noncommutative extreme values based on the Ando-max

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Classical

f, g independent r.v.

$f \vee g$ (i.e. $\max(f,g)$)

$F_f(t) = \mu_f((-\infty, t]) = \Pr(f \leq t)$

distribution function

$$F_{f \vee g} = F_f \cdot F_g$$

limit distributions of

$$\frac{f_1 \vee \cdots\cdots \vee f_n - B_n}{A_n}, \quad n \to \infty$$

$f_1, f_2, \cdots$ i.i.d.

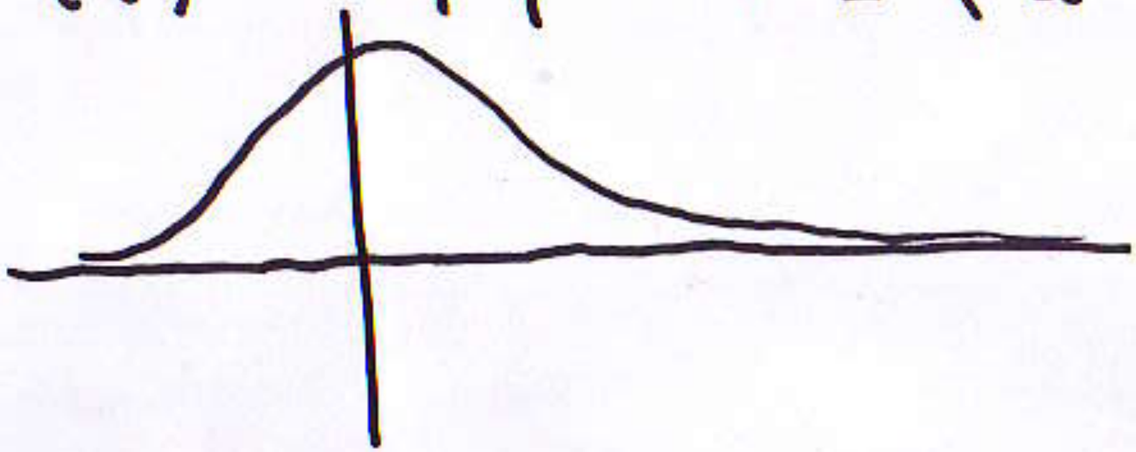
Max-stable distributions

(Insurance, Withstanding Floodings, Earthquakes ...)

Classification

1° Gumbel

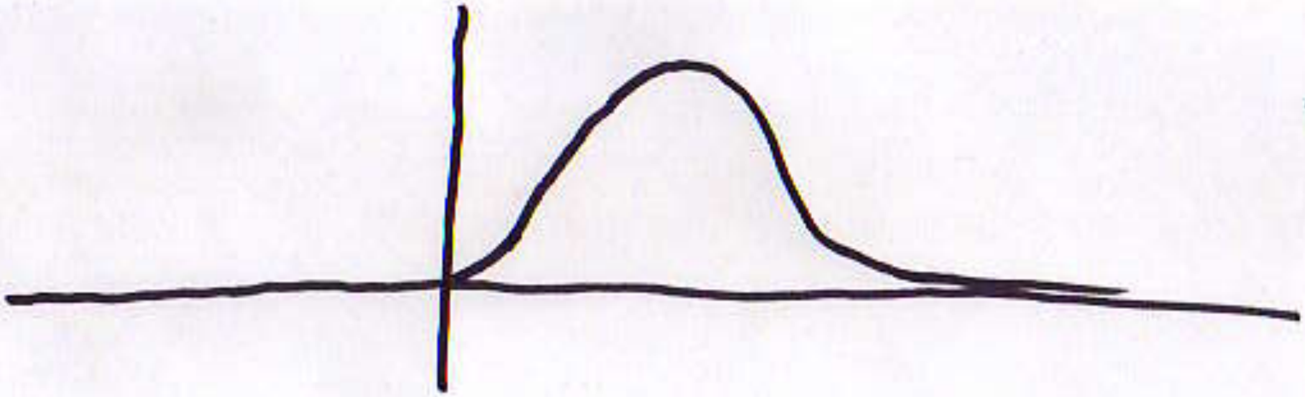
$$F(t) = \exp\left\{-\exp\left[-\left(\frac{t-b}{a}\right)\right]\right\} \quad \begin{matrix} -\infty < t < \infty \\ a > 0 \\ b \in \mathbb{R} \end{matrix}$$



