Extending the Borcea-Brändén Characterization

Jonathan Leake

Department of Mathematics
UC Berkeley

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Stable Polynomials

**Definition**

For \( p \in \mathbb{C}[x] \equiv \mathbb{C}[x_1, \ldots, x_m] \), we say \( p \) is \( S\)-stable whenever \( p(x) \neq 0 \) for \( x \in S \). If \( p \in \mathbb{R}[x] \) and \( S = \mathcal{H}^+_m \), we call \( p \) real stable.

- Root bounds: mixed characteristic polynomial, additive convolution
- Combinatorics: matroids, coefficient data
- Optimization: hyperbolic polynomials

objects \( \rightarrow \) multivariate polynomials \( \rightarrow \) apply operators \( \rightarrow \) information

Borcea-Brändén: complete characterization of linear operators preserving real stability and \( C^m \)-stability (for any open circular region \( C \)).
(BB) Multivariate matching polynomial $= \text{MAP}(\prod_{(i,j) \in E}(1 - x_i x_j))$

- $(1 - x_i x_j)$ is real stable, products are real stable.
- MAP = “Multi-Affine Part” preserves real-stability.
- Plug in $x$ for all variables $\rightarrow$ univariate matching poly is real-rooted.

(Gurvits) Doubly stochastic matrix $M \rightarrow \prod_{r \in \text{rows}} r \cdot x$

- $p_M(x) := \prod_i \sum_j m_{ij} x_j$ is real stable.
- (coefficient of $x_1 x_2 \cdots x_n$) $= \partial_{x_1} \cdots \partial_{x_n} p$ is the permanent of $M$.
- Can we obtain a bound on the permanent by analyzing $\partial_{x_k}$?

Both cases: want to determine properties of some linear operator on polynomials.
This Talk

Algebraic explanation/framework for the BB characterization.
- Explanation of why the BB characterization works out so well.
- Extensions which immediately follow from the new point of view.
- Unification of many of the BB results.

So why do we care? One application: capacity of a polynomial.
- Yields a theory of capacity-preserving operators.
- Application is straightforward, using similar techniques as above.
- Suggests a way forward for generalizing recent uses of capacity ideas. (e.g. operator scaling, coefficient optimization results)

Main thesis: This is the right way to think about preservation properties of linear operators on polynomials.
Throughout we will use the following shorthand:

- $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{x}^{\mu} = \prod_k x_k^{\mu_k}$, $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_m]$
- $+, -, >, \text{ etc. are element-wise, e.g. } x > 0 \text{ iff } \forall k, x_k > 0$
- $\mathbb{C}_\lambda[\mathbf{x}] = \{\text{polys in } \mathbb{C}[\mathbf{x}] \text{ of degree at most } \lambda_k \text{ in the variable } x_k\}$
- $\mu! = \prod_k \mu_k!$, $\binom{\lambda}{\mu} = \frac{\lambda!}{\mu!(\lambda-\mu)!}$
- $\mathcal{H}_+ = \text{upper half-plane}$, $\mathcal{H}_- = \text{lower half-plane}$
- $\mathbb{CP}^1$ refers to the Riemann sphere; $\mathbb{C} \subset \mathbb{CP}^1$ as usual by stereographic projection
- $S^c$ is set complement, $\overline{S}$ is set closure, usually as a subset of $\mathbb{CP}^1$ (roughly, OK to think $\mathbb{C}$ instead)
The BB Characterization

**Theorem (Borcea-Brändén)**

Let $T : \mathbb{C}_\lambda[x] \to \mathbb{C}_\gamma[x]$ be a linear operator with $\dim(\text{Im}(T)) > 1$. Then $T$ preserves $\mathcal{H}^m_+-$stability iff $\text{Symb}_{BB}(T)$ is $\mathcal{H}^{2m}_+$-stable.

**Theorem (Borcea-Brändén)**

Let $T : \mathbb{R}_\lambda[x] \to \mathbb{R}_\gamma[x]$ be a linear operator with $\dim(\text{Im}(T)) > 2$. Then $T$ preserves real stability iff one of $\text{Symb}_{BB}(T)(z, \pm x)$ is real stable.

