Extreme Eigenvalue Distributions of Sparse Erdős-Rényi Graphs

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Joint work with Benjamin Landon, Horng-Tzer Yau

IPAM Workshop: Random Matrices and Free Probability Theory

Random graph models

- Erdős-Rényi Graphs $G(N, p)$: each edge selected independently with probability p. The average degree is pN.

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- We are interested in the sparse random Erdős-Rényi graphs:

$$p \ll 1.$$

Random graph models

For a random graph $\mathcal{G}$ on N vertices, denote its adjacency matrix by

$$A_{ij} = \mathbf{1}_{\{i\sim j\}}.$$

We rescale it such that each entry has variance $1/N$, and consider the rescaled adjacency matrix

$$\frac{A}{\sqrt{Np(1-p)}},$$

and denote its eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ and corresponding normalized eigenvectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$.

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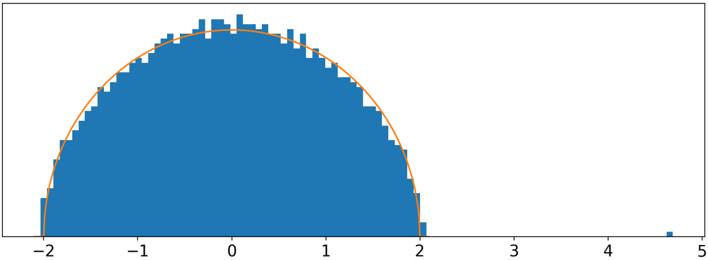
Fundamental question:

What are the probability distributions of the eigenvalues and eigenvectors?

Empirical eigenvalue distribution with $Np \gg 1$

The empirical eigenvalue distribution $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 2000$ and $p = 0.01$:



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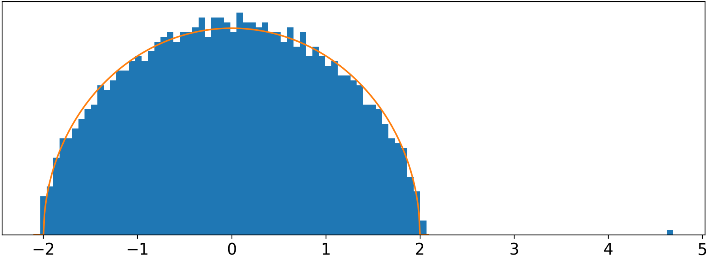

- The empirical eigenvalue distribution converges to the semicircle distribution (Wigner, 1950s)

$$\rho_{\mathrm{sc}}(x) = \frac{\sqrt{4 - x^2}}{2\pi}.$$

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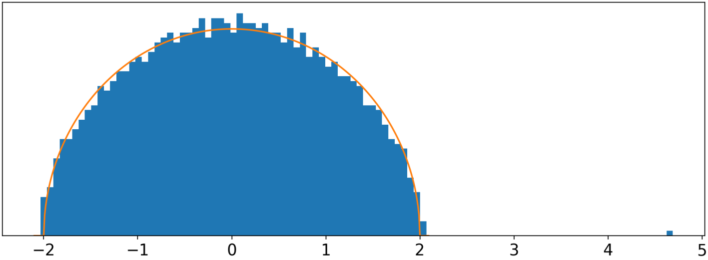

- The rescaled adjacency matrix is a rank one perturbation of a mean zero matrix (Füredi-Komlós, 1980; Féral-Péché, 2008). The largest eigenvalue concentrates around $\sqrt{Np/(1-p)} + \sqrt{(1-p)/Np} \approx 4.72$ (Erdős-Knowles–Yau–Yin, 2013).

