Random Matrix Theory beyond the Mean-Field Class

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Gaussian Orthogonal Ensemble (GOE):
$H = (h_{ij})_{i,j=1}^{N}$, $h_{jk} = h_{kj}$ are $N(0, 1/N)$, i.e., $\mathbb{E}h_{jk} = 0$, $\mathbb{E}|h_{jk}|^2 = \frac{1}{N}$.

Wigner ensembles: h_{ij} are independent (not necessarily normal).
The e-values $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$ are order one.
Wigner semicircle law: Density of eigenvalues $\rho(x) \sim \sqrt{4 - x^2}$

Wigner-Dyson-Mehta conjecture

For Wigner ensembles, $\mathbb{P}\left(\lambda_i \sim E + \frac{x_1}{N}, \lambda_j \sim E + \frac{x_2}{N}\right)$ are independent of the matrix law.

Life after the Wigner-Dyson-Mehta conjecture.

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Wigner's vision concerning the universality of random matrix statistics

The local spectral statistics of highly correlated quantum systems are given by the random matrix statistics of the same symmetry type.
Random matrix statistics are "universal" probability laws for highly correlated systems.

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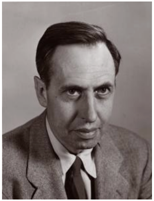
- The Wigner ensemble is a very special class of mean-field models. Other mean-field models:
  - sparsity : the adjacency matrices of Erdős-Rényi graphs.
  - correlations : d-regular graphs or other models with summable decay of correlation functions.
- Beyond mean-field: random Schrödinger operators (Anderson models) and random band matrices.

The global laws (e.g., the semicircle) are model-dependent and have no physical meaning.

If you admit that the Wigner ensemble gives a completely wrong answer for the level density, why do you believe any of the other predictions of random-matrix theory?

George Uhlenbeck



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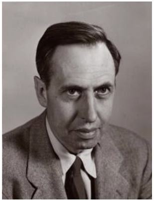


If you admit that the Wigner ensemble is a mean-field model, why do you believe that it models non-mean field systems?

Universality for non mean field models.

Eigenvalues of Δ on a domain (or random Schrödinger operators)

- GOE statistics in large energy limit if the corresponding classical dynamics are chaotic (Quantum Chaos conjecture Bohigas-Giannoni-Schmit, 1984)
- Poisson statistics if the corresponding classical dynamics are integrable—- Berry-Tabor (1977)
- **Anderson (1958):** $H = -\Delta + \lambda V_\omega, V_\omega$ random potential on $\mathbb{Z}^d$. localization regime: Frohlich-Spencer, Minami, Aizenman-Molchanov.

No results for the GOE statistics.

Band matrix, a variation of the Anderson model

Band matrix: $H = (h_{ij})_{i,j=1}^N$. With periodic distance:

$$h_{ij} = 0, |i-j| > W/2; \quad \mathbb{E}h_{ij} = 0,\ \mathbb{E}h_{ij}^2 = \frac{1}{W}$$

E.g., 4×4 band matrix: symmetric matrix of the form

$$H = \begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{21} & h_{22} & h_{23} & 0 \\ 0 & h_{32} & h_{33} & h_{34} \\ h_{41} & 0 & h_{43} & h_{44} \end{pmatrix}$$

Band matrices are not mean-field models. Mean-field: h_{ij} are of the same order for essentially all i, j.

Conjecture (Fyodorov-Mirlin): GOE statistics if $W \gg \sqrt{N}$ and Poisson statistics if $W \ll \sqrt{N}$. In dimension $d \geq 3$, GOE statistics hold if W is large enough independent of N.

Theorem [Bourgade-Erdos-Y-Yin, 2015] Local statistics in the bulk are given by the GOE statistics if N/W is bounded as $N \to \infty$ $(d = 1)$.

Theorem [Bourgade-Y-Yin, 2018]

Suppose the distribution of h_{ij} satisfies some mild technical conditions. Then the local statistics in the bulk are given by the GOE statistics if $W > N^{3/4}$ as $N \to \infty$ $(d = 1)$. The eigenvectors are delocalized and satisfy a version of probabilistic quantum unique ergodicity.

