The large-N limits of conditional expectation

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Introduction

N: size of the matrices
$X_N = X_N^*$: **Gaussian matrix** from the GUE
$Y_N = Y_N^*$: **independent matrix** with converging eigenvalue measure to a compactly supported measure

Consequence of Voiculescu's asymptotic freeness

There exists a noncommutative probability space $(\mathcal{A}, \tau)$ containing **freely independent** variables $X = X^*$, $Y = Y^*$ such that, $\forall P \in \mathbb{C}[x]$,

$$P(X_N + Y_N) \xrightarrow[N\to\infty]{distr} P(X+Y),$$

$$\mathbb{E}\left[P(X_N + Y_N) | Y_N\right] \xrightarrow[N\to\infty]{distr} \tau\left[P(X+Y) | Y\right].$$

Introduction

Indeed,

$$\mathbb{E}\left[P(X_N+Y_N)|Y_N\right]^k = \mathbb{E}\left[P(X_N^{(1)}+Y_N)\cdots P(X_N^{(k)}+Y_N)\Big|\, Y_N\right]$$

$$\tau\left[P(X+Y)|Y\right]^k = \tau\left[P(X^{(1)}+Y)\cdots P(X^{(k)}+Y)\Big|\, Y\right].$$

if $X_N^{(1)},\dots,X_N^{(k)}$ are independent copies of X_N,
and $X^{(1)},\dots,X^{(k)}$ are freely independent copies of X.

Moreover $(X_N^{(1)},\dots,X_N^{(k)},Y_N)$ converges to $(X^{(1)},\dots,X^{(k)},Y)$.

Heat kernel point of view

The conditional expectation $\mathbb{E}\left[P(X_N+Y_N)|Y_N\right]$ is given by the heat-kernel semigroup $(e^{\frac{t}{2}\Delta_N})_{t\geq 0}$ at time 1:

$$\begin{array}{ccc} \mathbb{E}\left[P(X_N+Y_N)|Y_N\right] & = & \left[e^{\frac{1}{2}\Delta_N}P\right](Y_N) \\ \Big\downarrow {\scriptstyle N\to\infty}\ {\scriptstyle distr} & & \\ \tau\left[P(X+Y)|Y\right] & = & ?? \end{array}$$

What is the limit of $e^{\frac{1}{2}\Delta_N}$ as $N\to\infty$?

Limit of $e^{\frac{1}{2}\Delta_N}$ when $N \to \infty$?

$\mathbb{H}_N$: space of Hermitian matrices

Let $P \in \mathbb{C}[x]$ and consider it as a map $P : \mathbb{H}_N \to \mathbb{M}_N$.

Example: If $P(Y_N) = Y_N^3$,

$$e^{\frac{1}{2}\Delta_N} P(Y_N) = Y_N^3 + Y_N + \frac{1}{2N}\operatorname{Tr}(Y_N).$$

However, $e^{\frac{1}{2}\Delta_N} P : \mathbb{H}_N \to \mathbb{M}_N$ **is not a polynomial** nor a function given by functional calculus.

Question: Is it possible to make sense of $\lim_N e^{\frac{1}{2}\Delta_N}$?

Yes, if we enlarge the space of functional calculus.

- Consider the normalized trace $\mathrm{tr} = \frac{1}{N}\mathrm{Tr}$.
- Consider **trace polynomials**,
  i.e. polynomials in Y_N and normalized traces of power of Y_N.

Example: $P(Y_N) = Y_N^3 + Y_N \,\mathrm{tr}(Y_N) + \mathrm{tr}(Y_N^2)\,\mathrm{tr}(Y_N^3)$.

- Δ_N and $e^{\frac{1}{2}\Delta_N}$ leaves invariant the set $\mathbb{C}\{x\}$ of **trace polynomials**.