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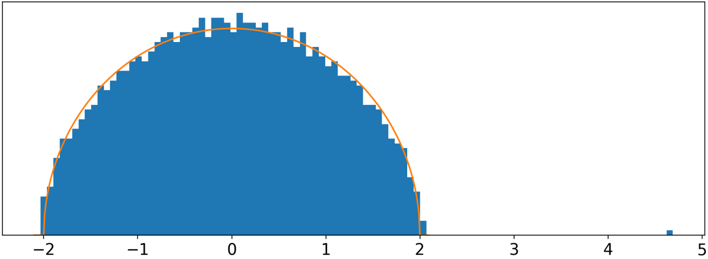
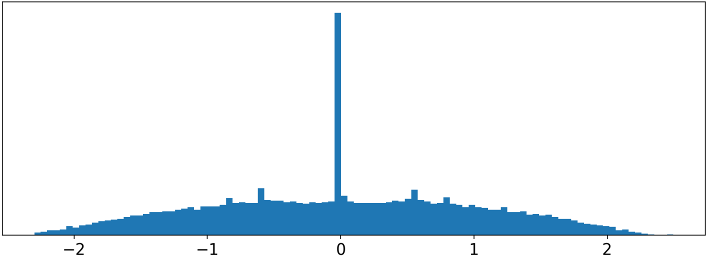


- The second largest eigenvalue concentrates around 2 (Khorunzhiy, 2001; Vu, 2007; Erdős-Knowles-Yau-Yin, 2013; Benaych-Georges-Bordenave-Knowles, 2017).

Empirical eigenvalue distribution with $Np \asymp 1$

The empirical eigenvalue distribution $\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}$ of a random Erdős-Rényi Graph with $N = 3000$ and $p = 0.001$:



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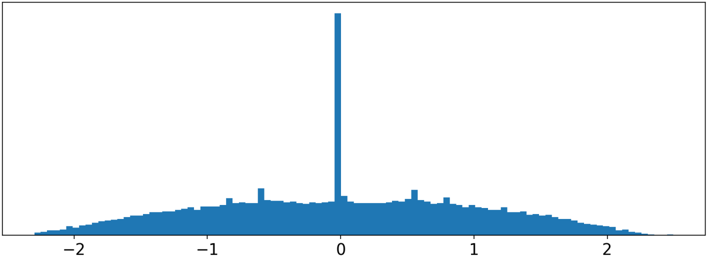
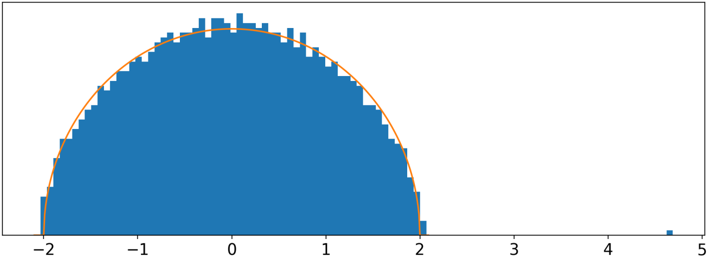


- The adjacency matrix is singular, there are many zero eigenvalues (Bordenave-Sen-Virág, 2013).
- The empirical eigenvalue density is not compactly supported (Khorunzhiy, 2001; Bordenave-Sen-Virág, 2013; Benaych-Georges-Bordenave-Knowles, 2017).

Growing average degree case: $pN \gg 1$



Empirical eigenvalue distribution of Erdős-Rényi Graphs $G(2000, 0.01)$.

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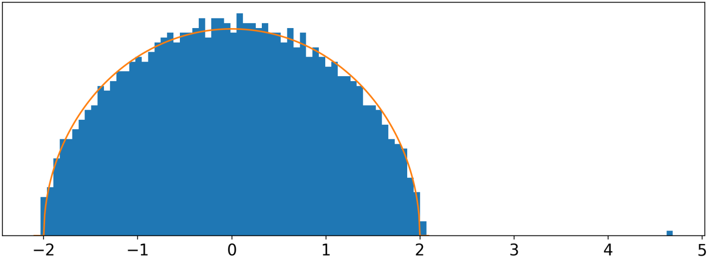

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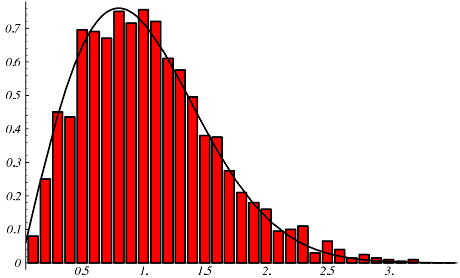
Main goal:

To understand the local statistics of eigenvalues and eigenvectors.