Shcherbina (2014, 2017), Bao-Erdos (2015), Disertori-Pinson-Spencer : supersymmetric method; for complex Gaussian (but not real) band matrices.

Three step strategy for the universality of mean field models.

1. **A priori estimate, a local laws (“local” means $\eta \sim N^{-1+\varepsilon}$):**

$$G_{aa}(z) := \left(\frac{1}{H-z}\right)_{aa} = \sum_j \frac{|u_j(a)|^2}{\lambda_j - E - i\eta},$$

$$m_N(z) := \frac{1}{N} \operatorname{Tr} G(z),\ z = E + i\eta.$$

2. **Universality for Gaussian divisible ensembles:** $H(t) = A + \sqrt{t}$ GOE
   Statistics of $H(t)$ and $H(\infty)$ coincide for $t \gg N^{-1}$.

3. **Stability or Comparison:** statistics of $H(0)$ and $H(t)$ coincide for $t \ll N^{-1/2}$ (use the matrix structure).

Step 1: a local law for band matrix (EKYY 2012):

Stieltjes transform of the semicircle law: $m_{sc}(z) = \int \frac{\varrho_{sc}(x)\mathrm{d}x}{x-z}$

$$\max_{i,j} |G_{ij}(z) - \delta_{ij} m_{sc}(z)| \ll \frac{1}{\sqrt{W\eta}}, \text{ with high prob if } \eta \gg W^{-1}.$$

delocalization up to the scale W: w.h.p. $\max_k |v_i(k)| \leq \frac{C}{W}$

no good estimate for $\eta \leq 1/W$ for band matrices. Conjecture:

$$|G_{1x}|^2 \sim \frac{1}{N} \sum_{p \in \frac{2\pi}{N}\mathbf{Z}} \frac{e^{\mathrm{i}px}}{\eta + (Wp)^2}, \quad \text{random walks at the scale } W$$

$$\begin{aligned} |G_{1x}|^2 &\sim \frac{1}{W\sqrt{\eta}} \mathbf{1}(|x| \leq W/\sqrt{\eta}) \quad \text{if } \eta \geq (W/N)^2, \\ &\sim 1/N\eta \qquad\qquad\qquad\qquad \text{if } \eta < (W/N)^2 \end{aligned}$$

Step 2: Matrix Brownian Motion

$$\mathrm{d}H_t = \frac{1}{\sqrt{N}}\mathrm{d}\mathcal{B}_t, \quad \mathcal{B}_{ij}(t) : \text{symm. indep. BM}, \quad H_t \sim H_0 + \sqrt{t}\,\text{GOE}$$

$$\mathrm{d}\lambda_k = \frac{\mathrm{d}B_k}{\sqrt{N}} + \frac{1}{N}\sum_{\ell \neq k} \frac{1}{\lambda_k - \lambda_\ell}\mathrm{d}t$$

Dyson Brownian Motion (1962): Time to local equilibrium is N^{-1}. [Erdos-Schlein-Y-Yin, Landon-Y].

Step 3 (Continuity of DBM or Comparison): Consider Wigner matrices with different second moments. For reasonable test function f,

$$\mathbb{E}f(m(z,t)) - \mathbb{E}f(m(z,0)) \sim tO(N^{2-2\times 1/2}) = tO(N).$$

Since $t \gtrsim 1/N$, $\mathbb{E}f(m(z,t)) - \mathbb{E}f(m(z,0)) \not\leq o(1)$.

Direct comparison method will not work for the universality of band matrices.

Question: What is the main mechanism for the universality of non-mean field models (e.g., band matrices, Anderson models)?

Quantum unique ergodicity conjecture (QUE) [Rudnick-Sarnak]:

$(\psi_k)_{k\geqslant 1}$ eigenfunctions for any negatively curved compact Riemannian manifolds are equidistributed for any large energy limit.

$$\int a(x)|\psi_j(x)|^2 \mathrm{dVol}(x) \to \int a(x)\mathrm{dVol}(x)$$

Quantum Ergodicity Theorem: Averaged version of QUE [Snirelman, Colin de Verdiere, Zelditch], Anantharaman. Arithmetic QUE (Rudnick, Sarnak, Bourgain, Lindenstrauss, Soundararajan)

QUE hold for GUE/GOE with high probability because eigenvectors are uniformly distributed on $O(N)$.

