Small perturbations of non-Hermitian matrices

Ofer Zeitouni
Based on joint work with Anirban Basak and Elliot Paquette

IPAM
May 2018
**An empirical fact**

\[ J_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}, \quad P_N(z) = \det(zI - J_N) = z^N, \quad \text{roots}=0. \]
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\end{pmatrix} \]

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\[ \hat{T}_N := U_N T_N U_N^* \] where \( U_N \) is random unitary matrix, Haar-distributed. Of course, \( \text{Spec}(\hat{T}_N) = \text{Spec}(T_N) \).
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Goes back to Trefethen et al.s - pseudo-spectrum.
A – matrix with singular values denoted $\sigma_1(A) \geq \sigma_2(A) \geq \ldots$,

$$\|A\| = \sup_{\|v\|_2 = 1} \|Av\|_2 = \sigma_1(A).$$
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If $A$ is symmetric then $\sigma_i(A) = |\lambda_i(A)|$ and $\|A\| = \max(\lambda_{\text{max}}, -\lambda_{\text{min}}).$
Background: Spectrum stability for symmetric matrices

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**Weyl inequalities**: $\sigma_{i+j-1}(A + B) \leq \sigma_i(A) + \sigma_j(B)$.

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Small Perturbations
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If \( A, B \) are Hermitian and \( \|B\| < \epsilon \) then \( |\lambda_i^{A+B} - \lambda_i^A| \leq \epsilon \).
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If \( A, B \) are Hermitian and \( \|B\| < \epsilon \) then \( |\lambda_i^{A+B} - \lambda_i^A| \leq \epsilon \).

In particular, if \( W \) is a symmetric matrix with i.i.d. centered standard Gaussian entries on and above diagonal (**Wigner matrix**), then \( \lambda_{\text{max}}(N^{-1/2}W) \to 2 \), and if \( \gamma > 1/2 \) then

\[
|\lambda_i(A_N + N^{-\gamma}W) - \lambda_i(A_N)| \to_{N\to\infty} 0.
\]
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|\lambda_i(A_N + N^{-\gamma}W) - \lambda_i(A_N)| \rightarrow_{N \rightarrow \infty} 0.
\]

No such control holds for eigenvalues of non-Hermitian matrices.
Background II: Ginibre matrices

Denote by $G_N$ matrix with i.i.d. standard complex Gaussian entries, and set $g_N = N^{-1/2} G_N$.

$\lambda_i$ - eigenvalues of $g_N$.

$L^g_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ - empirical measure of eigenvalues.
Denote by $G_N$ matrix with i.i.d. standard complex Gaussian entries, and set $g_N = N^{-1/2}G_N$.

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$L_{N}^{g} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ - empirical measure of eigenvalues.

**Theorem**

$L_{N}^{g}$ converges to the uniform measure on the unit disc.
Background II: Ginibre matrices

In fact, also

\[ \|g_N\| \to \sqrt{2} \]
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Thus, \( \|N^{-\gamma} G_N\| \to 0 \) if \( \gamma > 1/2 \).
Consider the nilpotent $N$-by-$N$ matrix

$$J_N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \\ \end{pmatrix}$$
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Eigenvalues $\lambda_i = 0$, empirical measure $n^{-1} \sum \delta_{\lambda_i} = \delta_0$. 
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\end{pmatrix}$$

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Regularization by noise II

Set $\gamma > 1/2$. 

Theorem (Guionnet-Wood-Z. '14)

Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L_A N$. Then $L_A N$ converges weakly to the uniform measure on the unit circle in the complex plane. Thus, $L_J N = \delta_0$ but for a vanishing perturbation, $L_A N$ has different limit.

Earlier version - Davies-Hager '09 (Generalization to i.i.d. $G_N$: Wood '15.)
Set $\gamma > 1/2$.