Bulk universality

Gap distribution for the bulk eigenvalues $N(\lambda_i - \lambda_{i+1})$ is expected to be universal, the Gaudin-Mehta distribution (Gap distribution for GOE), approximately given by the Wigner Surmise

$$p(s) \approx \frac{\pi s}{2} e^{-\frac{\pi}{4}s^2}.$$



Eigenvalue gaps for Erdős-Rényi Graphs $G(2000, 0.01)$.

Bulk Universality

Theorem (Erdős-Knowles–Yau–Yin, 2011)

Let $\varepsilon > 0$ and $pN \geqslant N^{2/3+\varepsilon}$. Then in the bulk, the Erdős-Rényi graphs $G(N, p)$ obey the same local eigenvalue statistics as Gaussian Orthogonal Ensemble.

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Theorem (H.–Landon–Yau, 2015)

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Theorem (Bourgade-H.-Yau, 2016)

Let $\varepsilon > 0$ and $pN \geqslant N^{\varepsilon}$. The bulk eigenvectors are asymptotically normal,

$$N|\boldsymbol{u}_i(j)|^2 \to \mathcal{N}(0,1)^2,$$

where $\mathcal{N}$ is the standard normal random variable.

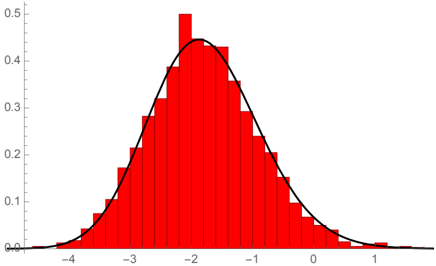
- Eigenvector Flow (Bourgade-Yau, 2013).

Edge universality

Distribution of the second largest eigenvalue is expected to be given by Tracy-Widom $\beta = 1$ distribution (largest eigenvalue of GOE).

$$N^{2/3}(\lambda_2 - E_*) \to TW_1.$$

(The largest eigenvalue is trivial and roughly given by the expected degree.)



Tracy Widom $\beta = 1$ distribution.

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Theorem (Lee-Schnelli, 2016)

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$$N^{2/3}(\lambda_2 - E_*) \to TW_1, \quad E_* = 2 + \frac{1}{Np}.$$

Gaussian Fluctuation

We normalized the entries of the adjacency matrix A to have mean zero and variance $1/N$,

$$H = \frac{A - p\mathbf{1}\mathbf{1}^*}{\sqrt{Np(1-p)}},$$

and denote its eigenvalues by $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_N$.

Theorem (H.-Landon-Yau, 2017)

Let $\varepsilon > 0$ and $N^{2/9+\varepsilon} \leqslant pN \leqslant N^{1/3-\varepsilon}$. The largest few eigenvalues of H have Gaussian fluctuation, i.e. for any fixed $k \geqslant 1$

$$\sqrt{p/2}N(\mu_k - E_*) \to \mathcal{N}(0,1), \quad E_* = 2 + \frac{1}{pN} - \frac{5}{4(pN)^2}.$$

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- We can explicitly identify the fluctuation

$$\mu_k - E_* = \mathcal{X} + \text{error}, \quad \mathcal{X} = \frac{1}{N}\left(\sum_{ij} h_{ij}^2 - N^{-1}\right).$$

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- By Cauchy Interlacing Theorem, the eigenvalues of $A/\sqrt{Np(1-p)}$ and H are interlaced, i.e. $\mu_2 \leqslant \lambda_2 \leqslant \mu_1$. $\sqrt{p/2}N(\lambda_2 - E_*) \to \mathcal{N}(0,1)$.

Gaussian Fluctuation

- For $Np \gg N^{1/3}$, λ_2 has Tracy-Widom $\beta = 1$ distribution asymptotically.
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Why is there a transition at $Np \asymp N^{1/3}$?

Theorem (Shcherbina-Tirozzi, 2011)

For $1/N \ll p \ll 1$, under mild conditions for the test function f, the linear statistics of H are asymptotically Gaussian

$$\sqrt{p}\left(\sum_i f(\mu_i) - \mathbb{E}\sum_i f(\mu_i)\right) \to \mathcal{N}(0, \sigma_f^2).$$

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- For Wigner matrices, the normalization factor is 1.
- The eigenvalues of H behave like an oscillating spring system, eigenvalues oscillate together on the scale $1/(N\sqrt{p})$, but the gaps are rigid.