Answer: When restricted to $\mathbb{C}\{x\}$, $\Delta_N = \Delta_\infty + O\left(\frac{1}{N^2}\right)$, with an operator $\Delta_\infty : \mathbb{C}\{x\} \to \mathbb{C}\{x\}$ **independent from** N.

$\lim_N e^{\frac{1}{2}\Delta_N} = \lim_N e^{\frac{1}{2}\Delta_\infty + O\left(1/N^2\right)} = e^{\frac{1}{2}\Delta_\infty}$

The $\mathbb{C}\{x\}$-calculus is valid in any non-commutative probability space $(\mathcal{A}, \tau)$.

$$\begin{array}{ccc}
\mathbb{E}\left[P(X_N+Y_N)|Y_N\right] & = & \left[e^{\frac{1}{2}\Delta_N}P\right](Y_N) \\
N\to\infty \Big\downarrow distr & & N\to\infty \Big\downarrow distr \\
\tau\left[P(X+Y)|Y\right] & = & \left[e^{\frac{1}{2}\Delta_\infty}P\right](Y)
\end{array}$$

Wigner law revisited:

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}(P(X_N))\right] = \left[e^{\frac{1}{2}\Delta_N}(\mathrm{tr}\,P)\right](0) \xrightarrow[N\to\infty]{} \left[e^{\frac{1}{2}\Delta_\infty}(\mathrm{tr}\,P)\right](0) = \tau\left[P(X)\right].$$

Brownian motions on Lie groups

A **Brownian motion** $(U_t)_{t\geq 0}$ on the unitary group $\mathbb{U}_N$ is a Markov process starting at I_N **whose generator is the Laplacian** $\frac{1}{2}\Delta_{\mathbb{U}_N}$ for a certain metric.

Because $\Delta_{\mathbb{U}_N} = \Delta_{\mathbb{U}} + O\left(\frac{1}{N^2}\right)$ on $\mathbb{C}\{x\}$, we have

$$\mathbb{E}\left[\frac{1}{N}\operatorname{Tr}(P(U_t))\right] = e^{\frac{t}{2}\Delta_{\mathbb{U}_N}}(\operatorname{tr}(P))(1) \underset{N\to\infty}{\longrightarrow} e^{\frac{t}{2}\Delta_{\mathbb{U}}}(\operatorname{tr}(P))(1).$$

Theorem (Biane, Rains, Xu 1997)

For all $t \geq 0$, and all polynomial P,

$$\frac{1}{N}\operatorname{Tr}(P(U_t)) \text{ converges almost surely as } N \to \infty.$$

The same proof gives similar results for Brownian motions on other Lie groups.

- **Lévy (2011)**: for the orthogonal group $\mathbb{O}_N$, and for the symplectic group $\mathbb{S}p(N)$

$$\Delta_{\mathbb{O}_N} = \Delta_{\mathbb{U}} + O\left(1/N\right).$$

- **C. (2013)**, **Kemp (2013)**: for the general linear group GL_N

$$\Delta_{GL_N} = \Delta_{GL} + O(1/N^2).$$

- **Ulrich (2015)**: for the M^2 blocks of size $N \times N$ of a Brownian motion on $\mathbb{U}_{NM}$ (when $N \to \infty$)

$$\Delta_{\mathbb{U}_{NM}} = \Delta_{\mathbb{U},M} + O(1/N^2).$$

$\mathbb{C}\{x\}$ must be replaced by the space $\mathbb{C}\{x_{ij} : 1 \leq i, j \leq M\}$ of trace polynomials in M^2 variables.

It is also possible to consider more general situations.