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Set $A_N = J_N + N^{-\gamma} G_N$, empirical measure of eigenvalues $L_N^A$. Then $L_N^A$ converges weakly to the uniform measure on the unit circle in the complex plane.
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Thus, $L_N^{J_N} = \delta_0$ but for a vanishing perturbation, $L_N^A$ has different limit. Earlier version - Davies-Hager ’09
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Theorem (Guionnet-Wood-Z. ’14)

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Thus, $L^{J_N}_N = \delta_0$ but for a vanishing perturbation, $L^A_N$ has different limit.

Earlier version - Davies-Hager ’09
(Generalization to i.i.d. $G_N$: Wood ’15.)
What is going on?

\[ J^\delta_N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \vdots & \vdots & \vdots & \vdots & 0 & 1 \\
\delta_N & \vdots & \vdots & \vdots & \vdots & \vdots & 0
\end{pmatrix} \]

Characteristic polynomial:
\[ P_N(z) = \det(zI - J^\delta_N) = z^N \pm \delta_N. \]

Roots:
\[ \{ \delta_n^i / N \}_{n=1}^N. \]

If \( \delta_N = 0 \) then
\[ L^{J^\delta_N} = \delta_0. \]

If \( \delta_N \to 0 \) polynomially slowly then
\[ L^{J^\delta_N} \] converges to uniform on circle.

Why is this particular perturbation picked up? (General criterion - Guionnet, Wood, Z.)
What is going on?

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0 & 0 & 1 & 0 & \cdots & 0 \\
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\delta_N & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}
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\[ J_{\delta}^N = \begin{pmatrix}
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\end{pmatrix} \]

Characteristic polynomial:

\[ P_N(z) = \det(zI - J_{\delta}^N) = z^N \pm \delta_N. \]

Roots: \( \{\delta_{\delta}^{1/N} e^{2\pi i/N}\}_{i=1}^N \).
What is going on?

$$J_N^\delta = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & 1 \\
\delta_N & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}$$

Characteristic polynomial:

$$P_N(z) = \det(zI - J_N^\delta) = z^N \pm \delta_N.$$ 

Roots: \(\{\delta_N^{1/N} e^{2\pi i/N}\}_{i=1}^N\).

If \(\delta_N = 0\) then \(L_N^{J_N^\delta} = \delta_0\).
What is going on?

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\[ J_N^\delta = \begin{pmatrix}
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Roots: \( \{\delta_N^{1/N} e^{2\pi i/N}\}_{i=1}^N \).

If \( \delta_N = 0 \) then \( L_{N\delta_N}^N = \delta_0 \).

If \( \delta_N \to 0 \) polynomially slowly then \( L_{N\delta_N}^N \) converges to uniform on circle.

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Noise Stability-Block Nilpotent

A generalization: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N) \log N$, eigenvalue $c_i(N)$. 

\[
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_\ell(N)
\end{bmatrix}
\]
A generalization: $B^i = B^i(N)$ - Jordan blocks, dimension $a_i(N) \log N$, eigenvalue $c_i(N)$.

$$A_N = \begin{bmatrix}
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Simulations...
$A_N$ block matrix, each block of size $a_i \log N$. 
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$B_N = A_N + N^{-\gamma} G_N$. 

Theorem (Feldheim, Paquette, Z. '15) 
For $\gamma > 1$ and $\ell(N) = o(N)$, 
$d(\mathbb{L}_{B_N}, \mu_N) \to 0$ as $N \to \infty$ 
Analogous result for $\gamma \in (1/2, 1]$ if collection of circles “does not spread too much” (e.g., olympics rings example OK).
$A_N$ block matrix, each block of size $a_i \log N$. $c_i$ on diagonal.

$B_N = A_N + N^{-\gamma}G_N$.

Define $r_i(N) = e^{(-\gamma + 1/2)/a_i} \leq 1$. Set $\mu_N = \frac{1}{N} \sum_{i=1}^{\ell(N)} a_i \log N \nu_{c_i,r_i}$ where $\nu_{c,r}$ uniform on circle of radius $r$ centered on $c$. 

