Heavy-tail phenomena in large deviations of random matrices

Fanny Augeri

May 18, 2018

מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Unitary invariant models

Let V be some potential.

$$X \propto e^{-n\mathrm{tr}V(H)}dH,$$

Unitary invariant models

Let V be some potential.

$$X \propto e^{-n\mathrm{tr} V(H)} dH,$$

Joint law of the spectrum

$$\mathbb{P}^n_{V,\beta} \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n \sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

Unitary invariant models

Let V be some potential.

$$X \propto e^{-n\mathrm{tr}V(H)} dH,$$

Joint law of the spectrum

$$\mathbb{P}^n_{V,\beta} \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

If $$\liminf_{|x|\to+\infty} \{V(x) - 2\log|x|\} > -\infty.$$

(Ben-Arous - Guionnet). then μ_X follows a LDP at speed n^2,

Unitary invariant models

Let V be some potential.

$$X \propto e^{-n\mathrm{tr} V(H)} dH,$$

Joint law of the spectrum

$$\mathbb{P}^n_{V,\beta} \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n \sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

If $$\liminf_{|x|\to+\infty} \{V(x) - 2\log|x|\} > -\infty.$$

(Ben-Arous - Guionnet). then μ_X follows a LDP at speed n^2,

$$I_\beta(\mu) = \int V(x)d\mu(x) + \frac{\beta}{2}\int \log|x-y|^{-1} d\mu(x)d\mu(y) - c_\beta,$$

vanishes at a unique σ^V_β equilibrium measure of V.

Empirical means

$X_1, ..., X_n$ i.i.d random variables, X_i with no exponential moments.

$$\log \mathbb{P}(\pm X_i > t) \sim -at^{\alpha}, \ \alpha \in (0,1), a > 0.$$

Empirical means

$X_1, \ldots, X_n$ i.i.d random variables, X_i with no exponential moments.

$$\log \mathbb{P}(\pm X_i > t) \sim -at^{\alpha}, \ \alpha \in (0,1), a > 0.$$

$$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

(Nagaev): S_n follows a LDP with speed n^{α} and rate function,

$$I_{\alpha}(x) = a|x - m|^{\alpha}, \ m = \mathbb{E}X_i.$$

Empirical means

$X_1, ..., X_n$ i.i.d random variables, X_i with no exponential moments.

$$\log \mathbb{P}(\pm X_i > t) \sim -at^{\alpha},\ \alpha \in (0,1), a > 0.$$

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(Nagaev): S_n follows a LDP with speed n^{α} and rate function,

$$I_{\alpha}(x) = a|x-m|^{\alpha},\ m = \mathbb{E}X_i.$$

same rate function as $m + \frac{X_1}{n}$.

"heavy-tail phenomenon"

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

Assume that

$$V(x) = b|x|^\alpha + w(x), \;\; \alpha \geq 2,$$

w convex and $w(x) = o(|x|^\alpha)$. Let $p > \alpha$,

$$m_{n,p} = \frac{1}{n} \sum_{i=1}^n \lambda_i^p.$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

Assume that

$$V(x) = b|x|^\alpha + w(x), \;\; \alpha \geq 2,$$

w convex and $w(x) = o(|x|^\alpha)$. Let $p > \alpha$,

$$m_{n,p} = \frac{1}{n}\sum_{i=1}^n \lambda_i^p.$$

(A): Under $\mathbb{P}^n_{V,\beta}$, $m_{n,p}$ follows a LDP with speed $n^{1+\frac{\alpha}{p}}$ and rate function J_p,

$$J_{2k}(s) = \begin{cases} b\big(s - \sigma^V_\beta(x^p)\big)^{\frac{\alpha}{p}} & \text{if } s \geq \sigma^V_\beta(x^p), \\ +\infty & \text{otherwise,} \end{cases}$$

$$J_{2k+1}(s) = b\big|s - \sigma^V_\beta(x^p)\big|^{\frac{\alpha}{p}}.$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