2°. Frechet

$$F(t) = \begin{cases} 0 & t \le b \\ \exp\left\{-\left(\frac{t-b}{a}\right)^{-\alpha}\right\}, & t > b \end{cases}$$

$a > 0,\ b \in \mathbb{R},\ \alpha > 0$

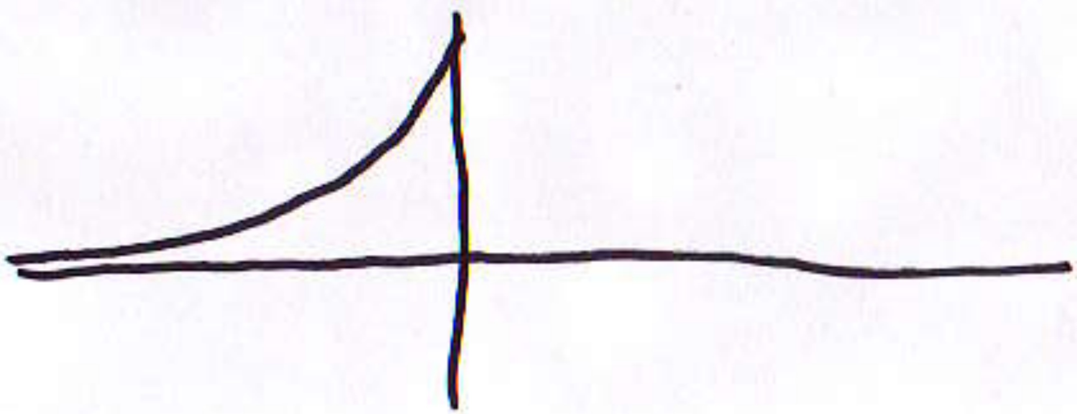


Frechet

3° Weibull

$$F(t) = \begin{cases} \exp\left\{-\left[-\left(\frac{t-b}{a}\right)^{\alpha}\right]\right\} & t < b \\ 1 & t \geq b \end{cases}$$

$a > 0, b \in \mathbb{R}, \alpha > 0$



Free (Ben Arous – V.)

$(\mathcal{A}, \varphi)$ v. Neumann alg. with normal state

P, Q hermitian projections in $\mathcal{A}$

$P \wedge Q$ = proj. onto $P\mathcal{H} \cap Q\mathcal{H}$

$P \vee Q$ = proj. onto $\overline{P\mathcal{H} + Q\mathcal{H}}$

P, Q free $\Longrightarrow$ $\varphi(P \wedge Q) = (\varphi(P) + \varphi(Q) - 1)_+$

$\varphi(P \vee Q) = \min(\varphi(P) + \varphi(Q), 1)$

Ando Spectral Order

replace $X = X^*$ in $(\mathcal{A}, \varphi)$ by projection-valued process

$$(-\infty, \infty) \ni a \longrightarrow E(X; (-\infty, a])$$

$$X \prec Y \overset{\text{def}}{\Longleftrightarrow} E(X; (-\infty; a]) \geq E(Y; (-\infty, a])$$

Ando - max

$$E(X \vee Y; (-\infty, a]) = E(X; (-\infty, a]) \wedge E(Y; (-\infty, a])$$

Distribution of X is

$$\mu_X(\omega) = \varphi(E(X;\omega)), \ \underset{\text{Borel}}{\omega \subset \mathbb{R}}$$

μ_X probability measure on $\mathbb{R}$

Free Max-convolution X, Y free

$$\mu_X \boxtimes \mu_Y = \mu_{X \vee Y}$$

If $F_\mu(t) = \mu((-\infty, t])$

$$F \boxtimes G(t) = (F(t) + G(t) - 1)_+$$

operation on distribution functions.

