Extreme Eigenvalue Distributions of Sparse Erdős-Rényi Graphs

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IPAM Workshop: Random Matrices and Free Probability Theory
Random graph models

- **Erdős-Rényi Graphs** $G(N, p)$: each edge selected independently with probability $p$. The average degree is $pN$.
Random graph models

- **Erdős-Rényi Graphs** $G(N, p)$: each edge selected independently with probability $p$. The average degree is $pN$.

- We are interested in the **sparse** random Erdős-Rényi graphs:

  $$p \ll 1.$$
For a random graph $G$ on $N$ vertices, denote its adjacency matrix by

$$A_{ij} = 1_{\{i \sim j\}}.$$ 

We rescale it such that each entry has variance $1/N$, and consider the rescaled adjacency matrix

$$A / \sqrt{Np(1-p)},$$

and denote its eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ and corresponding normalized eigenvectors $u_1, \ldots, u_N$. 

Fundamental question: What are the probability distributions of the eigenvalues and eigenvectors?
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**Fundamental question:**

What are the probability distributions of the eigenvalues and eigenvectors?
The empirical eigenvalue distribution $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 2000$ and $p = 0.01$: 

![Graph showing empirical eigenvalue distribution](image-url)
Empirical eigenvalue distribution with $Np \gg 1$

The empirical eigenvalue distribution $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 2000$ and $p = 0.01$:

- The empirical eigenvalue distribution converges to the semicircle distribution (Wigner, 1950s)

\[ \rho_{sc}(x) = \frac{\sqrt{4 - x^2}}{2\pi}. \]
The empirical eigenvalue distribution $\frac{1}{N} \sum_{i=1}^{N} \delta \lambda_i$ of a random Erdős-Rényi Graph with $N = 2000$ and $p = 0.01$:

The rescaled adjacency matrix is a rank one perturbation of a mean zero matrix (Füredi-Komlós, 1980; Féral-Péché, 2008). The largest eigenvalue concentrates around $\sqrt{Np/(1 - p)} + \sqrt{(1 - p)/Np} \approx 4.72$ (Erdős-Knowles–Yau–Yin, 2013).
Empirical eigenvalue distribution with $Np \gg 1$

The empirical eigenvalue distribution $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 2000$ and $p = 0.01$:

Empirical eigenvalue distribution with $Np \approx 1$

The empirical eigenvalue distribution is given by:

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$$

of a random Erdős-Rényi Graph with $N = 3000$ and $p = 0.001$: 

![Graph showing the empirical eigenvalue distribution](image)
Empirical eigenvalue distribution with $Np \approx 1$

The empirical eigenvalue distribution $\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 3000$ and $p = 0.001$:

- The adjacency matrix is singular, there are many zero eigenvalues (Bordenave-Sen-Virág, 2013).
- The empirical eigenvalue density is not compactly supported (Khorunzhiy, 2001; Bordenave-Sen-Virág, 2013; Benaych-Georges-Bordenave-Knowles, 2017).
Growing average degree case: $pN \gg 1$

Empirical eigenvalue distribution of Erdős-Rényi Graphs $G(2000, 0.01)$. 
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Empirical eigenvalue distribution of Erdős-Rényi Graphs $G(2000, 0.01)$.

Main goal:
To understand the **local statistics** of eigenvalues and eigenvectors.
Bulk universality

Gap distribution for the bulk eigenvalues $N(\lambda_i - \lambda_{i+1})$ is expected to be universal, the Gaudin-Mehta distribution (Gap distribution for GOE), approximately given by the Wigner Surmise

$$p(s) \approx \frac{\pi s}{2} e^{-\frac{\pi}{4} s^2}.$$
Theorem (Erdős-Knowles–Yau–Yin, 2011)

Let $\varepsilon > 0$ and $pN \geq N^{2/3+\varepsilon}$. Then in the bulk, the Erdős-Rényi graphs $G(N, p)$ obey the same local eigenvalue statistics as Gaussian Orthogonal Ensemble.
**Theorem (Erdős-Knowles–Yau–Yin, 2011)**

