Developments with $b-$additive and $q-$multiplicative finite free convolutions

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Outline

1. An Introduction to the Classical Convolutions
   - Real Rooted Polynomials
   - The Classical Convolutions
2. The $b-$additive Convolution
   - Mesh Considerations
   - Finite Difference Operators
   - Main Result
   - Current Work
   - Future Directions

Introduction

- Real rooted polynomials used to encode combinatorial objects using coefficients and roots

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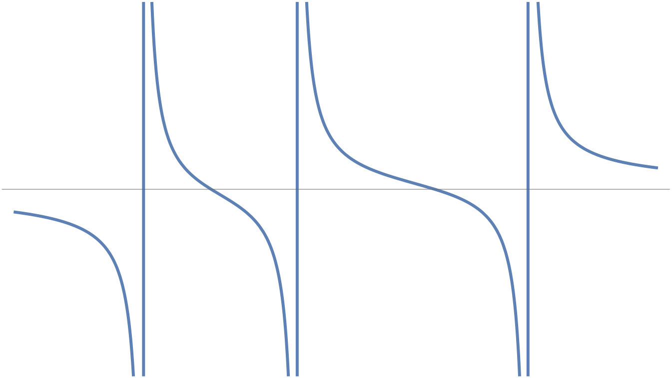

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- $D : p \mapsto p'$ preserves real rootedness by Rolle's Theorem
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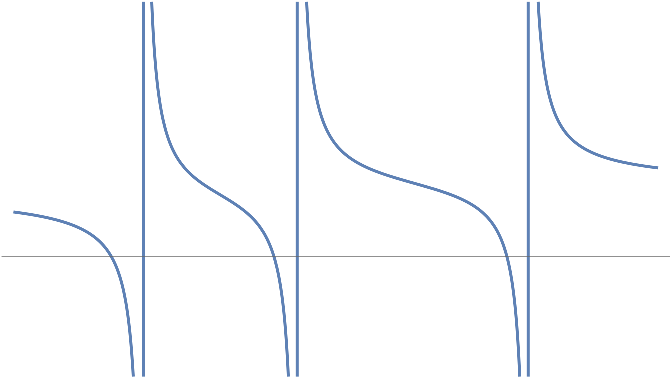

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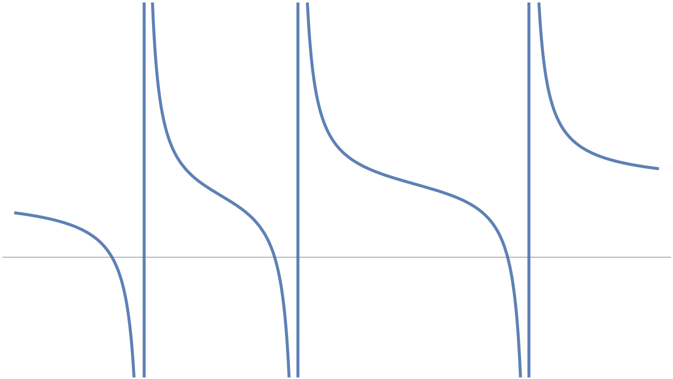
Lemma

The map $p \mapsto p' - \alpha p$ for real α preserves real rootedness



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Question

Which operators $\sum a_k D^k$ preserve real rootedness?

History of R. R. Preserving Operators

Theorem

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Theorem (Polya-Schur)

A diagonal linear operator $T(x^k) = a_k x^k$ *preserves the set of all real rooted polynomials if and only if* $T(e^x)$ *is the limit of either positive roote polynomials or negative rooted polynomials.*

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Theorem (Borcea-Brendan 2008)

For all non-degenerate linear transformations, real rooted preserving if and only if $T(e^{xz})$ or $T(e^{-xz})$ is real-stable.

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Example

If $p(x) = x^2 + 2x + 2$ then $p(D) = D^2 + 2D + 2I$. p is not real rooted, but $p(D)$ preserves all real rooted quadratic polynomials.