Surprising: a given operator $T$ preserves stability exactly when a single polynomial $\text{Symb}_{BB}(T)$ is stable.

**Remark**

Mobius transforms and various versions of $\text{Symb}_{BB}$ allow different stability regions.
An Explicit Example

Definition

Given a linear operator \( T : \mathbb{C}_\lambda[x] \to \mathbb{C}_\gamma[x] \) define:

\[
\text{Symb}_{BB}(T)(z, x) := T[(z + x)^\lambda] = \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu} z^{\lambda - \mu} T(x^\mu)
\]

Fix real-rooted \( p \) (with roots \( a_k \)) and consider the additive convolution:

\[
T_p(r) = p \boxplus^n r = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{k=1}^n (x - a_k - b_{\sigma(k)})
\]

- \( \text{Symb}_{BB}(T_p) = \prod_k (x + z - a_k) \) is real-stable.
- BB: \( T_p \) preserves real-rootedness.
Another Example

Recall: \( Symb_{BB}(T)(z, x) := T[(z + x)^\lambda] = \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu} z^{\lambda - \mu} T(x^\mu) \)

Consider \( MAP \) as discussed above:

\[
Symb_{BB}(MAP) = MAP[(z + x)^\lambda] = \prod_{k} MAP[(z_k + x_k)^{\lambda_k}]
\]

\[
= \prod_{k} (z_k^{\lambda_k} + \lambda_k z_k^{\lambda_k-1} x_k)
\]

\[
= z^{\lambda - 1} \prod_{k} (z_k + \lambda_k x_k)
\]

\( Symb_{BB}(MAP) \) is real stable \( \Rightarrow \) \( MAP(\prod_{(i,j) \in E}(1 - x_i x_j)) \) is real stable.
The Symbol

Where does the symbol come from?

\[ \text{bilinear form } \langle \cdot, \cdot \rangle : \mathbb{C}_\lambda[x] \otimes \mathbb{C}_\lambda[x] \to \mathbb{C} \]

\[ \iff \]

\[ \text{Hom}(\mathbb{C}_\lambda[x], \mathbb{C}_\gamma[x]) \cong \mathbb{C}_\lambda[x]^* \otimes \mathbb{C}_\gamma[x] \cong \mathbb{C}_\lambda[x] \otimes \mathbb{C}_\gamma[x] \cong \mathbb{C}((\lambda, \gamma))[z, x] \]

Definition

The symbol map \( \text{Symb} \) corresponding to \( \langle \cdot, \cdot \rangle \) is given for any \( p, x \) as:

\[ T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle \]

- \( \text{Symb}(T) \) “acts on” \( p \) via \( \langle \cdot, \cdot \rangle \) to get \( T(p) \).
- \( \text{Symb}(T) \) encodes all info about what \( T \) does.

Which bilinear form?
The Apolarity Form

\[ \langle p, q \rangle := \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu}^{-1} (-1)^\mu p_\mu q_{\lambda-\mu} \] (notice: coeff. ↔ evaluation)

Remark

This is the unique \( SL_2^m \)-invariant (variable-wise Mobius transformations) bilinear form on polynomials (up to scalar).

Lemma

The Symb map corresponding to the apolarity form is:

\[ \text{Symb}(T)(z, x) = T[(x - z)^\lambda] = \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu} (-z)^{\lambda-\mu} T(x^\mu) \]

Properties of the apolarity form (denote \( \langle \cdot, \cdot \rangle \)):

- Provides stability information (classical Grace’s theorem)
- Symmetry properties (avoids Mobius transformations)
- Spaces of polynomials are now \( SL_2^m \)-modules
Example Revisited

A quick aside: Why the algebraic mindset?

Consider $\boxplus^n$ as a map with a single input in $\mathbb{C}_n[x] \otimes \mathbb{C}_n[x] \cong \mathbb{C}_{(n,n)}[x_1, x_2]$.