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**Theorem (H.–Landon–Yau, 2015)**

Let $\varepsilon > 0$ and $pN \geq N^\varepsilon$. Then in the bulk, the Erdős-Rényi graphs $G(N, p)$ obey the same local eigenvalue statistics as Gaussian Orthogonal Ensemble.
Eigenvector Universality

Theorem (Bourgade-H.-Yau, 2016)

Let $\epsilon > 0$ and $pN \geq N^{-\epsilon}$. The bulk eigenvectors are asymptotically normal, $N^{-1}u_i(j) \rightarrow N(0, 1)^2$, where $N$ is the standard normal random variable.

Eigenvector Flow (Bourgade-Yau, 2013).
Theorem (Bourgade-H.-Yau, 2016)

Let $\varepsilon > 0$ and $pN \geq N^\varepsilon$. The bulk eigenvectors are asymptotically normal, $N|u_i(j)|^2 \to \mathcal{N}(0, 1)^2$, where $\mathcal{N}$ is the standard normal random variable.

Distribution of the second largest eigenvalue is expected to be given by the Tracy-Widom $\beta = 1$ distribution (largest eigenvalue of GOE).

$$N^{2/3}(\lambda_2 - E_*) \to TW_1.$$ 

(The largest eigenvalue is trivial and roughly given by the expected degree.)
Theorem (Erdős-Knowles–Yau–Yin, 2011)

Let $\varepsilon > 0$ and $pN \geq N^{2/3+\varepsilon}$. The second largest eigenvalue of Erdős-Rényi graphs $G(N, p)$ obeys the Tracy-Widom $\beta = 1$ distribution, i.e.

$$N^{2/3}(\lambda_2 - 2) \to TW_1.$$
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Theorem (Lee-Schnelli, 2016)

Let $\varepsilon > 0$ and $pN \geq N^{1/3+\varepsilon}$. The second largest eigenvalue of Erdős-Rényi graphs $G(N, p)$ obeys the Tracy-Widom $\beta = 1$ distribution, i.e.

$$N^{2/3}(\lambda_2 - E_*) \rightarrow TW_1, \quad E_* = 2 + \frac{1}{Np}.$$
We normalized the entries of the adjacency matrix $A$ to have mean zero and variance $1/N$,

$$H = \frac{A - p11^*}{\sqrt{Np(1 - p)}},$$

and denote its eigenvalues by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$.

**Theorem (H.-Landon-Yau, 2017)**

Let $\varepsilon > 0$ and $N^{2/9 + \varepsilon} \leq pN \leq N^{1/3 - \varepsilon}$. The largest few eigenvalues of $H$ have Gaussian fluctuation, i.e. for any fixed $k \geq 1$

$$\sqrt{p/2N}(\mu_k - E_*) \to \mathcal{N}(0, 1), \quad E_* = 2 + \frac{1}{pN} - \frac{5}{4(pN)^2}.$$
Gaussian Fluctuation


Let $\varepsilon > 0$ and $N^{2/9+\varepsilon} \leq pN \leq N^{1/3-\varepsilon}$. The largest few eigenvalues of $H = (h_{ij})_{i,j=1}^N$ have Gaussian fluctuation, i.e. for any fixed $k \geq 1$

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\]

- We can explicitly identify the fluctuation

\[
\mu_k - E_* = \chi + \text{error}, \quad \chi = \frac{1}{N} \left( \sum_{ij} h_{ij}^2 - N^{-1} \right).
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- By Cauchy Interlacing Theorem, the eigenvalues of $A/\sqrt{Np(1-p)}$ and $H$ are interlaced, i.e. $\mu_2 \leq \lambda_2 \leq \mu_1$. $\sqrt{p/2N}(\lambda_2 - E_*) \to \mathcal{N}(0, 1)$. 
Gaussian Fluctuation

- For $Np \gg N^{1/3}$, $\lambda_2$ has Tracy-Widom $\beta = 1$ distribution asymptotically.
- For $Np \ll N^{1/3}$, $\lambda_2$ has Gaussian distribution asymptotically.
Gaussian Fluctuation

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Why is there a transition at $Np \asymp N^{1/3}$?
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Why is there a transition at $Np \asymp N^{1/3}$?