Assume that

$$V(x) = b|x|^\alpha + w(x), \ \alpha \geq 2,$$

w convex and $w(x) = o(|x|^\alpha)$. Let $p > \alpha$,

$$m_{n,p} = \frac{1}{n}\sum_{i=1}^n \lambda_i^p.$$

(A): Under $\mathbb{P}^n_{V,\beta}$, $m_{n,p}$ follows a LDP with speed $n^{1+\frac{\alpha}{p}}$ and rate function J_p,

$$J_{2k}(s) = \begin{cases} b\left(s - \sigma^V_\beta(x^p)\right)^{\frac{\alpha}{p}} & \text{if } s \geq \sigma^V_\beta(x^p), \\ +\infty & \text{otherwise,} \end{cases}$$

$$J_{2k+1}(s) = b\left|s - \sigma^V_\beta(x^p)\right|^{\frac{\alpha}{p}}.$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n \sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i .$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i .$$

σ_β^V the equilibrium measure, $|\lambda_1^*| \geq ... \geq |\lambda_n^*|$,

$$m_{p,n} = \frac{1}{n} \sum_{i=1}^k \lambda_i^{*p} + \frac{1}{n} \sum_{i=k+1}^n \lambda_i^{*p} .$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

σ^V_β the equilibrium measure, $|\lambda^*_1| \geq \ldots \geq |\lambda^*_n|$,

$$m_{p,n} = \frac{1}{n}\sum_{i=1}^k \lambda_i^{*p} + \frac{1}{n}\sum_{i=k+1}^n \lambda_i^{*p}.$$

Deviations due to extreme particles $k = \log n$

$$m_{p,n} \simeq \frac{1}{n}\sum_{i=1}^k \lambda_i^{*p} + \sigma^V_\beta(x^p).$$

Moments of β-ensembles

$$d\mathbb{P}^n_{V,\beta} = \frac{1}{Z^n_\beta} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-n\sum_{i=1}^n V(\lambda_i)} \prod_{i=1}^n d\lambda_i.$$

σ^V_β the equilibrium measure, $|\lambda^*_1| \geq ... \geq |\lambda^*_n|$,

$$m_{p,n} = \frac{1}{n}\sum_{i=1}^k \lambda_i^{*p} + \frac{1}{n}\sum_{i=k+1}^n \lambda_i^{*p}.$$

Deviations due to extreme particles $k = \log n$

$$m_{p,n} \simeq \frac{1}{n}\sum_{i=1}^k \lambda_i^{*p} + \sigma^V_\beta(x^p).$$

Logarithmic interaction negligible,

$$m_{p,n} \simeq \frac{1}{n}\sum_{i=1}^k X_i{}^p + \sigma^V_\beta(x^p),$$

$X_1, ..., X_n$ i.i.d $\propto e^{-nV(x)}dx$.

Motivation

Let X be a Gaussian Wigner matrix,
such that $\mathbb{E}X = 0$ and $\mathbb{E}|X_{1,2}|^2 = 1$.

Find a LDP for $\dfrac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$, for $p \geq 3$

and for $\dfrac{1}{n}\mathrm{tr}P(X_1/\sqrt{n}, ..., X_m/\sqrt{n})$,

when $P \in \mathbb{R}\langle X_1, ..., X_m\rangle$,
and $X_1, ..., X_m$ are i.i.d Gaussian Wigner matrices.

Motivation

Let X be a Gaussian Wigner matrix,
such that $\mathbb{E}X = 0$ and $\mathbb{E}|X_{1,2}|^2 = 1$.

Find a LDP for $\frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$, for $p \geq 3$

and for $\frac{1}{n}\mathrm{tr}P(X_1/\sqrt{n}, ..., X_m/\sqrt{n})$,

when $P \in \mathbb{R}\langle X_1, ..., X_m \rangle$,
and $X_1, ..., X_m$ are i.i.d Gaussian Wigner matrices.

(Meckes-Szarek): Concentration inequality for $Z = \frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$.

$$\mathbb{P}(|Z - \mathbb{E}Z| > t) \leq C \exp\left(- c \min\left(n^2 t^2, n^{1+2/p} t^{2/p}\right)\right).$$

LDP for Wiener chaoses

LDP for Wiener chaoses

$E = C_0([0,1])$, $\mathcal{H} = W_0^2([0,1])$, $\mu =$ Wiener measure.