Free Max-stable Laws

equivalent

$$\underbrace{\mu \boxtimes \cdots \boxtimes \mu}_{n} = T_* \mu$$

$T(t) = a_n t + b_n, \quad a_n > 0$

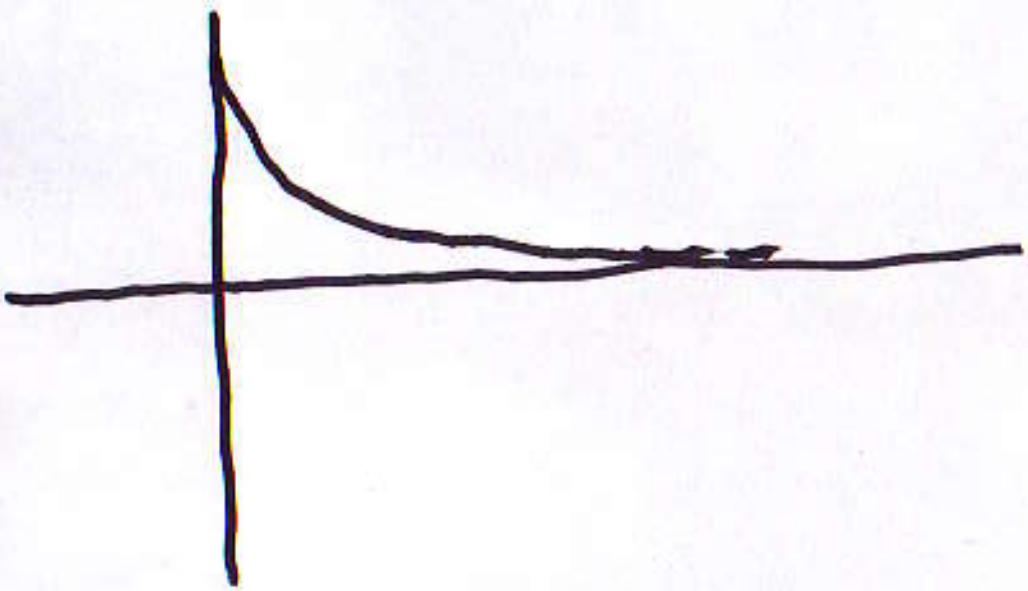
Classification analogous to classification of classical (with some differences ...)

so-called generalized Pareto distributions

The free Max-stable Laws are affine transforms $(at+b,\ a>0)$ of :