The Classical Convolutions

Definition

Given $p = \sum a_k x^k, r = \sum b_k x^k$, we define the (Walsh) additive convolution and (Grace-Szego) multiplicative convolution

$$(p \boxplus^n r)(x) = \sum_k D^k p(0) D^{n-k} r(x)$$

$$(p \boxtimes^n r)(x) = \sum_k \binom{n}{k}^{-1} a_k b_k x^k$$

Classical Convolution Intuitions

Proposition (Root Intepretation)

Given $p = \prod(x - a_i), r = \prod(x - b_i)$

$$(p \boxplus^n r)(x) = \sum_{\sigma \in S_n} \prod (x - a_i - b_{\sigma(i)})$$

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Theorem (Random Matrix Interpretation)

Given positive definite matrices A, B such that $\chi_A, \chi_B = p, r$, we have

$$\mathbb{E}_{O_n} \chi(A + PBP^T) = p \boxplus^n r$$

$$\mathbb{E}_{O_n} \chi(APBP^T) = p \boxtimes^n r$$

Characterization of Fixed Degree R.R. Preservers

Theorem

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Theorem (Borcea-Brendan 2008)

A nondegenerate linear operator T preserves real rooted polynomials of degree at most n if and only if $T[(x+z)^n]$ is real stable.

Mesh and Interlacing

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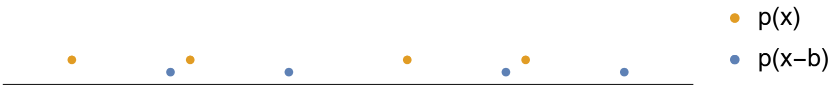
- $p(x)$ is $b-$mesh if and only if $p(x-b)$ interlaces $p(x)$



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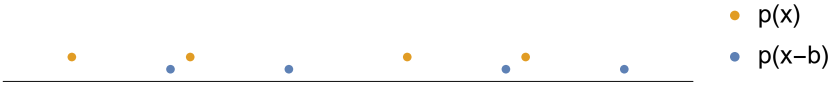

Definition

We call the minimal ratio between two roots of a positive rooted polynomial its *log mesh*. We say a polynomial is $q-$*log mesh* if its positive rooted with log mesh at least q.

Mesh Properties of Classical Convolutions

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Classical Log Mesh Preservation

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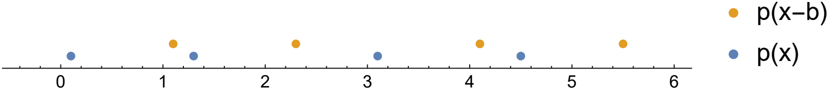
Question

What other operators preserve $b-$mesh?