LDP for Wiener chaoses

$E = C_0([0,1]),\ \mathcal{H} = W_0^2([0,1]),\ \mu =$ Wiener measure.

Ψ homogeneous Wiener chaos of degree d taking values in E.

LDP for Wiener chaoses

$E = C_0([0,1])$, $\mathcal{H} = W_0^2([0,1])$, $\mu =$ Wiener measure.

Ψ homogeneous Wiener chaos of degree d taking values in E.

$$\Psi = \int_0^{\cdot} \int_0^{t_1} \ldots \int_0^{t_{d-1}} k(t_1, \ldots, t_d) dw_{t_1} \ldots dw_{t_d}, \; k \in L^2(\ell_d).$$

LDP for Wiener chaoses

$$E = C_0([0,1]),\ \mathcal{H} = W_0^2([0,1]),\ \mu = \text{Wiener measure.}$$

Ψ homogeneous Wiener chaos of degree d taking values in E.

$$\Psi = \int_0^{\cdot} \int_0^{t_1} ... \int_0^{t_{d-1}} k(t_1, ..., t_d) dw_{t_1} ... dw_{t_d},\ k \in L^2(\ell_d).$$

(Borell|Ledoux). $\varepsilon^d \Psi$ follows a LDP with speed ε^{-2} and rate function

$$\mathcal{I}_\Psi(s) = \inf\left\{\frac{1}{2}|h|^2 : s = \Psi^{(d)}(h)\right\},$$

$$\Psi^{(d)}(h) = \int \Psi(w+h) d\mu(w),$$

$$|h| = \left(\int_0^1 |\dot{h}|^2 dt\right)^{1/2}.$$

LDP for Wiener chaoses

LDP for Wiener chaoses

Uniform deterministic equivalent

$$||\varepsilon^d \Psi(w + \varepsilon^{-1} h) - \Psi^{(d)}(h)|| \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

uniformly in $h \in \mathcal{O}$, unit ball of $\mathcal{H}$, in probability.

LDP for Wiener chaoses

Uniform deterministic equivalent

$$||\varepsilon^d \Psi(w + \varepsilon^{-1} h) - \Psi^{(d)}(h)|| \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

uniformly in $h \in \mathcal{O}$, unit ball of $\mathcal{H}$, in probability.

Estimate of translates (Cameron-Martin formula)

$$\mu(E + \varepsilon^{-1} h) \geq e^{-\frac{\varepsilon^{-2}}{2}|h|^2 + o(\varepsilon^{-2})},$$

for $\mu(E) \geq 1/2$.

LDP for Wiener chaoses

Uniform deterministic equivalent

$$||\varepsilon^d \Psi(w + \varepsilon^{-1}h) - \Psi^{(d)}(h)|| \underset{\varepsilon \to 0}{\longrightarrow} 0,$$

uniformly in $h \in \mathcal{O}$, unit ball of $\mathcal{H}$, in probability.

Estimate of translates (Cameron-Martin formula)

$$\mu(E + \varepsilon^{-1}h) \geq e^{-\frac{\varepsilon^{-2}}{2}|h|^2 + o(\varepsilon^{-2})},$$

for $\mu(E) \geq 1/2$.

Gaussian isoperimetric inequality

$$\mu(x \notin E + \sqrt{2r}\mathcal{O}) \leq e^{-r},$$

as soon as $\mu(E) \geq 1/2$.

Non-linear large deviations for the Gaussian measure

Take $X \in \mathbb{R}^n$ standard Gaussian random variable.

Look at the large deviations of $f_n(X)$ at speed v_n.

Non-linear large deviations for the Gaussian measure

Take $X \in \mathbb{R}^n$ standard Gaussian random variable.

Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0, \delta > 0,$

$$\mathbb{P}\Big(\sup_{|h|^2 \leq 2r v_n} \big| f_n(X+h) - \mathbb{E} f_n(X+h) \big| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

Non-linear large deviations for the Gaussian measure

Take $X \in \mathbb{R}^n$ standard Gaussian random variable.

Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0, \delta > 0$,

$$\mathbb{P}\Big(\sup_{|h|^2 \leq 2r v_n} \big| f_n(X+h) - \mathbb{E} f_n(X+h) \big| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

Then $f_n(X)$ follows a LDP with speed v_n and rate function

$$I(x) = \inf_{n \in \mathbb{N}} \Big\{ \frac{1}{2v_n} |h|^2 : x = \mathbb{E} f_n(X+h), h \in \mathbb{R}^n \Big\}.$$

Examples and counter-examples

The low complexity condition:

$$\mathbb{P}\Big(\sup_{|h|^2 \leq 2 r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

holds for

- Traces of Gaussian Wigner matrices $f_n = \frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$
- Non-commutative polynomial $\frac{1}{n}\mathrm{tr}P(X_1/\sqrt{n}, ..., X_m/\sqrt{n})$.

does not for

- Empirical spectral measure $\mu_{X/\sqrt{n}}$
- Largest eigenvalue $\lambda_{X/\sqrt{n}}$...

Traces of Gaussian Wigner matrices

Traces of Gaussian Wigner matrices

Let X be a centered Gaussian Wigner matrix, $\mathbb{E}|X_{1,2}|^2 = 1$.

(A). For $p \geq 3$. $\frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$ follows a LDP with speed $n^{1+\frac{2}{p}}$, and rate function J_p,

Traces of Gaussian Wigner matrices

Let X be a centered Gaussian Wigner matrix, $\mathbb{E}|X_{1,2}|^2 = 1$.

(A). For $p \geq 3$. $\frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$ follows a LDP with speed $n^{1+\frac{2}{p}}$, and rate function J_p,

$$J_{2k}(s) = \begin{cases} c_p\big(s - \mu_{sc}(x^p)\big)^{\frac{2}{p}} & \text{if } s \geq \mu_{sc}(x^p), \\ +\infty & \text{otherwise,} \end{cases}$$

$$J_{2k+1}(s) = c_p|s|^{\frac{2}{p}}.$$

Traces of Gaussian Wigner matrices

Let X be a centered Gaussian Wigner matrix, $\mathbb{E}|X_{1,2}|^2 = 1$.
(A). For $p \geq 3$. $\frac{1}{n}\mathrm{tr}(X/\sqrt{n})^p$ follows a LDP with speed $n^{1+\frac{2}{p}}$, and rate function J_p,

$$J_{2k}(s) = \begin{cases} c_p\big(s - \mu_{sc}(x^p)\big)^{\frac{2}{p}} & \text{if } s \geq \mu_{sc}(x^p), \\ +\infty & \text{otherwise,} \end{cases}$$

$$J_{2k+1}(s) = c_p|s|^{\frac{2}{p}}.$$

Variational formula

$$X \propto e^{-q(Y)}dY, \ q \text{ quadratic form.}$$

$$J_p(s) = \inf_{n\in\mathbb{N}}\{q(H) : s = \mu_{sc}(x^p) + \mathrm{tr}H^p\}.$$

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||_{\ell^\alpha}^{\alpha}} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||_{\ell^\alpha}^\alpha} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0$, $\delta > 0$,

$$\mathbb{P}\Big(\sup_{||h||_{\ell^\alpha}^\alpha \leq r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||_{\ell^\alpha}^\alpha} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0$, $\delta > 0$,

$$\mathbb{P}\Big(\sup_{||h||_{\ell^\alpha}^\alpha \leq r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

and L_2 the ℓ^2-Lipschitz constant of f_n satisfies

$$L_2 \ll v_n^{-1/2}$$

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||_{\ell^\alpha}^\alpha} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0$, $\delta > 0$,

$$\mathbb{P}\Big(\sup_{||h||_{\ell^\alpha}^\alpha \le r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

and L_2 the ℓ^2-Lipschitz constant of f_n satisfies

$$L_2 \ll v_n^{-1/2}$$

Then $f_n(X)$ satisfies a LDP with speed v_n and rate function

$$I_\alpha(x) = \inf \Big\{ \frac{||h||_{\ell^\alpha}^\alpha}{v_n} : \mathbb{E} f_n(X+h) = x \Big\}.$$