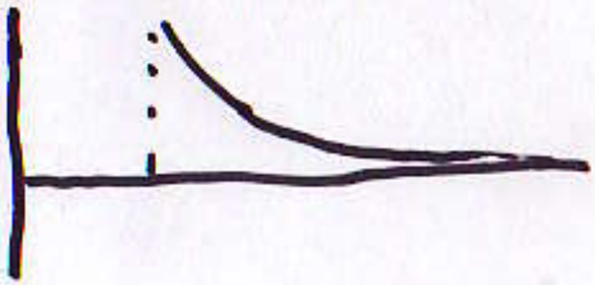
1°. exponential distribution

$$F(t) = (1 - e^{-t})_+$$



2°. Pareto distribution

$$F(t) = (1 - t^{-\alpha})_+ , \quad \alpha > 0$$

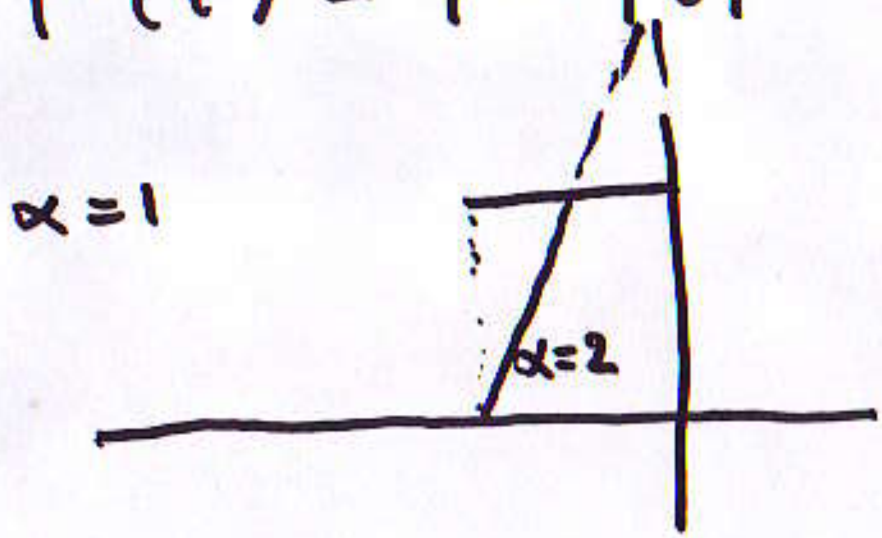


$$\alpha t^{\alpha - 1}, \ t \geq 1$$

3°. Beta law

$$F(t) = 1 - |t|^{\alpha} \text{ if } -1 \leq t \leq 0$$

$$\alpha > 0$$



$$\alpha |t|^{\alpha - 1}$$

- <u>Max-domains of attraction</u>:
  Classical and Free coincide
  (analogue of Bercovici-Pata)
- <u>Factorization Correspondence</u> (Hasebe-Simon-Wang)
  roughly: classical = free $\times$ $\underbrace{\text{some } \Gamma \text{ related}}_{\text{classical indep}}$
  (analogue of Hasebe-Kuznetsov)

- Free Max-stable laws $\;\;\;\;$ <u>Free</u>
  $\|$
  limit laws in Peaks over Thresholds $\;\;\;\;$ <u>Classical</u>
  (de Haan and Balkema
  "Residual lifetime at great age")

X r.v., u threshold $\Pr(X > u) > 0$

$(X - u \mid X \geq u)$ limit of affine transforms of distribution as $u \uparrow$

$$F^{[u]}(x) = \frac{F(u+x) - F(u)}{1 - F(u)}, \quad x > 0$$

limit of
$F^{[u]}(a_u x + b_u)$ as $F(u) \uparrow 1$.

Free Extremal Projection valued Process over a Set (particular case)

(M, τ) v. Neumann algebra with normal faithful tracial state

(Ω, Σ, μ) probability measure on Ω

1°. $\Sigma \ni \omega \longrightarrow P(\omega) \in \mathrm{Proj}(M)$

$\omega_1, \dots, \omega_n$ disjoint $\Rightarrow P(\omega_1), \dots, P(\omega_n)$ free

2°. $\omega = \bigcup_{1 \le j \le n} \omega_j \Rightarrow P(\omega) = \bigvee_{1 \le j \le n} P(\omega_j)$

3°. $\tau(P(\omega)) = \mu(\omega)$.

Realization

(M,τ) containing $L^\infty(\Omega,\Sigma,\mu)$

so that $\tau|L^\infty$ is $\int \cdot \, d\mu$

$C \in M$ circular element

$\{C, C^*\}$, $L^\infty(\Omega, \Sigma, \mu)$ freely indep.

$P(\omega)$ = range projection of $C\chi_\omega C^*$

$\omega \longrightarrow P(\omega)$

has properties $1^\circ - 3^\circ$.

<u>Remark</u> $\Pi(\omega) = C\chi_\omega C^*$, $\omega \in \Sigma$

free Poisson process over Ω ...

Boolean (J. Garza Vargas – V.)