Theorem (Shcherbina-Tirozzi, 2011)

For $1/N \ll p \ll 1$, under mild conditions for the test function $f$, the linear statistics of $H$ are asymptotically Gaussian

$$\sqrt{p} \left( \sum_i f(\mu_i) - \mathbb{E} \sum_i f(\mu_i) \right) \to \mathcal{N}(0, \sigma_f^2).$$
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- For Wigner matrices, the normalization factor is 1.
- The eigenvalues of $H$ behave like an oscillating spring system, eigenvalues oscillate together on the scale $1/(N\sqrt{p})$, but the gaps are rigid.

Let $\varepsilon > 0$ and $N^{2/9+\varepsilon} \leq pN$. Subject to a random shift, the largest few eigenvalues of $H = (h_{ij})_{i,j=1}^N$ have Tracy-Widom $\beta = 1$ distribution, i.e. for any fixed $k \geq 1$, the joint law

$$N^{2/3} \left( \mu_1 - E_* - \chi, \mu_2 - \mu_1, \ldots, \mu_k - \mu_{k-1} \right) \to TW_1,$$

where $\chi = \frac{1}{N} \left( \sum_{ij} h_{ij}^2 - N^{-1} \right) \asymp O \left( \frac{1}{N\sqrt{p}} \right).$
Tracy Widom $\beta = 1$ Distribution


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$$N^{2/3} (\mu_1 - E_* - \chi, \mu_2 - \mu_1, \ldots, \mu_k - \mu_{k-1}) \to TW_1,$$

where $\chi = \frac{1}{N} \left( \sum_{ij} h_{ij}^2 - N^{-1} \right) \lesssim O \left( \frac{1}{N\sqrt{p}} \right)$.

- For $pN \gg N^{1/3}$, $\chi \ll N^{-2/3}$ is negligible, the extreme eigenvalues have Tracy-Widom $\beta = 1$ distribution.

Let \( \varepsilon > 0 \) and \( N^{2/9+\varepsilon} \leq pN \). Subject to a random shift, the largest few eigenvalues of \( H = (h_{ij})_{i,j=1}^{N} \) have **Tracy-Widom \( \beta = 1 \) distribution**, i.e. for any fixed \( k \geq 1 \), the joint law

\[
N^{2/3} (\mu_1 - E_* - \mathcal{X}, \mu_2 - \mu_1, \cdots, \mu_k - \mu_{k-1}) \to TW_1,
\]

where \( \mathcal{X} = \frac{1}{N} \left( \sum_{ij} h_{ij}^2 - N^{-1} \right) \sim O \left( \frac{1}{N\sqrt{p}} \right) \).

- For \( pN \gg N^{1/3} \), \( \mathcal{X} \ll N^{-2/3} \) is negligible, the extreme eigenvalues have Tracy-Widom \( \beta = 1 \) distribution.
- For \( pN \ll N^{1/3} \), \( \mathcal{X} \gg N^{-2/3} \), the fluctuation of extreme eigenvalues is asymptotically Gaussian.

Let $\varepsilon > 0$ and $N^{2/9+\varepsilon} \leq pN$. Subject to a random shift, the largest few eigenvalues of $H = (h_{ij})_{i,j=1}^{N}$ have Tracy-Widom $\beta = 1$ distribution, i.e. for any fixed $k \geq 1$, the joint law

$$N^{2/3} \left( \mu_1 - E_* - \chi, \mu_2 - \mu_1, \cdots, \mu_k - \mu_{k-1} \right) \rightarrow TW_1,$$

where $\chi = \frac{1}{N} \left( \sum_{ij} h_{ij}^2 - N^{-1} \right) \sim O \left( \frac{1}{N \sqrt{p}} \right)$.