Examples

When $X \propto e^{-||h||_{\ell^\alpha}^\alpha} dh$ with $\alpha \in (0,2)$,
the conditions

$$\mathbb{P}\Big(\sup_{||h||_{\ell^\alpha}^\alpha \le r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n \to +\infty}{\longrightarrow} 0,$$

and $L_2 \ll v_n^{-1/2}$, hold for

- Traces of non-commutative polynomials $v_n = n^{\alpha(\frac{1}{2}+\frac{1}{d})}$
- Empirical spectral measure $v_n = n^{1+\alpha/2}$ (Bordenave-Caputo)
- Largest eigenvalue $v_n = n^{\alpha/2}$
- Last passage time $v_n = n^\alpha$, $\alpha < 1$,

but does not for the empirical mean $f_n = \frac{1}{n}\sum_{i=1}^n X_i$ when $\alpha \ge 1$...

Concentration for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in [1, 2]$.

Concentration for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^{\alpha}} dY$ with $\alpha \in [1, 2]$.

(Talagrand). Two-level deviations inequality

$$\mathbb{P}(X \notin V + \sqrt{r} B_{\ell^2} + r^{1/\alpha} B_{\ell^\alpha}) \le C e^{-cr},$$

for any $\mathbb{P}(X \in V) \ge 1/2$.

Concentration for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in [1, 2]$.

(Talagrand). Two-level deviations inequality

$$\mathbb{P}(X \notin V + \sqrt{r} B_{\ell^2} + r^{1/\alpha} B_{\ell^\alpha}) \leq C e^{-cr},$$

for any $\mathbb{P}(X \in V) \geq 1/2$.

(Lidskii). $\quad p \geq 1, \ \mathcal{W}_p(\mu_A, \mu_B) \leq \frac{1}{n^{1/p}} \left(\mathrm{tr}|A - B|^p\right)^{1/p}.$

Concentration for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in [1,2]$.

(Talagrand). Two-level deviations inequality

$$\mathbb{P}(X \notin V + \sqrt{r} B_{\ell^2} + r^{1/\alpha} B_{\ell^\alpha}) \leq C e^{-cr},$$

for any $\mathbb{P}(X \in V) \geq 1/2$.

(Lidskii). $\quad p \geq 1, \ \mathcal{W}_p(\mu_A, \mu_B) \leq \frac{1}{n^{1/p}} \left(\mathrm{tr}|A - B|^p\right)^{1/p}.$

$$p \leq 2, \qquad \mathrm{tr}|H|^p \leq ||H||_{\ell^p}^p.$$

Concentration for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in [1,2]$.

(Talagrand). Two-level deviations inequality

$$\mathbb{P}(X \notin V + \sqrt{r} B_{\ell^2} + r^{1/\alpha} B_{\ell^\alpha}) \leq C e^{-cr},$$

for any $\mathbb{P}(X \in V) \geq 1/2$.

(Lidskii). $\quad p \geq 1, \; \mathcal{W}_p(\mu_A, \mu_B) \leq \frac{1}{n^{1/p}} \left(\mathrm{tr}|A-B|^p\right)^{1/p}.$

$$p \leq 2, \qquad \mathrm{tr}|H|^p \leq ||H||_{\ell^p}^p.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-Lipschitz function.

$$\mathbb{P}(\mu_{X/\sqrt{n}}(f) - \mathbb{E}\mu_{X/\sqrt{n}}(f) > t) \leq A \exp\left(-\kappa \min\left(n^2 t^2, n^{1+\frac{\alpha}{2}} t^\alpha\right)\right).$$

LDP for the empirical spectral measure

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0,2)$.

(Bordenave-Caputo). $\mu_{X/\sqrt{n}}$ follows a LDP with speed $n^{1+\alpha/2}$ and rate function,

$$I(\mu) = \sup_{\delta>0} \inf_n \left\{ \frac{1}{n} ||H||_{\ell^\alpha}^\alpha : d(\mu, \mu_H \boxplus \mu_{sc}) < \delta, H \in \mathcal{H}_n \right\},$$

with $\boxplus$ the free convolution.

LDP for the empirical spectral measure

Let $X \propto e^{-||Y||^{\alpha}_{\ell^\alpha}} dY$ with $\alpha \in (0,2)$.