- $\text{Symb}_{BB}(\boxplus^n) = (x + z)^n \boxplus^n (x + t)^n = (x + z + t)^n$
- $\text{Symb}(\boxplus^n) = (x - z)^n \boxplus^n (x - t)^n = (x - z - t)^n$

Why not multivariate? $p \boxplus^\lambda q := \frac{1}{\lambda!} \sum_\mu \partial^\mu p(0) \partial^{\lambda - \mu} q(x)$

- $\text{Symb}_{BB}(\boxplus^\lambda) = (x + z)^\lambda \boxplus^\lambda (x + t)^\lambda = (x + z + t)^\lambda$
- $\text{Symb}(\boxplus^\lambda) = (x - z)^\lambda \boxplus^\lambda (x - t)^\lambda = (x - z - t)^\lambda$

Notice: $\boxplus^\lambda$ preserves real stability by the BB characterization.
Grace’s Theorem

**Theorem**

If \( p \) is \((\mathcal{H}_+ \cup \mathbb{R}_+)^m\)-stable and \( q \) is \((\mathcal{H}_- \cup \mathbb{R}_-)^m\)-stable then \( \langle p, q \rangle \neq 0 \).

**Corollaries:**

- \( SL_2^m \)-invariance \( \Rightarrow \) any circular regions with portion of boundary
- compactness of \( \mathbb{CP}^1 \) \( \Rightarrow \) closed and open (classical) circular regions

**Corollary (Grace, Borcea-Brändén)**

For any closed circular regions \( C_i \), if \( p \) is \((C_1 \times \cdots \times C_m)\)-stable and \( q \) is \((C_1^c \times \cdots \times C_m^c)\)-stable, then \( \langle p, q \rangle \neq 0 \).

For input polynomials with given stability properties, the form is nonzero. This says that certain evaluations of \( T(p) \) are nonzero.
The Main Characterization

**Theorem**

$T$ maps $(C_1^c \times \cdots \times C_m^c)$-stable polynomials to $S$-stable polynomials iff $\text{Symb}(T)$ is $(C_1 \times \cdots \times C_m) \times S$-stable. Here $S$ can be any set.

**Proof.**

$(\Leftarrow)$ Recall that $T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle$.

$(\Rightarrow)$ Apolarity form dual of $C_i$ is $C_i^c$. ($\mathbb{CP}^1 \sim \{\text{linear polys}\}$)

- Output stability region is completely free.
- No need for Mobius transformations: all stability info included.
- Unifies the many BB characterization results.

**Corollary (Walsh)**

Let $a_k \in \mathbb{C}$ denote the roots of $p$. If $r$ has all its roots in $C$, then $p \boxplus^n r$ has all its roots in $\bigcup_k (C + a_k)$. 
Possible to use other stability regions besides circular regions?

**Theorem**

\[ T \text{ preserves the set of polynomials with roots in } C^\circ \cup \gamma \text{ iff } \text{Symb}(T) \text{ is } (C^\circ \cup \gamma) \times (C^\circ \cup \gamma)^c\text{-stable, for } \gamma \text{ connected portion of the boundary of } C. \]

What about real intervals and rays?

**Corollary**

Let \( T : \mathbb{R}_n[x] \to \mathbb{R}_m[x] \) be a linear operator with \( \dim(\text{Im}(T)) > 2 \), and let \( I, J \) be real intervals. Then \( T \) preserves real-rootedness and maps \( I \)-rooted polynomials to \( J \)-rooted polynomials iff \( \text{Symb}(T) \) is either \( (\mathcal{H}^- \cup I) \times (\overline{\mathcal{H}^+ \setminus J})\text{-stable or } (\mathcal{H}^- \cup I) \times (\overline{\mathcal{H}^- \setminus J})\text{-stable.} \)
Relation to BB Characterization

How does this mesh with BB characterization (e.g. set complements)?

