\Big[E(k) - D(k) - W(k)\Big] + \sum_{k\in\Gamma^{\text{nor}}}\sum_{\alpha,\beta} \mathfrak{K}(k)_{\alpha,\beta}\, b^*_{k,\alpha} b_{k,\beta} + \text{corrections}$$

Finally, explicit computation shows that

$$\text{tr}\Big[E(k) - D(k) - W(k)\Big] \simeq \frac{1}{\pi}\int_0^\infty \log\Big[1 + 2\pi\kappa_0 \hat{V}(k)(1 - \lambda\arctan(1/\lambda))\Big] d\lambda - \frac{\pi}{2}\kappa_0 \hat{V}(k)$$

An additional challenge: From

$$\mathbb{H}_0\, b^*_{r,\alpha}\Omega \simeq 2\kappa_0\varepsilon\,(\widehat{w}_\alpha\cdot r)\, b^*_{r,\alpha}\Omega \quad \not\Rightarrow \quad \mathbb{H}_0 \simeq 2\kappa_0\varepsilon \sum_{r\in\mathbb{Z}^3}(\widehat{w}_\alpha\cdot r)\, b^*_{r,\alpha}b_{r,\alpha} =: \mathbb{D}_B$$

Instead, we only obtain that

$$\Big[\mathbb{H}_0-\mathbb{D}_B, b_{r,\alpha}\Big], \Big[\mathbb{H}_0-\mathbb{D}_B, b^*_{r,\alpha}\Big] \simeq 0 \quad \Rightarrow \quad T^*(\mathbb{H}_0-\mathbb{D}_B)T \simeq (\mathbb{H}_0-\mathbb{D}_B)$$

Therefore,

$$\begin{aligned}\mathcal{L}_N &\simeq \mathcal{E}_{\mathsf{HF}}(\omega_F) + \mathbb{H}_0 + Q_B \\ &\simeq \mathcal{E}_{\mathsf{HF}}(\omega_F) + (\mathbb{H}_0 - \mathbb{D}_B) + 2\kappa_0\varepsilon \sum_{k\in\Gamma^{\mathrm{nor}}} |k|\mathfrak{h}(k)\end{aligned}$$

and

$$\begin{aligned}T^*\mathcal{L}_N T \simeq \mathcal{E}_{\mathsf{HF}}(\omega_F) &+ 2\kappa_0\varepsilon \sum_{k\in\Gamma^{\mathrm{nor}}} |k|\,\mathrm{tr}\,\Big[E(k) - D(k) - W(k)\Big] \\ &+ (\mathbb{H}_0 - \mathbb{D}_B) + 2\kappa_0\varepsilon \sum_{k\in\Gamma^{\mathrm{nor}}} |k| \sum_{\alpha,\beta} \mathfrak{K}(k)_{\alpha,\beta}\, b^*_{k,\alpha}b_{k,\beta}\end{aligned}$$

To control $-\mathbb{D}_B$, we use a **second** almost bosonic **Bogoliubov transformation**, diagonalizing $\mathfrak{K}$.

Dynamics of excitations

Time evolution of almost bosonic excitations can be described by **Bogoliubov Hamiltonian** $\mathfrak{K}(k)$.

Schrödinger equation: consider solution of

$$i\varepsilon\partial_t\psi_{N,t} = \left[-\sum_{j=1}^{N} \varepsilon^2\Delta_{x_j} + \frac{1}{N}\sum_{i<j}^{N} V(x_i - x_j)\right]\psi_{N,t}$$

with **initial data** of the form $\psi_{N,0} = RT\xi \in L_a^2(\Lambda^N)$, with

$$\xi = \frac{1}{Z}b^*(\varphi_1)\dots b^*(\varphi_m)\Omega, \qquad \text{with } b^*(\varphi) = \sum_{k\in\Gamma^{\text{nor}}}\sum_{\alpha\in\mathcal{I}_k}\varphi_\alpha(k)b^*_{k,\alpha}$$

Then

$$T^*R^*\psi_{N,t} = T^*R^*e^{-iH_Nt/\varepsilon}RT\xi$$

$$= e^{-iT^*R^*H_NRTt/\varepsilon}\xi = e^{-iT^*\mathcal{L}_NTt/\varepsilon}\xi$$

From approximation

$$T^*\mathcal{L}_N T = \mathcal{E}_{\mathsf{HF}}(\omega_F) + 2\varepsilon\kappa_0 \sum_{k\in\Gamma^{\mathsf{nor}}} |k| \,\mathrm{tr}\Big[E(k) - D(k) - W(k)\Big]$$
$$+ (\mathbb{H}_0 - \mathbb{D}_B) + 2\kappa_0\varepsilon \sum_{k\in\Gamma^{\mathsf{nor}}} |k| \sum_{\alpha,\beta} \mathfrak{K}(k)_{\alpha,\beta}\, b^*_{k,\alpha} b_{k,\beta}$$

we are led to **excitation dynamics**

$$\xi_t = \frac{1}{Z} b^*(\varphi_{1,t}) \dots b^*(\varphi_{m,t})\Omega$$

where

$$\varphi_t = e^{-ih_B t/\varepsilon}\varphi, \quad \text{with } h_B = 2\kappa_0\varepsilon \bigoplus_{k\in\Gamma^{\mathsf{nor}}} |k|\mathfrak{K}(k)$$

Theorem [Benedikter, Nam, Porta, S., Seiringer]: many-body evolution can be **approximated in norm** by

$$\|e^{-iH_N t/\varepsilon} RT\xi - e^{-i(\mathcal{E}_{\mathsf{HF}}(\omega_F)+E_{\mathsf{corr}})t/\varepsilon} RT\xi_t\| \leq C|t|\varepsilon^{1/15}$$