(Bordenave-Caputo). $\mu_{X/\sqrt{n}}$ follows a LDP with speed $n^{1+\alpha/2}$ and rate function,

$$I(\mu) = \sup_{\delta>0} \inf_{n} \left\{ \frac{1}{n} ||H||^{\alpha}_{\ell^\alpha} : d(\mu, \mu_H \boxplus \mu_{sc}) < \delta, H \in \mathcal{H}_n \right\},$$

with $\boxplus$ the free convolution.

$$\mathcal{D}(I) \subset \{\nu \boxplus \mu_{sc}, \nu(|x|^\alpha) < +\infty\}.$$

Heuristics for the lower bound

Let $X \propto e^{-\|Y\|_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0,2)$

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \mu_A),$$

with $A \in \mathcal{H}_m$ such that $\mu_A \simeq \nu$.

Heuristics for the lower bound

Let $X \propto e^{-\|Y\|_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0,2)$

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \mu_A),$$

with $A \in \mathcal{H}_m$ such that $\mu_A \simeq \nu$.

$$X_{(m)} = (X_{i,j} \mathbb{1}_{i,j \le m})_{i,j}, \quad X^{(m)} = X - X_{(m)}.$$

Heuristics for the lower bound

Let $X \propto e^{-\|Y\|_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0, 2)$

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \mu_A),$$

with $A \in \mathcal{H}_m$ such that $\mu_A \simeq \nu$.

$$X_{(m)} = (X_{i,j} \mathbb{1}_{i,j \le m})_{i,j}, \quad X^{(m)} = X - X_{(m)}.$$

By independence,

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(X_{(m)}/\sqrt{n} \simeq A) \times \mathbb{P}(\mu_{X^{(m)}/\sqrt{n}+A} \simeq \mu_{sc} \boxplus \mu_A).$$

Heuristics for the lower bound

Let $X \propto e^{-\|Y\|_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0,2)$

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \mu_A),$$

with $A \in \mathcal{H}_m$ such that $\mu_A \simeq \nu$.

$$X_{(m)} = (X_{i,j} \mathbb{1}_{i,j \leq m})_{i,j}, \quad X^{(m)} = X - X_{(m)}.$$

By independence,

$$\begin{aligned} \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(X_{(m)}/\sqrt{n} \simeq A) \\ \times \mathbb{P}(\mu_{X^{(m)}/\sqrt{n}+A} \simeq \mu_{sc} \boxplus \mu_A). \end{aligned}$$

From the asymptotic freeness of $X/\sqrt{n}$ and A,

$$\mathbb{P}(\mu_{X^{(m)}/\sqrt{n}+A} \simeq \mu_{sc} \boxplus \mu_A) \underset{n \to +\infty}{\longrightarrow} 1.$$

Heuristics for the lower bound

Let $X \propto e^{-||Y||_{\ell^\alpha}^\alpha} dY$ with $\alpha \in (0,2)$

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \mu_A),$$

with $A \in \mathcal{H}_m$ such that $\mu_A \simeq \nu$.

$$X_{(m)} = (X_{i,j} \mathbb{1}_{i,j \leq m})_{i,j}, \quad X^{(m)} = X - X_{(m)}.$$

By independence,

$$\mathbb{P}(\mu_{X/\sqrt{n}} \simeq \mu_{sc} \boxplus \nu) \gtrsim \mathbb{P}(X_{(m)}/\sqrt{n} \simeq A) \times \mathbb{P}(\mu_{X^{(m)}/\sqrt{n}+A} \simeq \mu_{sc} \boxplus \mu_A).$$

From the asymptotic freeness of $X/\sqrt{n}$ and A,

$$\mathbb{P}(\mu_{X^{(m)}/\sqrt{n}+A} \simeq \mu_{sc} \boxplus \mu_A) \underset{n \to +\infty}{\longrightarrow} 1.$$

$$\mathbb{P}(X_{(m)}/\sqrt{n} \simeq A) \gtrsim e^{-n^{\alpha/2}||A||_{\ell^\alpha}^\alpha}.$$