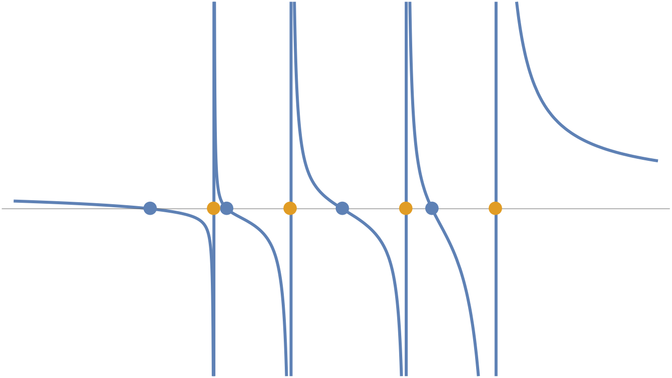
Other Mesh Preservers

We have $p(x)$ and $p(x-b)$ interlace



Other Mesh Preservers

Graph of $\frac{p(x)}{p(x-b)}$



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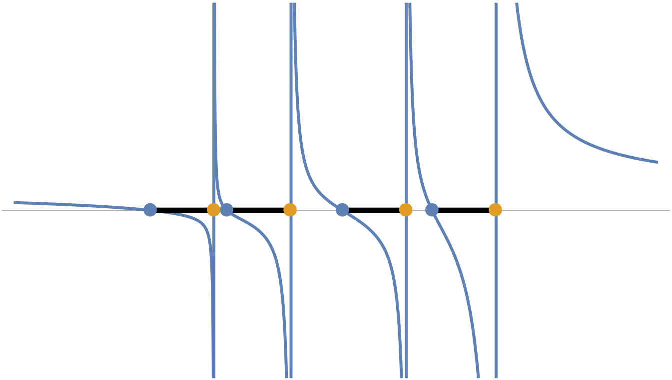
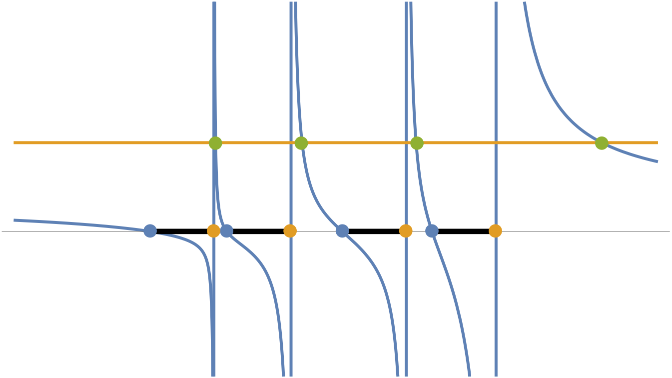


Other Mesh Preservers

Graph of $\frac{p(x)}{p(x-b)}$

Shifting graph: $\frac{p(x)}{p(x-b)} = \alpha \iff \frac{p(x)-\alpha p(x-b)}{p(x-b)} = 0$



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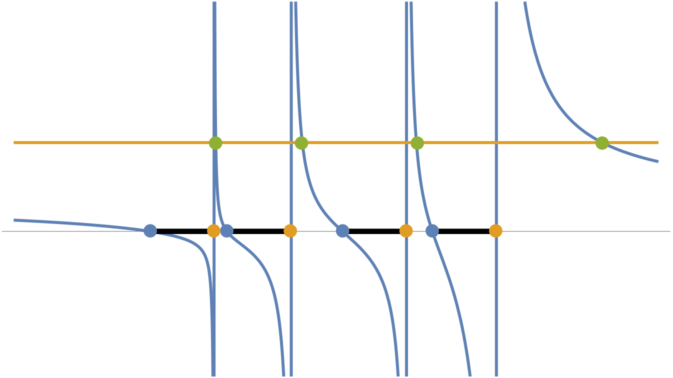

Lemma

$p(x) \mapsto p(x) - \alpha p(x - b)$ *preserves* $b-$*mesh when* $\alpha > 0$

Finite Difference Operator

Define $\Delta_b(p) := \frac{p(x)-p(x-b)}{b}$, $d_{q,n} := \frac{p(qx)-p(x)}{q^{1-n}(q^n-1)x}$, $d^*_{q,n} := \frac{p(qx)-q^n p(x)}{q^n-1}$.

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Theorem (Branden, Krasikov, Shapiro)

$p(\Delta_b) = \sum a_k \Delta_b^k$ preserves all $b-$mesh polynomials if and only if p is real rooted with roots in $(-\infty, b^{-1}]$

Finite Degree Case

Abstraction of the classical convolutions:

$$p \boxplus_b^n r := \sum_k \Delta_b^k p(0) \Delta_b^{n-k} r(x)$$

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Remark

This type of result first in 1976 with Suffridge and circle rooted polynomials.

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Theorem

Given $b-$mesh p, r, and s real rooted which is interlaced by r, then $p \boxplus_b^n s$ is interlaced by $p \boxplus_b^n r$

Corollary

Given $b-$mesh p, r (resp $q-$log mesh p, r) we have

$$mesh(p \boxplus_b^n r) \geq \max\{mesh(p), mesh(r)\}$$

$$logmesh(p \boxtimes_q^n r) \geq \max\{logmesh(p), logmesh(r)\}$$

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Linear operator T preserves interlacing with respect to f if for all g interlaced by f we have $T[g]$ is interlaced by $T[f]$ and vice versa.

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Given $f = (x - \alpha_1) \dots (x - \alpha_n)$*. Let* $f_{\alpha_k} = \frac{f(x)}{(x-\alpha_k)}$*. If* $g(x) = cf(x) + \sum c_{\alpha_i} f_{\alpha_i}$ *then* g *and* f *interlace if and only if all the* c_{α_k} *are the same sign.*

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Theorem

If $T[f_{\alpha_k}]$ *is interlaced by* $T[f]$ *for all* k *then* T *preserves interlacing with respect to* f.