- **C. (2016)**: for a Lévy process on $\mathbb{U}_N$ with generator

$$\mathcal{L}_{\mathbb{U}_N} = \mathcal{L}_{\mathbb{U}} + O(1/N).$$

- **Gabriel (2015)**: for a random walk $(S_N(t))_{t\geq 0}$ on the permutation group $\mathfrak{S}_N$ with generator

$$\mathcal{L}_{\mathfrak{S}_N} = \mathcal{L} + O(1/N)$$

The set of **trace polynomials** has to be replaced by the set of **traffic operations** (in the sense of Camille Male).

$$e^{\frac{1}{2}\Delta}P(z) = \int_{\mathbb{R}} P(x)e^{-\frac{(x-z)^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} P(x)\overline{\psi_z(x)}e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \langle P|\psi_z\rangle_{L^2(\mathbb{R},\gamma)}$$

where $\psi_z(x) := e^{-\frac{\bar{z}^2}{2}+\bar{z}x}$ and γ is the standard Gaussian law.

Theorem (Bargmann 1958, Segal 1958)

We have the resolution of the identity

$$Id_{L^2(\mathbb{R},\gamma)} = \int_{\mathbb{C}} |\psi_z\rangle\langle\psi_z| \;\mathrm{d}\gamma^{\mathbb{C}}(z),$$

(where $\mathrm{d}\gamma^{\mathbb{C}}$ is the complex Gaussian law), in the sense that

$$\langle P|P\rangle_{L^2(\mathbb{R},\gamma)} = \int_{\mathbb{C}} \langle P|\psi_z\rangle_{L^2(\mathbb{R},\gamma)}\langle\psi_z|P\rangle_{L^2(\mathbb{R},\gamma)} \;\mathrm{d}\gamma^{\mathbb{C}}(z).$$

The **Segal-Bargmann transform** maps P to $z \mapsto \langle P|\psi_z^q\rangle_{L^2(\mathbb{R},\gamma)}$.

q-Gaussian law (Bozejko and Speicher, 1991)

Interpolation

$q=0$ free case	$0 \leq q \leq 1$	classical case $q=1$
$\mathrm{d}\gamma_0$ semicircular law	$\mathrm{d}\gamma_q$	Gaussian law $\mathrm{d}\gamma$
$\mathrm{d}\gamma_0^{\mathbb{C}}$ length measure on $\mathbb{U}_1$	$\mathrm{d}\gamma_q^{\mathbb{C}}$	complex Gaussian law $\mathrm{d}\gamma^{\mathbb{C}}$
ψ_z^0	ψ_z^q	ψ_z

For $0 \leq q < 1$,

$$\mathrm{d}\gamma_q(x) = 1_{|x| \leq 2/\sqrt{1-q}} \frac{1}{\pi} \sqrt{1-q} \sin\theta \prod_{n=1}^{\infty} (1-q^n)|1-q^n e^{2i\theta}|^2 \mathrm{d}x$$

where $\theta \in [0,\pi]$ is such that $x = 2\cos(\theta)/\sqrt{1-q}$ and $\mathrm{d}\gamma_q^{\mathbb{C}}$ is a measure on $\mathbb{C}$ concentrated on a family of concentric circles.

q-deformation of the Segal-Bargmann transform

Theorem (van Leeuwen, Maassen 1995)

We have the resolution of the identity

$$Id_{L^2(\mathbb{R},\gamma_q)} = \int_{\mathbb{C}} |\psi_z^q\rangle\langle\psi_z^q| \,\mathrm{d}\gamma_q^{\mathbb{C}}(z).$$

The q-**deformed Segal-Bargmann transform**, which maps P to $z \mapsto \langle P|\psi_z^q\rangle_{L^2(\mathrm{d}\gamma_q)}$ can also be written as the application of a q-deformed heat kernel on P.

Theorem (C.-Ho 2017)

Let $0 \leq q \leq 1$. For any polynomial P, we have

$$\langle P|\psi_Z^q\rangle_{L^2(\mathrm{d}\gamma_q)} = \tau[P(X+Z)|Z],$$

where $X \sim \mathrm{d}\gamma_q$ and $Z \sim \mathrm{d}\gamma_q^{\mathbb{C}}$ are "q-independent".