Recall: \( \text{Symb}(T)(z, x) = T[(x - z)^\lambda] = \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu} (-z)^{\lambda - \mu} T(x^\mu) \)

Negate \( z \) to obtain \( BB \) symbol. \( (\text{Symb}_{BB}(T)(z, x) = T[(x + z)^\lambda]) \)

**Corollary**

\( T \) preserves \( \mathcal{H}_+^m \)-stable polynomials iff \( \text{Symb}_{BB}(T) \) is \( (\overline{\mathcal{H}_+^m} \times \mathcal{H}_+^m) \)-stable.

What about the set closure? And the lack of dimension condition?

- Zero polynomial does not count as stable here.
- Hurwitz’ theorem and extra arguments yield the BB results.
- Excluding root “edge cases” gives the clean characterization.
Next question: what about analytic information?

Recall $T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle$. So:

- Bounds on $\langle \cdot, \cdot \rangle$ transfer to bounds on $T[p](x)$.
- Analytic notion needs to relate to polynomial evaluation.

The point: the above expression equates evaluation of $T(p)$ to a bilinear form which explicitly refers to coefficients of $p$ and Symb$(T)$.

What analytic notion relates evaluation and coefficients?
Capacity

**Definition (Gurvits)**
Given a polynomial $p \in \mathbb{R}_\lambda^+[x]$ and $\alpha \in \text{Newt}(p)$, define
\[ \text{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x^\alpha} \] (where $x^\alpha := \prod_k x_k^{\alpha_k}$ as usual).

**Theorem (Gurvits)**

Let $p \in \mathbb{R}_\lambda^+[x]$ be $m$-homogeneous and real stable. For $c_k := \min(k, \lambda_k)$:

\[ p_{(1^m)} = \partial_{x_1} \cdots \partial_{x_m} p(0) \geq \text{Cap}_{(1^m)}(p) \prod_{k=2}^m \left( \frac{c_k - 1}{c_k} \right)^{c_k-1} \]

- Simple proof of permanent inequality for doubly stochastic matrices and related results (e.g. Schrijver bound, mixed discriminants).
- Progress measure for matrix/operator scaling algorithms: non-commutative symbolic matrix singularity problem $\in \mathcal{P}$. 

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Extending the BB Characterization
QLAWS1, 2018
A Recent Relevant Result

**Definition**

For $p \in \mathbb{C}_{(\lambda, \lambda)}[z, x]$, define $\langle p \rangle_{SO_2} := \sum_{\mu \leq \lambda} (\frac{\lambda}{\mu})^{-1} p_{\mu, \mu}$.

**Properties of $\langle \cdot \rangle_{SO_2}$:**

- $\langle p, q \rangle_{SO_2} := \langle p(z)q(x) \rangle_{SO_2} = \langle z^\lambda p(-1/z), q(z) \rangle$ is a bilinear form.
- If $p$ is $(\mathcal{H}_{-} \cup \mathbb{R}_{+})^m \times (\mathcal{H}_{+} \cup \mathbb{R}_{+})^m$-stable, then $\langle p \rangle_{SO_2} \neq 0$.
- $\langle \cdot, \cdot \rangle_{SO_2}$ is $SO_2^m$-invariant.

**Theorem (Anari-Gharan 2017)**

If $p \in \mathbb{R}_{(1^m, 1^m)}^+[z, x]$ is $(\mathcal{H}_{-}^m \times \mathcal{H}_{+}^m)$-stable ("bistable"), then:

$$\langle p \rangle_{SO_2} \geq \alpha^\alpha (1 - \alpha)^{1-\alpha} \text{Cap}_{(\alpha, \alpha)}(p)$$

Proof uses strongly Rayleigh inequalities for real stable $p \in \mathbb{R}_{(1^m)}^[+] [x]$. 