Extending Lamprecht's Proof

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- $\Delta^*_{b,n} := nf(x-b) - (x-b)\Delta_b f(x)$ doesnt preserve $b-$mesh
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Lemma

For fixed $b-$mesh f, for large enough t, $(t\Delta_b + \Delta^*_{b,n})f$ is $b-$mesh and $\Delta_b f$ is interlaced by $(t\Delta_b + \Delta^*_{b,n})f$

Transfer Results

Question

How to transfer results from multiplicative setting to additive setting?

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Additive Convolution	Multiplicative Convolution
Preserves Real Rootedness	Preserves Positive Rootedness
Triangle Inequality	Multiplicative Triangle Inequality
Preserves Mesh	Preserves Log Mesh

Connecting the multiplicative and additive convolution

Introduce a new variable q
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\begin{aligned}
&(1-q)^n[p(\frac{1-x}{1-q}) \boxtimes^n r(\frac{1-x}{1-q})]((1-q)x+1) \\
&= (1-q)^{-n} \sum \prod (x + 1 - (1-(1-q)a_i)(1-(1-q)b_{\sigma(i)})) \\
&= \sum_{S_n} \prod (x - a_i - b_{\sigma(i)} + (1-q)a_i b_{\sigma(i)})
\end{aligned}
$$

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$$(1-q)^n[p(\frac{1-x}{1-q}) \boxtimes^n r(\frac{1-x}{1-q})]((1-q)x+1)$$
$$= (1-q)^{-n}\sum\prod(x+1-(1-(1-q)a_i)(1-(1-q)b_{\sigma(i)}))$$
$$= \sum_{S_n}\prod(x - a_i - b_{\sigma(i)} + (1-q)a_i b_{\sigma(i)})$$

Theorem

$\lim_{q\to 1}(1-q)^n[p(\frac{1-x}{1-q}) \boxtimes^n r(\frac{1-x}{1-q})](q^x) = p \boxplus^n r$

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$$(1-q)^n[p(\frac{1-x}{1-q}) \boxtimes^n r(\frac{1-x}{1-q})]((1-q)x+1)$$
$$= (1-q)^{-n}\sum\prod(x+1-(1-(1-q)a_i)(1-(1-q)b_{\sigma(i)}))$$
$$= \sum_{S_n}\prod(x - a_i - b_{\sigma(i)} + (1-q)a_i b_{\sigma(i)})$$

Theorem

$\lim_{q\to1}(1-q)^n[p(\frac{1-x}{1-q}) \boxtimes^n r(\frac{1-x}{1-q})](q^x) = p \boxplus^n r$

Allows us to get root information from multiplicative to additive

Toy Example

Theorem

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- Limiting yields $p \boxplus^n r$ is real rooted

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- Proved on basis using $q-$Lagrange interpolation formula.

Lemma (Q-Lagrange Interpolation)

$$(-1)^j q^{-\binom{j}{2}} \cdot [t^j]p(t) = \sum_{i=0}^{j} p((i)_{q^{-1}}) \frac{(-1)^i}{(i)_q!(j-i)_q!} q^{\binom{i}{2}}$$

Bounds on the Roots of the Additive Convolution

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- Strengthening of triangle inequality:

Theorem (Marcus, Spielman, Srivastava)

Given real rooted p, r and positive number α, define $U_\alpha = I - \alpha D$ then

$$\lambda_{max}(U_\alpha(p \boxplus^n r)) + n\alpha \leq \lambda_{max}(U_\alpha(p)) + \lambda_{max}(U_\alpha(r))$$

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Conjecture

Given $b-$mesh p, r and $U_\alpha = I - \alpha\tilde{\Delta}_b$, for $\alpha > 0$ we get

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Remark

Limiting $\alpha \to 0$ yields the triangle inequality while limiting $b \to 0$ yields the classical MSS result.

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- Is there a Random Matrix Model which interprets this convolution?