- For $pN \gg N^{1/3}$, $\chi \ll N^{-2/3}$ is negligible, the extreme eigenvalues have Tracy-Widom $\beta = 1$ distribution.
- For $pN \ll N^{1/3}$, $\chi \gg N^{-2/3}$, the fluctuation of extreme eigenvalues is asymptotically Gaussian.
- For $pN \asymp N^{1/3}$, $\chi \asymp N^{-2/3}$, the fluctuation of extreme eigenvalues is a combination of Gaussian and Tracy-Widom $\beta = 1$ distribution.
Basic tools

Stieltjes transform of a measure $\varrho$:

$$m_{\varrho}(z) = \int \frac{\varrho(x)dx}{x - z}, \quad z \in \mathbb{C}_+, \quad (1)$$

$$\varrho(E) = \frac{1}{\pi} \lim_{\eta \to 0^+} \text{Im}[m_{\varrho}(E + i\eta)]. \quad (2)$$

The Stieltjes transform contains info for eigenvalues: Writing $z = E + i\eta$, we have

$$\text{Im}[m_N(z)] = \text{Im} \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} \right] = \frac{1}{N} \sum_{i=1}^{N} \frac{\eta}{(\lambda_i - E)^2 + \eta^2} \quad (3)$$

and $\eta$ is the spectral resolution.
The Stieltjes transform can be used to detect the locations of extreme eigenvalues.

**Proposition**

If for some \( z = E + i\eta \in \mathbb{C}_+ \), such that \( \text{Im}[m_N(z)] \ll 1/N\eta \), then there is no eigenvalue in the interval \([E - \eta, E + \eta]\).
Basic tools

The Stieltjes transform can be used to detect the locations of extreme eigenvalues.

**Proposition**

If for some $z = E + i\eta \in \mathbb{C}_+$, such that $\text{Im}[m_N(z)] \ll 1/N\eta$, then there is no eigenvalue in the interval $[E - \eta, E + \eta]$.

**Proof.**

We prove by contradiction. If there is an eigenvalue $\lambda_k \in [E - \eta, E + \eta]$, then

$$
\frac{1}{N\eta} \gg \text{Im}[m_N(z)] = \text{Im}\left[\frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z}\right] = \frac{1}{N} \sum_{i=1}^{N} \frac{\text{Im}[z]}{|\lambda_i - z|^2}
$$

$$
\gg \frac{1}{N} \frac{\text{Im}[z]}{|\lambda_k - z|^2} = \frac{1}{N} \frac{\eta}{(E - \lambda_k)^2 + \eta^2} \geq \frac{1}{2N\eta}.
$$
Self-Consistent Equation of $m_N(z)$

Example (Gaussian Orthogonal Ensemble $H = (h_{ij})_{i,j=1}^N$)

The Stieltjes transform of the semicircle distribution satisfies

$$m_{sc}(z)^2 + zm_{sc}(z) + 1 = 0.$$  

We denote the Green's function of $H$ as $G(z) = (H - z)^{-1}$ and its Stieltjes transform $m_N(z) = \frac{\text{Tr} G(z)}{N}$, then

$$E[1 + zm_N + m_N^2] \approx 0.$$  

$$E[1 + zm_N] = \frac{1}{N} E[\text{Tr}(I + zG)] = \frac{1}{N} E[\text{Tr}((H - z)G + zG)] = \frac{1}{N} E[\text{Tr}(HG)] = \frac{1}{N} E[\sum h_{ij} G_{ji}] = -\frac{1}{N^2} E[\sum G_{ii} G_{jj} + G_{ji} G_{ji}] \approx -\frac{1}{N^2} E[\sum G_{ii} G_{jj}].$$

Using the same trick, the higher moment $E[|1 + zm_N + m_N^2|^r]$ is small, and by Markov's inequality, $1 + zm_N + m_N^2 \approx 0$ with high probability. By comparing with $1 + zm_{sc} + m_{sc}^2 = 0$, we can get

$$|m_N(z) - m_{sc}(z)| \approx 0.$$
Self-Consistent Equation of $m_N(z)$

Example (Gaussian Orthogonal Ensemble $H = (h_{ij})_{i,j=1}^N$)

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Self-Consistent Equation of \( m_N(z) \)