$(\mathcal{H}_1, \xi_1)\ (X_i)_{i\in I}$, $(\mathcal{H}_2, \xi_2)\ (Y_j)_{j\in J}$

$\mathcal{H} = (\mathcal{H}_1 \ominus \mathbb{C}\xi_1) \oplus \mathbb{C}\xi \oplus (\mathcal{H}_2 \ominus \mathbb{C}\xi_2)$

V_k isometry $\mathcal{H}_k \to (\mathcal{H}_k \ominus \mathbb{C}\xi_k) \oplus \mathbb{C}\xi \hookrightarrow \mathcal{H}$

$(V_1 X_i V_1^*)_{i\in I}$, $(V_2 Y_j V_2^*)_{j\in J}$

Boolean independent in $(B(\mathcal{H}), \langle \cdot \xi, \xi \rangle)$

(Speicher - Woroudi, Bercovici)

$$E(V_1 X V_1^*; (-\infty,t]) = \begin{cases} V_1 E(X;(-\infty,t]) V_1^* & \text{if } t<0 \\ V_1 E(X;(-\infty,t]) V_1^* + P_{\mathcal{H}_2 \ominus \mathbb{C}\xi_2} & \text{if } t \geq 0 \end{cases}$$

restrict $\wedge$ to positive operators

restrict definition of Boolean max-stable laws to laws on $[0,\infty)$

restrict affine rescalings to dilations

Boolean max-convolution

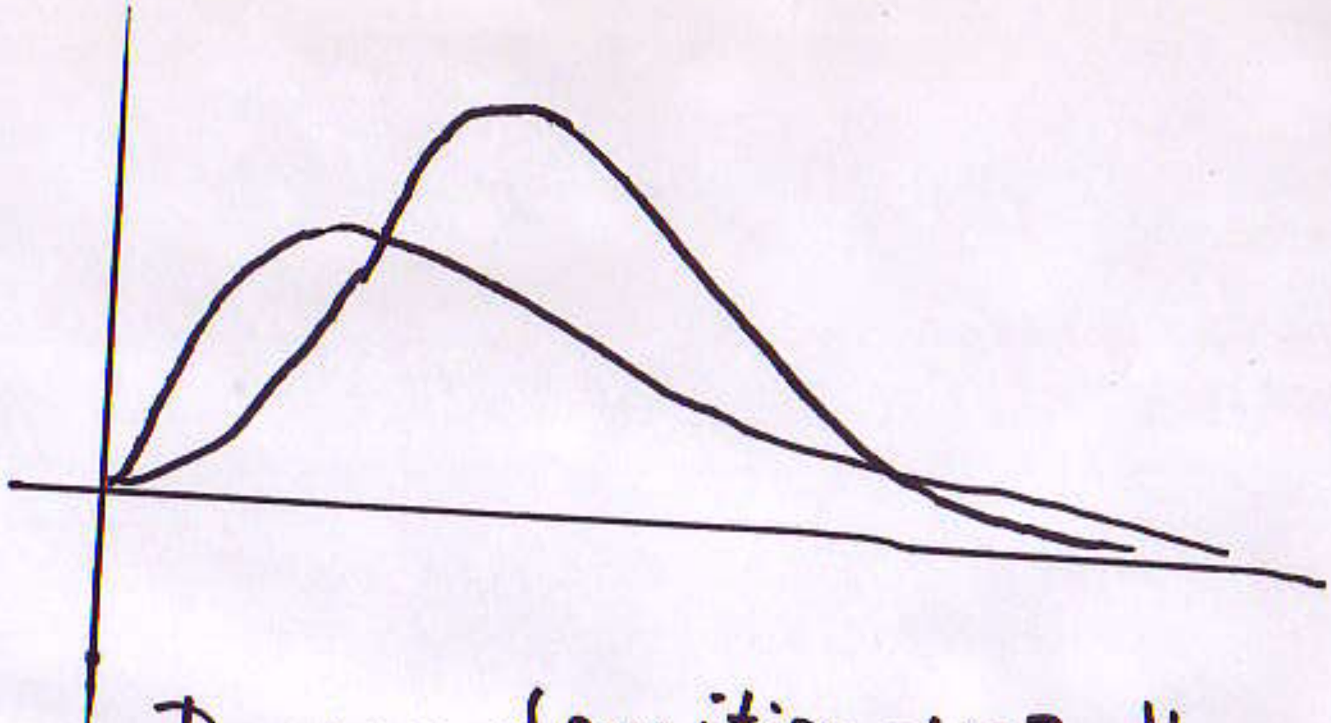
$$(F \,\textcircled{\vee}\, G)(t)^{-1} = F(t)^{-1} + G(t)^{-1} - 1.$$