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||_{\ell^\alpha}^\alpha} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0$, $\delta > 0$,

$$\mathbb{P}\Big(\sup_{||h||_{\ell^\alpha}^\alpha \leq r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n\to+\infty}{\longrightarrow} 0,$$

and L_2 the ℓ^2-Lipschitz constant of f_n satisfies

$$L_2 \ll v_n^{-1/2}$$

Then $f_n(X)$ satisfies a LDP with speed v_n and rate function

$$I_\alpha(x) = \inf\Big\{ \frac{||h||_{\ell^\alpha}^\alpha}{v_n} : \mathbb{E} f_n(X+h) = x \Big\}.$$

The transportation method

μ satisfies a transportation-entropy inequality with cost c $(\mathcal{T}_c)$ if

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\mu)} \int c(x-y) d\pi(x,y) \leq H(\nu|\mu).$$

The transportation method

μ satisfies a transportation-entropy inequality with cost c ($\mathcal{T}_c$) if

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\mu)} \int c(x-y) d\pi(x,y) \leq H(\nu|\mu).$$

$\mathcal{T}_c$ implies deviations inequalities with enlargements w.r.t c

$$\mu\big(x : \inf_{y \in V} c(x-y) > r\big) \leq e^{-r\mu(V)}.$$

The transportation method

μ satisfies a transportation-entropy inequality with cost c ($\mathcal{T}_c$) if

$$\forall \nu, \inf_{\pi \in \Pi(\nu,\mu)} \int c(x-y) d\pi(x,y) \leq H(\nu|\mu).$$

$\mathcal{T}_c$ implies deviations inequalities with enlargements w.r.t c

$$\mu\big(x : \inf_{y \in V} c(x-y) > r\big) \leq e^{-r\mu(V)}.$$

Tensorization: if μ_i satisfies $\mathcal{T}_{c_i}$, then $\mu_1 \otimes \mu_2$ satisfies $\mathcal{T}_{c_1 \oplus c_2}$.

The transportation method

μ satisfies a transportation-entropy inequality with cost c ($\mathcal{T}_c$) if

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\mu)} \int c(x-y) d\pi(x,y) \leq H(\nu|\mu).$$

$\mathcal{T}_c$ implies deviations inequalities with enlargements w.r.t c

$$\mu\big(x : \inf_{y \in V} c(x-y) > r\big) \leq e^{-r\mu(V)}.$$

Tensorization: if μ_i satisfies $\mathcal{T}_{c_i}$, then $\mu_1 \otimes \mu_2$ satisfies $\mathcal{T}_{c_1 \oplus c_2}$.

$\mathcal{T}_c$ implies dimension free concentration inequalities

$$\mu^n\big(x : \inf_{y \in V} \sum_{i=1}^n c_i(x_i - y_i) > r\big) \leq e^{-r\mu^n(V)}.$$

The transportation method

μ satisfies a transportation-entropy inequality with cost c ($\mathcal{T}_c$) if

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\mu)} \int c(x-y)d\pi(x,y) \leq H(\nu|\mu).$$

$\mathcal{T}_c$ implies deviations inequalities with enlargements w.r.t c

$$\mu\big(x : \inf_{y \in V} c(x-y) > r\big) \leq e^{-r\mu(V)}.$$

Tensorization: if μ_i satisfies $\mathcal{T}_{c_i}$, then $\mu_1 \otimes \mu_2$ satisfies $\mathcal{T}_{c_1 \oplus c_2}$.

$\mathcal{T}_c$ implies dimension free concentration inequalities

$$\mu^n\big(x : \inf_{y \in V} \sum_{i=1}^n c_i(x_i - y_i) > r\big) \leq e^{-r\mu^n(V)}.$$

(Gozlan) : Dimension free concentration is equivalent to $\mathcal{T}_c$.

Examples

$\alpha \in (0,2],\ \nu_\alpha \propto e^{-|x|^\alpha} dx$.

(Talagrand). ν_2 satisfies $\mathcal{T}_c$ with $c = |x-y|^2$.