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Corollary

If \( p \in \mathbb{R}^+_{(\lambda, \lambda)}[z, x] \) is \((\mathcal{H}_-^m \times \mathcal{H}_+^m)\)-stable ("bistable"), then:

\[
\langle p \rangle_{SO_2} \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_{(\alpha, \alpha)}(p)
\]

For \( T \) and \( p \) with desired stability properties and any \( x > 0 \):

\[
T[p](x) = \langle \text{Symb}_{SO_2}(T)(z, x), p(z) \rangle_{SO_2} \\
\geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_\alpha(p) \text{Cap}_\alpha(\text{Symb}_{SO_2}(T)(\cdot, x))
\]

Divide by \( x^\beta \) and take \( \inf_{x>0} \) on both sides (recall \( \text{Cap}_\beta(p) := \inf_{x>0} \frac{p(x)}{x^\beta} \)):

\[
\text{Cap}_\beta(T[p]) \geq \frac{\alpha^\alpha (\lambda - \alpha)^{\lambda-\alpha}}{\lambda^\lambda} \text{Cap}_\alpha(p) \text{Cap}_{(\alpha, \beta)}(\text{Symb}_{SO_2}(T))
\]
Theorem

Let $T : \mathbb{R}^+_{\lambda}[z] \to \mathbb{R}^+_{\gamma}[z]$ be such that $\text{Symb}_{SO_2}(T)(z, x)$ is real stable in $z$ for every $x > 0$ ("semistable"). For any real stable $p \in \mathbb{R}^+_{\lambda}[x]$:

$$\frac{\text{Cap}_\beta(T[p])}{\text{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha(\lambda - \alpha)^{\lambda - \alpha}}{\lambda^\lambda} \text{Cap}(\alpha, \beta)(\text{Symb}_{SO_2}(T))$$

Moreover, this bound is tight for any fixed $\alpha$, $\beta$, and $T$.

Lemma

Given $T : \mathbb{R}^+_{\lambda}[z] \to \mathbb{R}^+_{\gamma}[z]$, we have $\text{Symb}_{SO_2}(T) = T[(xz + 1)^\lambda]$. Tightness is demonstrated by considering $p(z) = (xz + 1)^\lambda$ for fixed $x > 0$. 
Gurvits’ Theorem

Let real stable \( p \in \mathbb{R}^+ \lambda [x] \) be \( m \)-homogeneous. Recall \((c_k := \min(k, \lambda_k)):\)

\[
p(1^m) = \partial_{x_1} \cdots \partial_{x_m} p(0) \geq \text{Cap}_{(1^m)}(p) \prod_{k=2}^{m} \left( \frac{c_k - 1}{c_k} \right)^{c_k-1}
\]

Consider \( T = \partial_{x_m}\big|_{x_m=0} \) and let \( \gamma \) denote \( \lambda \) without the \( m \)th coordinate.

- \( \text{Symb}_{SO_2}(T) = \partial_{x_m}(xz + 1) \lambda_{x_m=0} = \lambda_m z_m(xz + 1) \gamma \)
- \( \text{Cap}_{(1^m,1^m-1)}(\text{Symb}_{SO_2}(T)) = \lambda_m \frac{\gamma^\gamma}{(\gamma-1)^{\gamma-1}} \)
- \( \text{Cap}_{(1^m-1)}\left( \partial_{x_m} p\big|_{x_m=0} \right) \geq \left( \frac{\lambda_{m-1}}{\lambda_m} \right)^{\lambda_{m-1}} \text{Cap}_{(1^m)}(p) \)

\( \partial_{x_m} p\big|_{x_m=0} \) is homogeneous of degree \( m - 1 \) \( \Rightarrow \) induction
Applications of capacity-preservers, beyond differential operators?

Can we get similar bounds based only on the total degree of a given homogeneous polynomial?

SO\(_m\)-invariant form: \( \langle p, q \rangle_{SO_m} := \sum_{\mu} \binom{d}{\mu}^{-1} p_{\mu} q_{\mu} \)

Conjecture (Gurvits 2009)

For real stable \( d \)-homogeneous polynomials \( p, q \in \mathbb{R}^+ [x] \), we have:

\[
\langle p, q \rangle_{SO_m} \geq m^{-d} \text{Cap}_\alpha(p) \text{Cap}_\alpha(q)
\]

What about the matrix capacity case used in operator scaling result?

- Some bound on Frobenius inner product? Some other bilinear form?
- Possibly related to \( SO_m \) form above.