**Example (Gaussian Orthogonal Ensemble \( H = (h_{ij})_{i,j=1}^N \))**

- The Stieltjes transform of the semicircle distribution satisfies
  \[
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- We denote the Green’s function of \( H \) as \( G(z) = (H - z)^{-1} \) and its Stieltjes transform \( m_N(z) = \text{Tr} \( G(z) \)/N, then \( \mathbb{E}[1 + zm_N + m_N^2] \approx 0. \)

\[
\mathbb{E}[1 + zm_N] = \frac{1}{N} \mathbb{E}[\text{Tr}(I + zG)] = \frac{1}{N} \mathbb{E}[\text{Tr}((H - z)G + zG)] = \frac{1}{N} \mathbb{E}[\text{Tr}(HG)] = \frac{1}{N} \mathbb{E}\left[ \sum h_{ij} G_{ji} \right] = \frac{1 + \delta_{ij}}{N^2} \mathbb{E}\left[ \sum \partial_{h_{ij}} G_{ji} \right] = -\frac{1}{N^2} \mathbb{E}\left[ \sum G_{ii} G_{jj} + G_{ji} G_{jj} \right] \approx -\frac{1}{N^2} \mathbb{E}\left[ \sum G_{ii} G_{jj} \right] = -\mathbb{E}[m_N^2].
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- We denote the Green’s function of $H$ as $G(z) = (H - z)^{-1}$ and its Stieltjes transform $m_N(z) = \text{Tr} \, G(z)/N$, then $\mathbb{E}[1 + zm_N + m_N^2] \approx 0$.

$$\mathbb{E}[1 + zm_N] = \frac{1}{N} \mathbb{E}[\text{Tr}(I + zG)] = \frac{1}{N} \mathbb{E}[\text{Tr}((H - z)G + zG)]$$

$$= \frac{1}{N} \mathbb{E}[\text{Tr}(HG)] = \frac{1}{N} \mathbb{E} \left[ \sum h_{ij} G_{ji} \right] = \frac{1 + \delta_{ij}}{N^2} \mathbb{E} \left[ \sum \partial_{h_{ij}} G_{ji} \right]$$

$$= - \frac{1}{N^2} \mathbb{E} \left[ \sum G_{ii} G_{jj} + G_{ji} G_{ji} \right] \approx - \frac{1}{N^2} \mathbb{E} \left[ \sum G_{ii} G_{jj} \right] = - \mathbb{E}[m_N^2].$$

- Using the same trick, the higher moment $\mathbb{E}[|1 + zm_N + m_N^2|^{2r}]$ is small, and by Markov's inequality, $1 + zm_N + m_N^2 \approx 0$ with high probability. By comparing with $1 + zm_{sc} + m_{sc}^2 = 0$, we can get $|m_N(z) - m_{sc}(z)| \approx 0$. 

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Extreme Eigenvalues of ER graphs
IPAM Workshop 18 / 27
Construction of the Self-consistent Equation

We denote $q = (Np)^{1/2}$. For $H = (h_{ij})_{i,j=1}^{N}$, $\mathbb{E}[h_{ij}] = 0$, $\mathbb{E}[h_{ij}^2] = 1/N$, and the $k$-th cumulant of $h_{ij}$ is $\kappa_k(h_{ij}) = \frac{c_k}{Nq^{k-2}}$, for $k \geq 3$. 

Proposition (higher order self-consistent equation)

There exists a polynomial $P_0$ depending on the cumulants of $h_{ij}$

$P_0(z, m) = 1 + zm + m^2 + C^4 q^2 m^4 + C^6 q^4 m^6 + \cdots$

so that for $z = \mathbb{E} + i\eta$ with $\eta \gg 1/N$, the Stieltjes transform $m_N$ of $H$ satisfies

$\mathbb{E}[P_0(z, m_N)] \leq \mathbb{E}[\Im m_N(z)]N^{-1/2}$.