Random matrices and q-deformation

Random matrix model for the measure $\mathrm{d}\gamma_q$ (Śniady 2001):

- law γ_N on $\mathbb{H}_N$ such that $\gamma_N \sim X_N \xrightarrow[N\to\infty]{distr} \gamma_q$.
- law $\gamma_N^{\mathbb{C}}$ on $\mathbb{M}_N$ such that $\gamma_N^{\mathbb{C}} \sim Z_N \xrightarrow[N\to\infty]{distr} \gamma_q^{\mathbb{C}}$.

The model of Śniady is also a model of q-independence, and therefore

$$\mathbb{E}[P(X_N + Z_N)|Z_N] \xrightarrow[N\to\infty]{distr} \tau[P(X+Z)|Z].$$

Theorem (q=0 by Biane in 1997, 0<q<1 by C.-Ho in 2017)

The **classical Segal-Bargmann transform** $M \mapsto \langle P|\psi_M\rangle_{L^2(\mathbb{H}_N,\gamma_N)}$ converges to the q-**deformed Segal-Bargmann transform** $z \mapsto \langle P|\psi_z\rangle_{L^2(\mathbb{R},\gamma_q)}$ in the following sense:

$$\mathbb{E}\left[\left\|\langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{H}_N,\gamma_N)} - \langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|_2^2\right] \underset{N\to\infty}{\longrightarrow} 0$$

whenever Z_N is a random matrix of law $\gamma_N^{\mathbb{C}}$.

Sketch of proof: We have $\langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{H}_N,\gamma_N)} = \mathbb{E}[P(X_N+Z_N)|Z_N]$, and

$$\mathbb{E}\left[\left\|\mathbb{E}[P(X_N+Z_N)|Z_N] - \langle P|\psi_{Z_N}\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|_2^2\right]$$

converges to

$$\left\|\tau[P(X+Z)|Z] - \langle P|\psi_Z\rangle_{L^2(\mathbb{R},\gamma_q)}\right\|_2^2 = 0.$$

Another notion of conditional expectation

$\mathbb{M}_N$ set of $N \times N$ matrices
$(\mathbb{M}_N, \tau_N = \frac{1}{N}\operatorname{Tr})$ is a noncommutative probability space

We denote by D the **conditional expectation** given the set of diagonal matrices $\mathbb{D}_N$:

$$D[\,\cdot\,] = \tau_N[\,\cdot\,|\mathbb{D}_N]$$

$(\mathbb{M}_N, D, \mathbb{D}_N)$ is called an **operator-valued probability space**, and $D : \mathbb{M}_N \to \mathbb{D}_N$ is the diagonal map.

Freeness with amalgamation over the diagonal

Definition

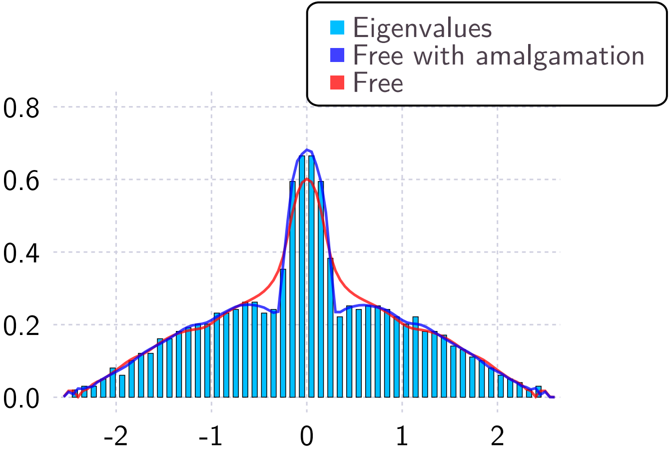
X_N and Y_N are **free with amalgamation over** $\mathbb{D}_N$ if

$$D\left[\left(P_{N,1} - D[P_{N,1}]\right) \cdots \left(P_{N,n} - D[P_{N,n}]\right)\right] = 0$$

for all $P_{N,1} \in \mathbb{D}_N\langle X_N\rangle$, $P_{N,2} \in \mathbb{D}_N\langle Y_N\rangle$, $P_{N,3} \in \mathbb{D}_N\langle X_N\rangle$...