Tracy Widom $\beta = 1$ Distribution

Theorem (H.-Landon-Yau, 2017)

Let $\varepsilon > 0$ and $N^{2/9+\varepsilon} \leqslant pN$. Subject to a random shift, the largest few eigenvalues of $H = (h_{ij})_{i,j=1}^N$ have Tracy-Widom $\beta = 1$ distribution, i.e. for any fixed $k \geqslant 1$, the joint law

$$N^{2/3}(\mu_1 - E_* - \mathcal{X}, \mu_2 - \mu_1, \cdots, \mu_k - \mu_{k-1}) \to TW_1,$$

where $\mathcal{X} = \frac{1}{N}\left(\sum_{ij} h_{ij}^2 - N^{-1}\right) \asymp \mathrm{O}\left(\frac{1}{N\sqrt{p}}\right)$.

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- For $pN \asymp N^{1/3}$, $\mathcal{X} \asymp N^{-2/3}$, the fluctuation of extreme eigenvalues is a combination of Gaussian and Tracy-Widom $\beta = 1$ distribution.

Basic tools

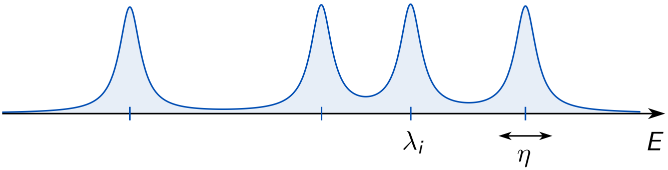
Stieltjes transform of a measure ϱ:

$$m_\varrho(z) = \int \frac{\varrho(x)\mathrm{d}x}{x-z}, \quad z \in \mathbb{C}_+,$$
$$\varrho(E) = \frac{1}{\pi} \lim_{\eta \to 0+} \mathrm{Im}[m_\varrho(E + \mathrm{i}\eta)].$$

The Stieltjes transform contains info for eigenvalues: Writing $z = E + \mathrm{i}\eta$, we have

$$\mathrm{Im}[m_N(z)] = \mathrm{Im}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z}\right] = \frac{1}{N}\sum_{i=1}^{N}\frac{\eta}{(\lambda_i - E)^2 + \eta^2}$$

and η is the spectral resolution.



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The Stieltjes transform can be used to detect the locations of extreme eigenvalues.

Proposition

If for some $z = E + \mathrm{i}\eta \in \mathbb{C}_+$, such that $\mathrm{Im}[m_N(z)] \ll 1/N\eta$, then there is no eigenvalue in the interval $[E - \eta, E + \eta]$.

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Proof.

We prove by contradiction. If there is an eigenvalue $\lambda_k \in [E-\eta, E+\eta]$, then

$$\frac{1}{N\eta} \gg \mathrm{Im}[m_N(z)] = \mathrm{Im}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\lambda_i - z}\right] = \frac{1}{N}\sum_{i=1}^{N}\frac{\mathrm{Im}[z]}{|\lambda_i - z|^2}$$
$$\geqslant \frac{1}{N}\frac{\mathrm{Im}[z]}{|\lambda_k - z|^2} = \frac{1}{N}\frac{\eta}{(E-\lambda_k)^2+\eta^2} \geqslant \frac{1}{2N\eta}.$$

□

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Example (Gaussian Orthogonal Ensemble $H = (h_{ij})_{i,j=1}^N$)

- The Stieltjes transform of the semicircle distribution satisfies

$$m_{sc}(z)^2 + zm_{sc}(z) + 1 = 0.$$

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- We denote the Green's function of H as $G(z) = (H - z)^{-1}$ and its Stieltjes transform $m_N(z) = \mathrm{Tr}\, G(z)/N$, then $\mathbb{E}[1 + zm_N + m_N^2] \approx 0$.