The first three terms of $P_0(z, m)$ recover the self-consistent equation of semi-circle distribution. The first four terms of $P_0(z, m)$ were derived by Lee and Schnelli, and used in their proof of Tracy-Widom distribution of extreme eigenvalues for $N_p \gg N^{1/3}$.
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so that for $z = E + i\eta$ with $\eta \gg 1/N$, the Stieltjes transform $m_N$ of $H$ satisfies

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Construction of the Self-consistent Equation

The same as in the GOE example:

\[
\mathbb{E}[1 + zm_N] = \frac{1}{N} \mathbb{E} \left[ \sum h_{ij} G_{ji} \right].
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Construction of the Self-consistent Equation

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By the cumulant expansion (here the independence is used)

$$\mathbb{E}[h_{ij} G_{ij}] = \sum_{k=1}^{\ell} \frac{C_{k+1}}{(k - 1)! N q^{k-1}} \mathbb{E}[\partial_{h_{ij}}^k G_{ji}] + O \left( \frac{1}{q^\ell} \right).$$
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By the cumulant expansion (here the independence is used)

$$\mathbb{E}[h_{ij} G_{ij}] = \sum_{k=1}^{\ell} \frac{C_{k+1}}{(k-1)!} \mathbb{E} [\partial^k h_{ij} G_{ji}] + O \left( \frac{1}{q^\ell} \right).$$

The derivatives of the resolvent entries, $\partial^k_{h_{ij}} G_{ij}$, are polynomials in terms of the Green’s function entries. Two kinds of terms might occur:

- terms containing off-diagonal entries are small.
- terms containing only diagonal entries can be iteratively rewritten as a polynomial of $m_N$, e.g.

$$N^{-1} \sum G^2_{ii} = N^{-2} \sum G_{ii} G_{jj} + \text{higher order terms} = m_N^2 + \text{higher order terms}.$$
Construction of the Self-consistent Equation

The same as in the GOE example:

\[
\mathbb{E}[1 + zm_N] = \frac{1}{N} \mathbb{E} \left[ \sum h_{ij} G_{ji} \right] = \mathbb{E} \left[ -m_N^2 - \frac{C_4}{q^2} m_N^4 - \frac{C_6}{q^4} m_N^6 + \cdots \right].
\]

By the cumulant expansion (here the independence is used)

\[
\mathbb{E}[h_{ij} G_{ij}] = \sum_{k=1}^{\ell} \frac{C_{k+1}}{(k-1)! N q^{k-1}} \mathbb{E}[\partial_{h_{ij}}^k G_{ji}] + O \left( \frac{1}{q^{\ell}} \right).
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Correction of the Self-Consistent Equation

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$$1 + zm_N = \frac{1}{N} \sum_{ij} h_{ij} G_{ij} = \frac{1}{N} \sum_{ij} h_{ij} \left[ - \sum_k G_{ii} h_{ik} G_{kj}^{(i)} \right],$$

where $G^{(i)}$ is the Green's function of the matrix $H^{(i)}$ with the $i$-th column and row setting to zero. Assume that $G_{ii}$ can be replaced by $m_N$ and the leading fluctuation is from terms with $j = k$.

$$- \frac{1}{N} \sum_{ij} h_{ij}^2 m_N^2 = -m_N^2 - \frac{1}{N} \sum_{ij} (h_{ij}^2 - N^{-1}) m_N^2 = -m_N^2 - X m_N^2$$
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The quantity $P_0(z, m_N)$ has fluctuation of order $O(1/N \sqrt{p})$. To cancel the fluctuation, we define the random polynomial $P(z, m)$ as

$$P(z, m) = P_0(z, m) + \chi m^2.$$
Construction of the Limit Measure

With the random polynomial $P(z, m)$

$$P(z, m) = P_0(z, m) + \mathcal{X} m^2,$$

We will prove that $P(z, m_N(z)) \approx 0$, and therefore $|m_N(z) - m_\rho(z)| \approx 0$, where $m_\rho(z)$ is the exact solution of this equation, $P(z, m_\rho(z)) = 0$. 