$\mathrm{GUE}_{\mathrm{vp}} + \mathrm{ER}$

X_{1000} Wigner matrix with variance profile,
Y_{1000} is the normalized adjacency matrix of a sparse Erdös-Rényi graph



Asymptotic freeness over the diagonal

Shlyakhtenko (1996): asymptotic freeness over the diagonal for certain Gaussian Wigner matrices with variance profile
Boedihardjo, Dykema (2017): certain random Vandermonde matrices are asymptotically $\mathcal{R}$-diagonal over the diagonal matrices.

Theorem (Au, C., Dahlqvist, Gabriel, Male 2018)

Let X_N and Y_N be two sequences of independent random matrices which are **bounded in operator norm** uniformly in N and **permutation invariant**.

Then X_N and Y_N are in probability **asymptotically free with amalgamation over** $\mathbb{D}_N$.

Asymptotic freeness over the diagonal

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Then X_N and Y_N are in probability **asymptotically free with amalgamation over** $\mathbb{D}_N$.

Namely, if

$$\epsilon_N := D\left[(P_{N,1} - D[P_{N,1}]) \cdots (P_{N,n} - D[P_{N,n}])\right]$$

where $P_{N,1} \in \mathbb{D}_N\langle X_N\rangle$, $P_{N,2} \in \mathbb{D}_N\langle Y_N\rangle$, $P_{N,3} \in \mathbb{D}_N\langle X_N\rangle$... have degrees and coefficients uniformly bounded, then for all $\epsilon > 0$,

$$\mathbb{P}\left[\|\epsilon_N\|_2 > \epsilon\right] \underset{N\to\infty}{\longrightarrow} 0.$$

Asymptotic freeness over the diagonal

Permutation invariance means

$$\{X_N(i,j)\}_{i,j} \overset{Law}{=} \{X_N(\sigma(i),\sigma(j))\}_{i,j}, \quad \forall \sigma \in S_N.$$

Examples:

- uniform permutation;
- $U_t D U_t^*$, where D is diagonal and U_t unitary BM starting at a uniform permutation.

What about Wigner matrices with variance profile or sparse Erdös-Rényi graphs?

Our assumption of **permutation invariance** can be relaxed by **multiplying all the entries by bounded coefficients**:

- Wigner with variance profile;
- percolated version (entries or blocks are randomly erased) of the previous models.

The boundedness of the norm can also be relaxed in the boundedness of the **traffic distribution**:

- Wigner with exploding moments;
- standardized adjacency matrix of a sparse Erdös-Rényi graph.

Asymptotic freeness over the diagonal

In the proof, we have to control **graph sums** as defined by Mingo and Speicher, or equivalently **traffic distribution**, as defined by Male:
for an oriented graph $G = (V, E)$ and a family $(A_N^{(e)})_{e\in E}$ of matrices among $\{X_N, Y_N\}$, we consider

$$\mathbb{E}\left[\sum_{i:V\to[N]} \prod_{e=(v,w)\in E} A_N^{(e)}(i(w), i(v))\right].$$

We prove that $\mathbb{E}[\frac{1}{N}\operatorname{Tr}(\epsilon_N\epsilon_N^*)]$ is a sum of such terms which tends to 0 as $N \to \infty$.