From QUE to Universality—Mean field reduction (BEYY)

Write $H_e = \begin{pmatrix} A & B^* \\ B & D-e \end{pmatrix}$ and $H = H_{e=0}$, where A a $W \times W$ Wigner matrix.

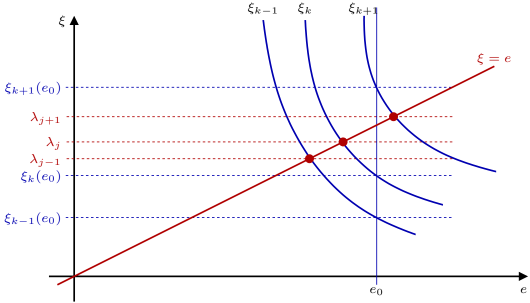
$$\begin{pmatrix} A & B^* \\ B & D \end{pmatrix} \psi_j = \lambda_j \psi_j := \lambda_j \begin{pmatrix} w_j \\ p_j \end{pmatrix} \Longrightarrow \left(A - B^* \frac{1}{D-\lambda_j} B\right) w_j = \lambda_j w_j$$

Consider the eigenvector eq.: $Q_e v = \left(A - B^* \frac{1}{D-e} B\right) v = \xi_k(e) v$

$$H\begin{pmatrix} w_j \\ p_j \end{pmatrix} = \lambda_j \begin{pmatrix} w_j \\ p_j \end{pmatrix},\ Q_e v = \Big(A - B^* \tfrac{1}{D-e} B\Big) v = \xi_k(e) v$$

$$\xi_k(e) = e \Longleftrightarrow \xi_k(e) = \lambda_j,\ \frac{\mathrm{d}\xi_k(e)}{\mathrm{d}e} = \frac{-\|p_j\|_2^2}{\|w_j\|_2^2} = \frac{W-N}{W} \text{ if QUE for } H_e \text{ holds.}$$

QUE for $H_e \Longrightarrow$ law of $\{\xi_k(e_0)\}$ = law of $\{\lambda_j\}$ up to a scaling.



mean field univ. + QUE $\Longrightarrow$ non-mean field (band matrix) univ.

How to prove QUE for H? Take A a $W \times W$ Wigner matrix.

$$\begin{pmatrix} A & B^* \\ B & D \end{pmatrix} \psi_j = \lambda_j \psi_j := \lambda_j \begin{pmatrix} w_j \\ p_j \end{pmatrix} \Longrightarrow \Big(A - B^* \frac{1}{D - \lambda_j} B \Big) w_j = \lambda_j w_j$$

$$Q_e v_j = \Big(A - B^* \frac{1}{D - e} B \Big) v_j = \xi_j v_j .$$

$\pi_{[1,W/2]}$: projection to the first $W/2$ coordinates.

Proof of QUE

- Suppose that QUE holds for Q_e with high probability.
- QUE for $Q_e \Longrightarrow \|\pi_{[1,W/2]} v_j\|_2 \sim \|\pi_{[W/2+1,W]} v_j\|_2$.
- Suppose that we can set $e = \lambda_j$ which is random.
- Patching the results $[\frac{nW}{2} + 1, (\frac{n}{2} + 1)W]$ implies ψ_j is flat.
- The total error is the sum of the individual QUE errors, i.e., $\frac{N}{W} \frac{1}{\sqrt{W}}$.

  Using $\sqrt{\frac{N}{W} \frac{1}{\sqrt{W}}}$, we determine the transition $W = \sqrt{N}$.

- We will need QUE for $Q_e = A - B^* \frac{1}{D-e} B$. But $\frac{1}{D-e}$ is singular.