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For $\gamma > 1$ and $\ell(N) = o(N)$,

$$d(L_N^{B_N}, \mu_N) \to_{N \to \infty} 0$$
Noise Stability

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For \( \gamma > 1 \) and \( \ell(N) = o(N) \),

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Analogous result for \( \gamma \in (1/2, 1] \) if collection of circles “does not spread too much” (e.g., olympics rings example OK).
More general models?

Figure: The eigenvalues of $J_N + J_N^2 + N^{-\gamma} G_N$, with $N = 4000$ and various $\gamma$. On left, actual matrix. On the right, $U_N(J_N + J_N^2)U_N^*$. 
More general models?

Figure: The eigenvalues of $D_N + J_N + N^{-\gamma} G_N$, with $N = 4000$ and various $\gamma$. Top: $D_N(i, i) = -1 + 2i/N$. Bottom: $D_N$ i.i.d. uniform on $[-2, 2]$. On left, actual matrix. On the right, $U_N(D_N + J_N)U_N^*$. 
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = D_N + J_N, \ M_N = T_N + N^{-\gamma} G_N, \ \gamma > 1/2. \]

d\_i iid uniform on \([-1, 1]\).
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\[ d_i \text{ iid uniform on } [-1, 1]. \]

Then \( L_N \to \mu, \mu \text{ explicit: log-potential of } \mu \text{ at } z \text{ is } (E \log |z - d_1|) \lor 0). \]
More general models

Theorem (Basak, Paquette, Z. ’17)

\[ T_N = \sum_{i=0}^{k} a_i J_N^i \] (Toeplitz, finite symbol, upper triangular). Then,

\[ L_N \to \text{Law of} \sum_{i=0}^{k} a_i U^i \]

where \( U \) is uniform on unit circle.
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where \( U \) is uniform on unit circle.

Extends to twisted Toeplitz \( T_N(i, j) = a_i(j/N), i = 1, \ldots, k, a_i \) continuous:

\[ L_N \rightarrow \int_{0}^{1} \text{Law of} \ \sum_{i=0}^{k} a_i(t) U^i \]
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Confirms simulations and predictions (based on pseudo-spectrum) of Trefethen et al.s. Some two-diagonal Toeplitz cases studied by Sjöstrand and Vogel (2016)
Recent extensions

• Non upper triangular models:

\[ \sum_{k=2}^{n} a_{i j} = J_N - a_{i j} N \]

Then,

\[ L_N \to \text{Law of } k \sum_{i=0}^{\infty} a_{i j} U_i \]

where \( U \) is uniform on unit circle.

Main issue - Toeplitz determinant of unperturbed matrix requires work, e.g. Widom's theorem.

• Extension to twisted Toeplitz - ??
Recent extensions

- Non upper triangular models:

**Theorem (Basak, Paquette, Z. ’18)**

\[ T_N = \sum_{i=-k_1}^{k_2} a_i J_N^i \quad (\text{Toeplitz, finite symbol, } J_N^{-1} = J_N^T.) \text{ Then,} \]

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*where* \( U \) *is uniform on unit circle.*
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• Extension to twisted Toeplitz - ??
Outliers

\[ J_N + N^{-\gamma} G_N \]

\[ J_N + J_N^2 + N^{-\gamma} G_N \]

Outliers are random. What is structure of outliers?

- \( J_N + N^{-\gamma} G_N \): outliers are zeros of a limiting Gaussian field, all inside disc.
- \( J_N + J_N^2 + N^{-\gamma} G_N \): Outliers are roots of a Gaussian field, limit of terms involving a single Gaussian in expansion of char. pol.
- \( \gamma = 0.75 \)
- \( \gamma = 1.75 \)
- \( \gamma = 4.00 \)
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\[ J_N + N^{-\gamma} G_N \]

\[ J_N + J_N^2 + N^{-\gamma} G_N \]
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\[
J_N + N^{-\gamma} G_N = J_N + J_N^2 + N^{-\gamma} G_N
\]
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- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.
- $J_N + J_N^2 + N^{-\gamma} G_N$: Write $z I + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N)$:
Outliers are random. What is structure of outliers?

- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.
- $J_N + J_N^2 + N^{-\gamma} G_N$: Write $zI + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N)$:
  - No outliers in $\{ z : |\lambda_i(z)| > 1, i = 1, 2 \}$
Outliers are random. What is structure of outliers?

- $J_N + N^{-\gamma} G_N$: outliers are zeros of a limiting Gaussian field, all inside disc.
- $J_N + J_N^2 + N^{-\gamma} G_N$: Write $z l + J_N + J_N^2 = (\lambda_1(z) - J_N)(\lambda_2(z) - J_N)$:
  - No outliers in $\{z : |\lambda_i(z)| > 1, i = 1, 2\}$
  - In $\{z : |\lambda_1(z)| > 1 > |\lambda_2(z)|\}$, outliers are roots of a Gaussian field, limit of terms involving a single Gaussian in expansion of char. pol.
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  - In $\{z : 1 > |\lambda_1(z)| > |\lambda_2(z)|\}$, outliers are roots of limit of terms involving a product of two Gaussians in expansion of char. pol.
Ongoing work - Outliers

Let $d = d(z)$ be such that $|\lambda_d| < 1 < |\lambda_{d+1}|$. Then outliers are zeros of the field determined by terms in char. pol. which are product of Gaussians.

Expect this to be a limiting Gaussian field.

• Expect to extend to general finite banded Toeplitz.

• Should depend on nature of noise.
Conjectures and ongoing

Ongoing work - Outliers

- Upper triangular, Toeplitz, finite symbol

\[ z + \sum_{i=0}^{k} a_i \lambda^i = \prod_{i=1}^{k} (\lambda(z) - \lambda), \quad |\lambda_i| \leq |\lambda_{i+1}| \]

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Conjectures and open problems - Spectrum limits

- General twisted Toeplitz symbol: Expect mixture as in upper triangular case.
- Main obstacle: compute determinant of twisted Toeplitz with non-zero winding number.
- Toeplitz with infinite symbol - depends on rate of decay?
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Śniady’s theorem

Assume $A_N \to^* a$. 

Theorem (Śniady '02)

$$\lim_{t \to 0} \lim_{N \to \infty} L_{A_N}(t) N = \nu_a.$$ 

In particular, some sequence of noise regularizes empirical measure to the Brown measure. 

Builds on regularization ideas of Haagerup. 

Main ingredient of proof compares the singular values $\Sigma_{A_N}(t) = (\sigma_{A_1},...,\sigma_{A_N})$ of $A_N + tN^{-1/2}G_N$ to the singular values $\Sigma_0(t) = (\sigma_1,...,\sigma_N)$ of $tN^{-1/2}G_N$; by coupling the SDEs for the evolution of $\Sigma_0$, $\Sigma_A$, for $f$ coordinate-wise increasing,

$$N^{-1} \text{tr}(f(\Sigma_{A}(t))) \geq N^{-1} \text{tr}(f(\Sigma_0(t))).$$

This gives required control of the determinant; Second part of theorem follows by diagonalization argument. 

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Several methods, all use Logarithmic potential:

\[ U_\mu(z) = \int \log |z - x| \mu(dx) \]

and Girko’s Hermitization (Girko, Bai, Gotze-Tikhomirov, Tao-Vu, ...)

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For \( L_{BN}^B \), \( U_{LB_N}^B(z) = \frac{1}{2N} \log \det(z - B_N)(z - B_N)^* \).
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Recall $T_N = M_N + N^{-\gamma} G_N$, $\gamma > 1/2$. 
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Write $zI - M_N = U\Sigma_N V^*$, $\Sigma_N$ - diagonal, singular values, arranged non-decreasing, and then

$$\Sigma = \Sigma_N = \begin{pmatrix} S_N & \\ & B_N \end{pmatrix}, \quad N^{-\gamma} G_N = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$ 

where $S_N$ has dimension $N^* \times N^*$. 