If $\mu = e^{-V} dx$ with V a convex function, then μ satisfies $\mathcal{T}_c$ with

$$c(x,y) = V(y) - V(x) - V'(x)(y-x).$$

If V is uniformly strictly convex, $V'' \geq \kappa$, then

$$\mu \text{ satisfies } \mathcal{T}_c \text{ with } c = \kappa \frac{|x-y|^2}{2}.$$

Given μ, among cost functions $c = c(x-y)$ the best one is

$$\Lambda_\mu^*(x) = \sup_\lambda \{ \langle \lambda, x \rangle - \Lambda_\mu(\lambda) \}.$$

(Latała-Wojtaszczyk): if $\mu \in \mathcal{P}(\mathbb{R})$ is log-concave and symmetric then it satisfies $\mathcal{T}_c$ with

$$c(x,y) = \Lambda_\mu^* \Big(\frac{|x-y|}{\kappa} \Big), \kappa \text{ universal}$$

The exponential measure

$\nu_1 \propto e^{-|x|}dx$. For any $\lambda \in (0,1)$, define the cost function

$$c_\lambda(x) = (\frac{1}{\lambda} - 1)(e^{-\lambda|x|} + \lambda|x| - 1).$$

(Talagrand). For any $\lambda \in (0,1)$, ν_1 satisfies $\mathcal{T}_{c_\lambda}$:

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\nu_1)} \int c_\lambda(x-y)d\pi(x,y) \leq H(\nu|\nu_1).$$

The exponential measure

$\nu_1 \propto e^{-|x|}dx$. For any $\lambda \in (0,1)$, define the cost function

$$c_\lambda(x) = \left(\frac{1}{\lambda} - 1\right)\left(e^{-\lambda|x|} + \lambda|x| - 1\right).$$

(Talagrand). For any $\lambda \in (0,1)$, ν_1 satisfies $\mathcal{T}_{c_\lambda}$:

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\nu_1)} \int c_\lambda(x-y)\,d\pi(x,y) \leq H(\nu|\nu_1).$$

Behavior of c_λ

$$c_\lambda(x) \underset{0}{\sim} \frac{\lambda(1-\lambda)}{2}x^2, \quad c_\lambda(x) \underset{+\infty}{\sim} (1-\lambda)|x|.$$

The exponential measure

$\nu_1 \propto e^{-|x|}dx$. For any $\lambda \in (0,1)$, define the cost function

$$c_\lambda(x) = \big(\frac{1}{\lambda} - 1\big)\big(e^{-\lambda|x|} + \lambda|x| - 1\big).$$

(Talagrand). For any $\lambda \in (0,1)$, ν_1 satisfies $\mathcal{T}_{c_\lambda}$:

$$\forall \nu, \quad \inf_{\pi \in \Pi(\nu,\nu_1)} \int c_\lambda(x-y)d\pi(x,y) \leq H(\nu|\nu_1).$$

Behavior of c_λ

$$c_\lambda(x) \underset{0}{\sim} \frac{\lambda(1-\lambda)}{2}x^2, \quad c_\lambda(x) \underset{+\infty}{\sim} (1-\lambda)|x|.$$

Two-level deviation inequality

$$\nu_1^n\big(x \notin V + \kappa_\lambda \sqrt{r} B_{\ell^2} + (1-\lambda) r B_{\ell^1}\big) \leq e^{-r\nu_1^n(V)}.$$

Transporting ν_1 to ν_α yields a similar deviation inequality.

Heavy-tail phenomena for super-Gaussian laws

Let $\alpha \in (0,2)$, $X \propto e^{-||h||^{\alpha}_{\ell^\alpha}} dh$.
Look at the large deviations of $f_n(X)$ at speed v_n.

(A). If for any $r > 0$, $\delta > 0$,

$$\mathbb{P}\Big(\sup_{||h||^{\alpha}_{\ell^\alpha} \leq r v_n} |f_n(X+h) - \mathbb{E} f_n(X+h)| > \delta \Big) \underset{n\to+\infty}{\longrightarrow} 0,$$

and L_2 the ℓ^2-Lipschitz constant of f_n satisfies

$$L_2 \ll v_n^{-1/2}$$

Then $f_n(X)$ satisfies a LDP with speed v_n and rate function

$$I_\alpha(x) = \inf \Big\{ \frac{||h||^{\alpha}_{\ell^\alpha}}{v_n} : \mathbb{E} f_n(X+h) = x \Big\}.$$

Thank you for your attention