Consider a Wigner matrices of size $n \times n$. Let S be a fixed subset of $[\![1, n]\!]$. Denote the eigenvector overlaps by

$$p_{ij} = \sum_{\alpha \in S} u_i(\alpha) u_j(\alpha), \quad i \neq j \in [\![1, n]\!]$$

$$p_{ii} = \sum_{\alpha \in S} \Big[u_i(\alpha)^2 - 1/n \Big], \qquad \mathbb{E} u_i(\alpha)^2 = 1/n \text{ if } u_i \text{ is flat.}$$

Probabilistic QUE [Bourgade-Y-Yin]

Suppose that there are constants $\mathfrak{c} > 0$ and $a > 0$ such that

$$\Big| \frac{1}{|S|} \sum_{i \in S} (Q_e - z)^{-1}_{ii} - \frac{1}{n} \mathrm{Tr}(Q_e - z)^{-1} \Big| \leqslant n^{-\mathfrak{c}}, \; \mathrm{Im}\, z \geq N^{-a},$$

with probability at least $1 - n^{-C}$ for any $C > 0$. Then with the same high probability we have

$$|p_{ii}| + |p_{ij}| \leq n^{-\mathfrak{c}}, \qquad \text{(We can choose } \mathfrak{c} = 1/2 - \varepsilon).$$

The E-vector equation from the matrix Brownian motion ($n = W$ in our application).

$$\mathrm{d}K_t = \frac{1}{\sqrt{n}}\mathrm{d}\mathcal{B}(t) \quad \mathcal{B}_{ij}(t): \text{symm. indep. BM}$$

E-vector equations depend on the e-values:

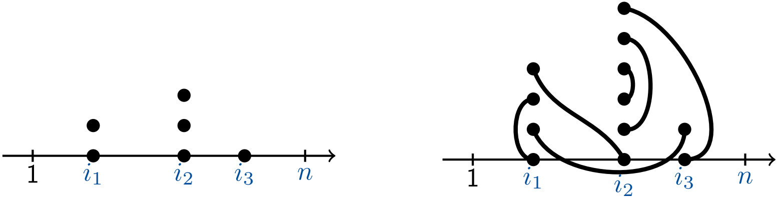
$$\mathrm{d}u_k = \frac{1}{\sqrt{n}}\sum_{\ell \neq k}\frac{u_\ell \, \mathrm{d}B_{k\ell}}{\lambda_k - \lambda_\ell} - \frac{1}{2n}\sum_{\ell \neq k}\frac{u_k \, \mathrm{d}t}{(\lambda_k - \lambda_\ell)^2}$$

$B_{k\ell}$ are independent BMs (up to symmetry).

Question: How to analyze this equation?
Answer: Create an interacting particle system from moments of eigenvectors.
Interpret “moments” as “particles”.

$\boldsymbol{\eta} : [\![1,n]\!] \to \mathbb{N}$. Denote by η_j the number of particles at j. Let $\mathcal{V}_{\boldsymbol{\eta}} = \{(i,a) : 1 \leqslant i \leqslant n, 1 \leqslant a \leqslant 2\eta_i\}$ be the vertex set and $\mathcal{G}_{\boldsymbol{\eta}}$ be the set of perfect matchings of the complete graph on $\mathcal{V}_{\boldsymbol{\eta}}$. Define

$$f_{\boldsymbol{\lambda},t}(\boldsymbol{\eta}) = \mathcal{M}(\boldsymbol{\eta})^{-1}\mathbb{E}\Big(\sum_{G\in\mathcal{G}_{\boldsymbol{\eta}}}\prod_{e\in\mathcal{E}(G)} p(e)\Big|\boldsymbol{\lambda}\Big), \quad \mathcal{M}(\boldsymbol{\eta}) = \prod_{i=1}^{n}(2\eta_i)!!,$$



(a) A configuration $\mathcal{N}(\boldsymbol{\eta}) = 6$, $\eta_{i_1} = 2$, $\eta_{i_2} = 3$, $\eta_{i_3} = 1$.

(b) A perfect matching $G \in \mathcal{G}_{\boldsymbol{\eta}}$: $P(G) = p_{i_1 i_1} p_{i_1 i_2} p_{i_2 i_2}^2 p_{i_2 i_3} p_{i_3 i_1}$.