Numerical computations

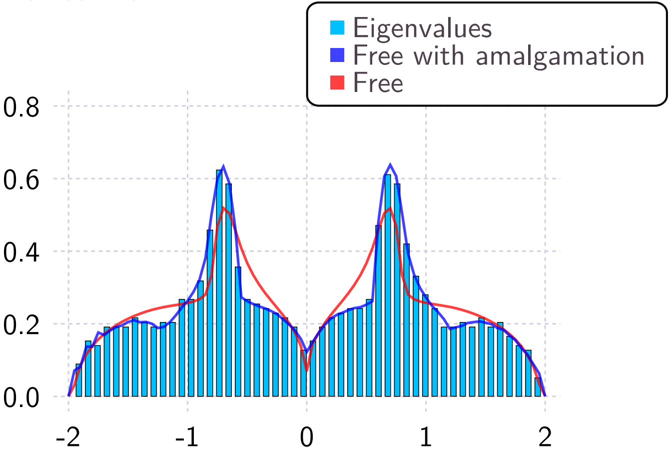
The joint law of (X_N, Y_N) is determined by the **operator-valued Stieltjes transforms** G_{X_N} and G_{Y_N}: the diagonal of the resolvent of our matrices

$$G_{X_N}(d) = D\left[(d - X_N)^{-1}\right], \quad \forall d \in \mathbb{D}_N, Im(d) > 0.$$

For numerical computation, the **algorithm of Belinschi, Mai and Speicher (2017)** is efficient to compute the eigenvalue law of $X_N + Y_N$.

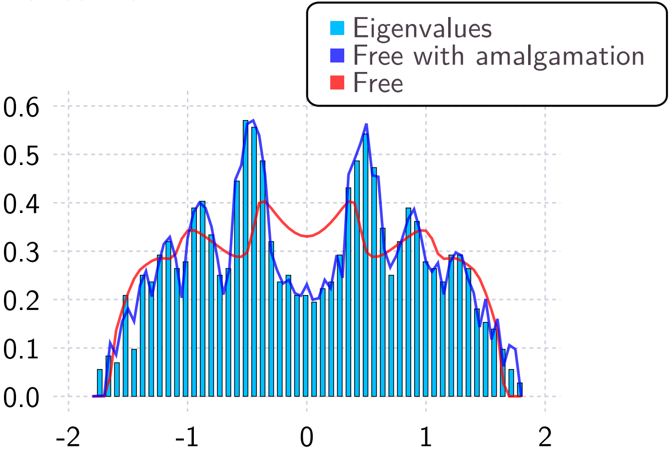
$\mathrm{GUE}_{\mathrm{vp}} + \mathrm{UBM}$

X_N Wigner matrix with variance profile,
$Y_N = U_{0.125} D U_{0.125}^*$ where U_t is unitary BM and D has $\{-1, +1\}$-Bernoulli law.



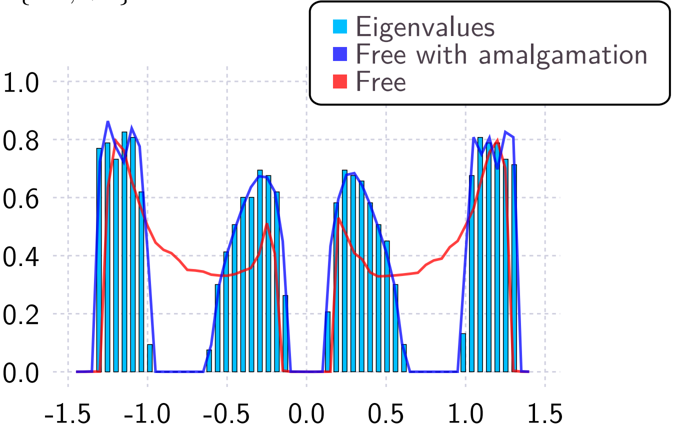
Perm + UBM

$X_N = S + S^*$, where S is a uniform permutation,
$Y_N = U_{0.125} D U_{0.125}^*$ where U_t is unitary BM and D has $\{-1, +1\}$-Bernoulli law.



UBM + UBM

$X_N = V_{0.125} C V_{0.125}^*$ where V_t is unitary BM,
$Y_N = U_{0.125} D U_{0.125}^*$ where U_t is unitary BM,
where C has symmetric constant two-edge support density and D has $\{-1, +1\}$-Bernoulli law.



Thank you!