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where \( S_N \) has dimension \( N^* \times N^* \).

Define \( N^* \) as

\[
\sup\{i \geq 1 : \Sigma_{ii}(Z) \leq \epsilon^{-1}_N N^{-\gamma}(N - i)^{1/2}\}, \quad \epsilon_N = N^{-\eta}
\]
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**Theorem (Basak-Paquette-Z. ’17 - Deterministic equivalence)**

If $N^* = o(N/\log N)$ then

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So only need to understand small singular values of $M_N$. 
The Toeplitz case - transfer matrices

Reformulation: \( M_N = \sum_{i=1}^{k} a_i J^i_N \), singular values?
The Toeplitz case - transfer matrices

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Set \( V = \{ x \in \mathbb{R}^N : ((M_N - zI_N)x)_j = 0, j = 1, \ldots, N - k \} \). Parametrized by \( x_1, \ldots, x_k \) and transfer matrices \( T_j(z) \)

\[(x_\ell)^{j+k}_{\ell=j+1} = T_j(z)(x_\ell)^{j+k-1}_{\ell=j}.\]
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$\mu_i(z)$ - Lyapunov exponents for $\prod_i T_i(z)$. If $T_i(z) = T(z)$ (Toeplitz) - modulii of eigenvalues of $T(z) = T_j(z)$, which are the roots of the symbol $P(z) = \sum_{i=0}^{k} a_i z^i$. The BPZ theorem can then be reformulated.

Theorem

$$U_{LN}(z) \to \frac{1}{2\pi} \int_0^{2\pi} \log |P(e^{i\theta}) - z| dz$$

and therefore

$$U_{LN}(z) \to \int_0^{2\pi} \log |P(e^{i\theta}) - z| dz = \log |a_k| + \sum_{i=1}^{k} [\log |\mu_i(z)|] \lor 0.$$

Ofer Zeitouni
BPZ - two diagonal case

Take $M_N = -zl + D_N + J_N$. 
Elements in proofs

BPZ - two diagonal case

Take $M_N = -zI + D_N + J_N$.

Lemma

Take $d_i = D_{ii}$ iid.

a) If $E \log |z - d_1| < 0$ then

$$N^{-1} \log \sigma_N(M_N) \to E \log |z - d_1|,$$

$$\sigma_{N-1}(M_N) \geq N^{-C}.$$  

b) If $E \log |z - d_1| > 0$ then $\sigma_N(M_N) \geq N^{-C}$.
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In particular $N^* = 1$. 
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Since

\[
N^{-1} \sum_{i=1}^{N} \log \sigma_i(M_N) = N^{-1} \log \det(M_N),
\]

get that log-potential converges to \((E \log |z - d_1|) \vee 0\).
Suppose $D_N = 0$, $d_i = -z$ is constant.
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Lemma

If $|z| \neq 1$, $\sigma_{N-1}(M_N) \geq C(z)(\log N)^{-1}$. 

This reconstructs the GWZ result!
Elements in proofs

BPZ - two diagonal case - \( M_N = -zI + D_N + J \)

Suppose \( D_N = 0, d_i = -z \) is constant.

**Lemma**

1. If \(|z| \neq 1\), \( \sigma_{N-1}(M_N) \geq C(z)(\log N)^{-1} \).
2. If \(|z| < 1\) then \( C_2 < \sigma_N(M_N)|z|^{-N} < C_1 \).