$$\begin{aligned}
\mathbb{E}[1 + zm_N] &= \frac{1}{N}\mathbb{E}[\mathrm{Tr}(I + zG)] = \frac{1}{N}\mathbb{E}[\mathrm{Tr}((H - z)G + zG)] \\
&= \frac{1}{N}\mathbb{E}[\mathrm{Tr}(HG)] = \frac{1}{N}\mathbb{E}\left[\sum h_{ij} G_{ji}\right] = \frac{1 + \delta_{ij}}{N^2}\mathbb{E}\left[\sum \partial_{h_{ij}} G_{ji}\right] \\
&= -\frac{1}{N^2}\mathbb{E}\left[\sum G_{ii} G_{jj} + G_{ji} G_{ji}\right] \approx -\frac{1}{N^2}\mathbb{E}\left[\sum G_{ii} G_{jj}\right] = -\mathbb{E}[m_N^2].
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- Using the same trick, the higher moment $\mathbb{E}[|1+zm_N+m_N^2|^{2r}]$ is small, and by Markov's inequality, $1+zm_N+m_N^2\approx 0$ with high probability. By comparing with $1+zm_{sc}+m_{sc}^2=0$, we can get $|m_N(z)-m_{sc}(z)|\approx 0$.

Construction of the Self-consistent Equation

We denote $q = (Np)^{1/2}$. For $H = (h_{ij})_{i,j=1}^N$, $\mathbb{E}[h_{ij}] = 0$, $\mathbb{E}[h_{ij}^2] = 1/N$, and the k-th cumulant of h_{ij} is $\kappa_k(h_{ij}) = \frac{\mathcal{C}_k}{Nq^{k-2}}$, for $k \geqslant 3$.

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Proposition (higher order self-consistent equation)

There exists a polynomial P_0 depending on the cumulants of h_{ij}

$$P_0(z, m) = 1 + zm + m^2 + \frac{\mathcal{C}_4}{q^2} m^4 + \frac{\mathcal{C}_6}{q^4} m^6 + \cdots ,$$

so that for $z = E + \mathrm{i}\eta$ with $\eta \gg 1/N$, the Stieltjes transform m_N of H satisfies

$$\mathbb{E}[P_0(z, m_N)] \leqslant \frac{\mathbb{E}[\mathrm{Im}[m_N(z)]]}{N\eta}.$$

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so that for $z=E+\mathrm{i}\eta$ with $\eta\gg 1/N$, the Stieltjes transform m_N of H satisfies

$$\mathbb{E}[P_0(z,m_N)]\leqslant\frac{\mathbb{E}[\mathrm{Im}[m_N(z)]]}{N\eta}.$$

The first three terms of $P_0(z,m)$ recover the self-consistent equation of semi-circle distribution. The first four terms of $P_0(z,m)$ were derived by Lee and Schnelli, and used in their proof of Tracy-Widom distribution of extreme eigenvalues for $Np\gg N^{1/3}$.

Construction of the Self-consistent Equation

The same as in the GOE example:

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The derivatives of the resolvent entries, $\partial_{h_{ij}}^k G_{ij}$, are polynomials in terms of the Green's function entries. Two kinds of terms might occur:

- terms containing off-diagonal entries are small.
- terms containing only diagonal entries can be iteratively rewritten as a polynomial of m_N, e.g.
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Correction of the Self-Consistent Equation

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$$1 + zm_N = \frac{1}{N}\sum_{ij} h_{ij} G_{ij} = \frac{1}{N}\sum_{ij} h_{ij}\Big[-\sum_k G_{ii} h_{ik} G_{kj}^{(i)}\Big],$$

where $G^{(i)}$ is the Green's function of the matrix $H^{(i)}$ with the i-th column and row setting to zero. Assume that G_{ii} can be replaced by m_N and the leading fluctuation is from terms with $j = k$.

$$-\frac{1}{N}\sum_{ij} h_{ij}^2 m_N^2 = -m_N^2 - \frac{1}{N}\sum_{ij}\left(h_{ij}^2 - N^{-1}\right) m_N^2 = -m_N^2 - \mathcal{X} m_N^2$$

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The quantity $P_0(z,m_N)$ has fluctuation of order $\mathrm{O}(1/N\sqrt{p})$. To cancel the fluctuation, we define the random polynomial $P(z,m)$ as

$$P(z,m) = P_0(z,m) + \mathcal{X} m^2.$$

Construction of the Limit Measure

With the random polynomial $P(z, m)$

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We will prove that $P(z, m_N(z)) \approx 0$, and therefore $|m_N(z) - m_\rho(z)| \approx 0$, where $m_\rho(z)$ is the exact solution of this equation, $P(z, m_\rho(z)) = 0$.