Boolean max-stable laws

$$F(t) = (1 + \lambda t^{-\alpha})^{-1}, \lambda > 0, \alpha > 0, t \geq 0$$

Dagum distributions (wealth distributions)

for domains of attractions Bercovici - Pata type result with domains for Frechet laws (classical)



Dagum densities $\alpha = 2, 4$

Bi-free

Systems of Left and Right variables

(A, φ), $\left((z_i)_{i\in I}, (z_j)_{j\in J}\right) \subset A$

left var. right var.

Simplest case: two-faced pair (a, b) left right

bi-partite $[a, b] = 0$

$a = a^*$, $b = b^*$ distribution of (a, b)

probability measure with compact-supp on $\mathbb{R}^2$.

bi-free independence (a.k.a. bi-freeness)

Like defining classical independence from tensor products of Hilbert spaces, using instead free products of Hilbert spaces with state vectors and the fact that on free products of Hilbert spaces there are left and right factorizations, hence left and right operators . . .

operation on probability measures on $\mathbb{R}^2$

bi-free max-convolution

$$\mu_{(a,b)} \boxtimes\!\!\boxtimes\, \mu_{(a',b')} = \mu_{(a\vee a',\, b\vee b')}$$

operation on bi-variate distribution functions

$$F_\mu(s,t) = \mu((-\infty,s]\times(-\infty,t])$$

$$F_\mu \boxtimes\!\!\boxtimes\, F_\nu = F_{\mu \boxtimes\!\!\boxtimes \nu}$$

F bi-variate, F_1 and F_2 marginals

$$H = F \boxed{\nabla}\boxed{\nabla} G$$

$$H_j = (F_j + G_j - 1)_+ \,, \quad j = 1,2$$

$$\frac{H_1(s)\,H_2(t)}{H(s,t)} - 1 =$$

$$= \left(\frac{F_1(s)\,F_2(t)}{F(s,t)} - 1\right) + \left(\frac{G_1(s)\,G_2(t)}{G(s,t)} - 1\right)$$

if $F(s,t) > 0$, $G(s,t) > 0$, $H_1(s) > 0$, $H_2(t) > 0$

and otherwise $H(s,t) = 0$.

Problems

1°. Bi-free Max-stable laws?
Bi-free Max-∞-divisible laws?

2°. Are there classical problems which give rise to bi-free Max-laws?
(like Peaks over Thresholds for free Max-stable)

3°. Applications? Free Floodings?
Bi-free Floodings??
Boolean Floodings??

1. G. Ben Arous, D.V. Voiculescu
Free Extreme Values
Ann. Probab. 34(5), 2037-2059 (2006)

2. D.V. Voiculescu
Free Probability for Pairs of Faces IV:
Bi-Free Extremes in the Plane
arXiv: 1505.05020 v3
to appear in J. Theoretical Probability

Ref. -2

3. G. Ben Arous, V. Kargin
Free Point Processes and Free Extreme Values
Probab. Theory Relat. Fields (2010) 147:161-183

4. F. Benaych-Georges, Th. Cabanal-Duvillard
A Matrix Interpolation between Classical and Free Max Operations I. The Univariate Case
J. Theoretical Probability (2010) 23: 447-465

Ref – 3

5. R.S. Hazra, K. Maulik
Free Subexponentiality
Ann. Probab. 41 (2), 961–988 (2013)

6. J. Grela, M. A. Nowak
On relations between extreme value statistics, extreme random matrices and Peak-Over-Threshold method.
preprint. arXiv: 1711.03459

7. J. Garza Vargas, D.V. Voiculescu
Boolean Extremes and Dagum Distributions
arXiv: 1711.06227

Ref -4

8. S. Chakraborty, R. S. Hazra

Boolean convolutions and regular variation.

arXiv: 1710.11402

9. T. Hasebe, T. Simon, M. Wang

Some properties of free stable distributions.

arXiv: 1805.0133