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Elements in proofs

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\[ \begin{align*}
\text{If } |z| \neq 1, \ & \sigma_{N-1}(M_N) \geq C(z)(\log N)^{-1}. \\
\text{If } |z| < 1 \ & \text{then } C_2 < \sigma_N(M_N)|z|^{-N} < C_1. \\
\text{If } |z| > 1 \ & \text{then } \sigma_N(M_N) > C/\log N.
\end{align*} \]

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BPZ - two diagonal case - \( M_N = -zI + J_N \)

Approximate singular vector construction
BPZ - two diagonal case - $M_N = -zI + J_N$

Approximate singular vector construction Assume $|z| < 1$. Set $v_1 = 1$, $v_k = zv_{k-1}$. Then $(M_Nv)_k = \begin{cases} 0, & k \leq N - 1, \\ -z^N, & k = N. \end{cases}$
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\[
\sigma_{N-1}(M_N) = \sup_w \inf_{x: \langle x, w \rangle = 0} \frac{\|M_Nx\|_2}{\|x\|_2} \geq \inf_{x: \langle x, v \rangle = 0} \frac{\|M_Nx\|_2}{\|x\|_2}
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$$\sigma_{N-1}(M_N) = \sup_w \inf_{x : \langle x, w \rangle = 0} \frac{\|M_Nx\|_2}{\|x\|_2} \geq \inf_{x : \langle x, v \rangle = 0} \frac{\|M_Nx\|_2}{\|x\|_2}$$

Let $\|x\|_2 = 1$, for $k \leq N - 1$, $x_{k+1} = zx_k + (M_Nx)_k$, hence for $a \in \mathbb{C}$,

$$x_k - av_k = (x_1 - av_1)z^{k-1} + \sum_{j=1}^{k-1} (M_Nx)_j z^{k-j}$$
Approximate singular vector construction Assume $|z| < 1$. Set $v_1 = 1$, $v_k = zv_{k-1}$. Then $(M_Nv)_k = \begin{cases} 0, & k \leq N - 1, \\ -z^N, & k = N. \end{cases}$

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But if $\|x\|_2 = 1$ and $\langle x, v \rangle = 0$, we have that $1 \leq \|x - av\|$. Choose $x_1 - av_1 = 0$, get a lower bound on $\|M_Nx\|$ in terms of $\|x - av\|$ of the form

$$\|M_Nx\| \geq C(z)/ \log N \Rightarrow \sigma_{N-1}(M_N) \geq C(z)/ \log N.$$
To control $\sigma_N(M_N)$, recall

$$\sigma_N(M_N) = \inf_x \frac{\|M_Nx\|_2}{\|x\|_2}$$
BPZ - two diagonal case - $M_N = -zI + D_N + J_N$

To control $\sigma_N(M_N)$, recall

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Write $x = av/\|v\|_2 + by$ with $\langle y, v \rangle = 0$, $\|y\|_2 = 1$. Then with $\pi$ the projection to first $N-1$ coordinates,

$$\|M_Nx\|_2^2 = \|\pi M_N y\|_2^2 |b|^2 + ((M_Nx)_N)^2.$$
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\]

Since \( a^2 + b^2 = 1 \), either \( b \) is large or \( M_Nv \) is small. Algebraic manipulations give

\[
\sigma_N(M_N) \geq C \inf_{x: \langle x, v \rangle = 0} \frac{\|\pi M_Nx\|_2}{\|x\|_2} \cdot \frac{\|M_Nv\|_2}{\|v\|_2} \geq \frac{c |Z|^N}{\log N}
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Upper bound: plug $x = v/\|v\|_2$. 
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Upper bound: plug $x = v/\|v\|_2$. Slowly varying diagonals: apply this argument in blocks.
Elements in proofs

BPZ - two diagonal case - $M_N = -zI + D_N + J_N$

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$$\sigma_N(M_N) \geq C \inf_{x: \langle x, v \rangle = 0} \frac{\|\pi M_Nx\|_2}{\|x\|_2} \cdot \frac{\|M_Nv\|_2}{\|v\|_2} \geq \frac{c|z|^N}{\log N}$$

Upper bound: plug $x = v/\|v\|_2$.

Slowly varying diagonals: apply this argument in blocks.

Multi diagonal: choose appropriate basis according to composition of eigenvalues of transfer matrix.