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Proposition (Existence of the limit measure)

There exists $m_\rho : \mathbb{C}_+ \to \mathbb{C}_+$, with $P(z, m_\rho(z)) = 0$ satisfying

- *m_ρ is the Stieltjes transform of a random probability measure ρ with $\operatorname{supp} \rho = [-L, L]$. The density ρ has a square root behavior at the edge.*
- *L has Gaussian fluctuation, explicitly*

$$L = E_* + \mathcal{X} + \mathrm{O}_\prec\left(\frac{1}{\sqrt{N} q^3}\right),$$

where E_* is deterministic depending on q and the cumulants of h_{ij}.

Higher moments

To show that $P(z, m_N(z))$ is small, we compute its higher moments. Since the spectral edge $L \approx E_* + \mathcal{X}$ fluctuates, we take z to be fixed with respect to the spectral edge.

Proposition (Higher moments estimate)

Fix $|\kappa| \ll 1$, *and* $\eta \gg 1/N$, *we take* $z = E_* + \mathcal{X} + \kappa + \mathrm{i}\eta$, *then*

$$\mathbb{E}[|P(z, m_N(z)|^{2r}] \leqslant \max_{s+t\geqslant 1} \mathbb{E}\left[\left(\left(\frac{1}{q^3} + \frac{1}{N\eta} \right) \left(\frac{\mathrm{Im}[m_N(z)]|P'(z, m_N(z))|}{N\eta} \right) \right)^{s/2} \times \left(\frac{\mathrm{Im}[m_N(z)]}{N\eta} \right)^t |P(z, m_N(z))|^{2r-s-t} \right].$$

Edge rigidity

Theorem (H.-Landon-Yau, 2017)

Let $\varepsilon > 0$ *and* $Np \geqslant N^{2/9+\varepsilon}$ *(weaker results for* $Np \leqslant N^{2/9}$*) and take* $z = E_* + \mathcal{X} + \kappa + \mathrm{i}\eta$*. Outside the spectrum, if* $\eta \sim N^{-2/3}$ *and* $\kappa \geqslant N^{-2/3}$*,*

$$|m_N(z) - m_\rho(z)| \ll \frac{1}{N\eta}.$$

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$$|m_N(z) - m_\rho(z)| \ll \frac{1}{N\eta}.$$

A corresponding estimate holds inside the spectrum.

- The spectral edges of $m_N(z)$ and $m_\rho(z)$ are the same with accuracy of order $N^{-2/3}$. Since the spectral edge of ρ is at $L \approx E_* + \mathcal{X}$, we have $\mu_k = E_* + \mathcal{X} + \mathrm{O}(N^{-2/3})$.
- In the regime $Np \ll N^{1/3}$, $\mathcal{X} \sim 1/(N\sqrt{p}) \gg N^{-2/3}$, this result implies that the fluctuation of extreme eigenvalues is governed by $\mathcal{X}$, which is asymptotically Gaussian, instead of Tracy-Widom $\beta = 1$ distribution.

Proof for the Tracy-Widom $\beta = 1$ Distribution

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$$H_t = e^{-t/2}H + \sqrt{1 - e^{-t}}GOE,$$

at the edge is reached at $t \geqslant N^{-1/3+\varepsilon}$. This is done via a purely Dyson Brownian motion argument (Landon-Yau 2017).

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  at the edge is reached at $t \geqslant N^{-1/3+\varepsilon}$. This is done via a purely Dyson Brownian motion argument (Landon-Yau 2017).
- Green function comparison method: It shows that the extreme eigenvalue distributions remain unchanged (up to a deterministic shift) for time $t \leqslant N^{-1/3+\varepsilon}$. This relies on the underlying matrix model.

Thank you!