Proposition (Existence of the limit measure)

There exists $m_\rho: C^+ \to C^+$, with $P(z, m_\rho(z)) = 0$ satisfying $m_\rho$ is the Stieltjes transform of a random probability measure $\rho$ with $\text{supp} \rho = [-L, L]$. The density $\rho$ has a square root behavior at the edge. $L$ has Gaussian fluctuation, explicitly $L = E^* + X + O \ll (1/\sqrt{N q^3})$, where $E^*$ is deterministic depending on $q$ and the cumulants of $h_{ij}$. 

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Construction of the Limit Measure

With the random polynomial $P(z, m)$

$$P(z, m) = P_0(z, m) + \lambda m^2,$$

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$$L = E_\ast + \lambda + O \left( \frac{1}{\sqrt{N}q^3} \right),$$

where $E_\ast$ is deterministic depending on $q$ and the cumulants of $h_{ij}$. 
Higher moments

To show that $P(z, m_N(z))$ is small, we compute its higher moments. Since the spectral edge $L \approx E_\ast + \lambda$ fluctuates, we take $z$ to be fixed with respect to the spectral edge.

**Proposition (Higher moments estimate)**

Fix $|\kappa| \ll 1$, and $\eta \gg 1/N$, we take $z = E_\ast + \lambda + \kappa + i\eta$, then

$$
\mathbb{E}[|P(z, m_N(z)|^{2r}] \leq \max_{s+t \geq 1} \mathbb{E} \left[ \left( \left( \frac{1}{q^3} + \frac{1}{N\eta} \right) \left( \frac{\text{Im}[m_N(z)]|P'(z, m_N(z))|}{N\eta} \right) \right)^{s/2} \times \left( \frac{\text{Im}[m_N(z)]}{N\eta} \right)^t |P(z, m_N(z))|^{2r-s-t} \right].
$$

Let $\varepsilon > 0$ and $Np \geq N^{2/9} + \varepsilon$ (weaker results for $Np \leq N^{2/9}$) and take $z = E_* + X + \kappa + i\eta$. Outside the spectrum, if $\eta \sim N^{-2/3}$ and $\kappa \geq N^{-2/3}$,

$$|m_N(z) - m_\rho(z)| \ll \frac{1}{N\eta}.$$  

A corresponding estimate holds inside the spectrum.

Let $\epsilon > 0$ and $Np \geq N^{2/9+\epsilon}$ (weaker results for $Np \leq N^{2/9}$) and take $z = E_* + \mathcal{X} + \kappa + i\eta$. Outside the spectrum, if $\eta \sim N^{-2/3}$ and $\kappa \geq N^{-2/3}$,

$$|m_N(z) - m_{\rho}(z)| \ll \frac{1}{N\eta}.$$  

A corresponding estimate holds inside the spectrum.

- The spectral edges of $m_N(z)$ and $m_{\rho}(z)$ are the same with accuracy of order $N^{-2/3}$. Since the spectral edge of $\rho$ is at $L \approx E_* + \mathcal{X}$, we have $\mu_k = E_* + \mathcal{X} + O(N^{-2/3})$.

- In the regime $Np \ll N^{1/3}$, $\mathcal{X} \sim 1/(N\sqrt{p}) \gg N^{-2/3}$, this result implies that the fluctuation of extreme eigenvalues is governed by $\mathcal{X}$, which is asymptotically Gaussian, instead of Tracy-Widom $\beta = 1$ distribution.
Proof for the Tracy-Widom $\beta = 1$ Distribution

- Using the higher order self-consistent equation to conclude the rigidity of the extreme eigenvalues, i.e., the eigenvalues of $H$ are close to the classical eigenvalue locations of the random measure $\rho$. 

Universality of extreme eigenvalue distributions for Gaussian divisible ensembles. Prove that the local equilibrium for $H_{t}=e^{-t/2}H+\sqrt{1-e^{-t}}\text{GOE}$, at the edge is reached at $t \geq N-1/3+\varepsilon$. This is done via a purely Dyson Brownian motion argument (Landon-Yau 2017).

Green function comparison method: It shows that the extreme eigenvalue distributions remain unchanged (up to a deterministic shift) for time $t \leq N-1/3+\varepsilon$. This relies on the underlying matrix model.
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Thank you!