For any given edge $e = \{(i_1,a_1),(i_2,a_2)\}$, we define $p(e) = p_{i_1,i_2}(u_t)$.

[Eigenvector moment flow]

$$\partial_t f_{\boldsymbol{\lambda},t} = \mathscr{L}(t) f_{\boldsymbol{\lambda},t}, \quad c_{k\ell}(t) = \frac{1}{n(\lambda_k(t)-\lambda_\ell(t))^2},$$

$$\mathscr{L}(t) f(\boldsymbol{\eta}) = \sum_{k\neq\ell} c_{k\ell}(t) 2\eta_k(1+2\eta_\ell)\big(f(\boldsymbol{\eta}^{k\ell}) - f(\boldsymbol{\eta})\big).$$

$\boldsymbol{\eta} = \{\eta_j\}_{j=1}^n, \quad \eta_j \in \mathbb{N}$

One particle case is a random walk (in the index j) in random environments.

$$\partial_t f(t,j) = \sum_{k\neq j} c_{jk}(t)(f(t,k) - f(t,j)), \quad c_{jk}(t) \text{ is non-local and singular.}$$

1. The eigenvalues are driven to local equilibrium by the DBM.
2. The maximum principle and other parabolic PDE methods (e.g., a finite speed of propagation estimate).
3. A priori estimate on the Green function of the associated matrix model is the key input. So our method is not a pure PDE argument.

How to get estimate on $G(z) = [Q_e - z]^{-1}, \quad Q_e = A - B^* \frac{1}{D-e} B$.

Problem: $\frac{1}{D-e}$ is singular and D is a band matrix.

Consider the generalized Green function. $I_1 = I_{W\times W}, I_2 = I_{(N-W)\times(N-W)}$.

$$G(z,e) := \left(H - \begin{bmatrix} zI_1 & 0 \\ 0 & eI_2 \end{bmatrix} \right)^{-1}, \; H = \begin{bmatrix} A & B^* \\ B & D \end{bmatrix},$$

$G(z,e)|_{W\times W} = (Q_e - z)^{-1}$. $\operatorname{Im} z = \eta \sim N^{-a}$ and e is real.

Idea: Derive a self-consistent equation for $|G_{ij}(z,e)|^2$ and use an expansion method (very complicated). We expect

$$|G_{ij}|^2 \sim \left(\frac{|m|^2 S}{1 - |m|^2 S} \right)_{ij} + \text{ errors}, \quad S_{ij} = E h_{ij}^2.$$

The constraint $\operatorname{Im} e = 0$ makes it difficult to estimate the generalized Green fn.

From the work of Erdos-Knowles-Y-Yin:

$$T_{iy} = |G_{iy}|^2 = |G_{ii}|^2 \sum_{k,l}^{(i)} h_{ik} G_{ky}^{(i)} G_{yl}^{(i)*} h_{li} \sim |m|^2 \sum_{k,l}^{(i)} h_{ik} G_{ky}^{(i)} G_{yl}^{(i)*} h_{li}$$

$$= |m|^2 \sum_{k}^{(i)} s_{ik} \left|G_{ky}\right|^2 + \text{ fluctuations and errors}, \; y \neq i$$

$$|G_{yy}|^2 = |m|^2 + \text{ fluctuations and errors.}$$

$$T = |m|^2 + |m|^2 ST + \text{ fluctuations and errors}$$

$$T = \frac{|m|^2}{1 - |m|^2 S} + \text{ fluctuations and errors}$$

Require $W \geq N^{1-c}$ for some c.

- A local theory for the bulk universality of mean-field models is generally understood. DBM converges to local equilibrium in a robust way.
- The basic mechanism that universality holds for non-mean field models is the mean-field reduction idea; QUE is the key input.
  The flatness of eigenvectors turns the non-mean field model into a mean-field one.
- Estimates on generalized Green's function is the basic input to QUE and universality of band matrices.
- QUE holds for mean field (and non-mean field) models in high probability by considering the associated moment flows, which is a reversible particle systems.
- Parallel results for band matrices in $d \geq 2$ should hold under some conditions on N and W. All results are not optimal in conditions on N and W.