Extending the Borcea-Brändén Characterization

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Stable Polynomials

Definition

For $p \in \mathbb{C}[x] \equiv \mathbb{C}[x_1, ..., x_m]$, we say p is *S-stable* whenever $p(x) \neq 0$ for $x \in S$. If $p \in \mathbb{R}[x]$ and $S = \mathcal{H}_+^m$, we call p *real stable*.

- Root bounds: mixed characteristic polynomial, additive convolution
- Combinatorics: matroids, coefficient data
- Optimization: hyperbolic polynomials

objects $\rightarrow$ multivariate polynomials $\rightarrow$ apply operators $\rightarrow$ information

Borcea-Brändén: complete characterization of linear operators preserving real stability and C^m-stability (for any open circular region C).

Two Brief Examples

(BB) Multivariate matching polynomial $= \text{MAP}(\prod_{(i,j)\in E}(1 - x_i x_j))$

- $(1 - x_i x_j)$ is real stable, products are real stable.
- MAP = "Multi-Affine Part" preserves real-stability.
- Plug in x for all variables $\to$ univariate matching poly is real-rooted.

(Gurvits) Doubly stochastic matrix $M \to \prod_{r \in \text{rows}} r \cdot x$

- $p_M(x) := \prod_i \sum_j m_{ij} x_j$ is real stable.
- (coefficient of $x_1 x_2 \cdots x_n$) $= \partial_{x_1} \cdots \partial_{x_n} p$ is the permanent of M.
- Can we obtain a bound on the permanent by analyzing ∂_{x_k}?

Both cases: want to determine properties of some linear operator on polynomials.

This Talk

Algebraic explanation/framework for the BB characterization.

- Explanation of why the BB characterization works out so well.
- Extensions which immediately follow from the new point of view.
- Unification of many of the BB results.

So why do we care? One application: capacity of a polynomial.

- Yields a theory of capacity-preserving operators.
- Application is straightforward, using similar techniques as above.
- Suggests a way forward for generalizing recent uses of capacity ideas. (e.g. operator scaling, coefficient optimization results)

Main thesis: This is the right way to think about preservation properties of linear operators on polynomials.

Notation

Throughout we will use the following shorthand:

- $x = (x_1, ..., x_m)$, $x^\mu = \prod_k x_k^{\mu_k}$, $\mathbb{C}[x] = \mathbb{C}[x_1, ..., x_m]$
- $+, -, >$, etc. are element-wise, e.g. $x > 0$ iff $\forall k, x_k > 0$
- $\mathbb{C}_\lambda[x] = \{$polys in $\mathbb{C}[x]$ of degree at most λ_k in the variable $x_k\}$
- $\mu! = \prod_k \mu_k!$, $\binom{\lambda}{\mu} = \frac{\lambda!}{\mu!(\lambda-\mu)!}$
- $\mathcal{H}_+ =$ upper half-plane, $\mathcal{H}_- =$ lower half-plane
- $\mathbb{CP}^1$ refers to the Riemann sphere; $\mathbb{C} \subset \mathbb{CP}^1$ as usual by stereographic projection
- S^c is set complement, $\overline{S}$ is set closure, usually as a subset of $\mathbb{CP}^1$ (roughly, OK to think $\mathbb{C}$ instead)

The BB Characterization

Theorem (Borcea-Brändén)

Let $T : \mathbb{C}_\lambda[x] \to \mathbb{C}_\gamma[x]$ *be a linear operator with* $\dim(\mathrm{Im}(T)) > 1$*. Then* T *preserves* $\mathcal{H}_+^m$*-stability iff* $\mathrm{Symb}_{BB}(T)$ *is* $\mathcal{H}_+^{2m}$*-stable.*

Theorem (Borcea-Brändén)

Let $T : \mathbb{R}_\lambda[x] \to \mathbb{R}_\gamma[x]$ *be a linear operator with* $\dim(\mathrm{Im}(T)) > 2$*. Then* T *preserves real stability iff one of* $\mathrm{Symb}_{BB}(T)(z, \pm x)$ *is real stable.*

Surprising: a given operator T preserves stability exactly when a *single polynomial* $\mathrm{Symb}_{BB}(T)$ is stable.

Remark

Mobius transforms and various versions of Symb_{BB} allow different stability regions.

An Explicit Example

Definition

Given a linear operator $T : \mathbb{C}_\lambda[x] \to \mathbb{C}_\gamma[x]$ define:

$$\mathrm{Symb}_{BB}(T)(z,x) := T[(z+x)^\lambda] = \sum_{0 \le \mu \le \lambda} \binom{\lambda}{\mu} z^{\lambda-\mu} T(x^\mu)$$

Fix real-rooted p (with roots a_k) and consider the additive convolution:

$$T_p(r) = p \boxplus^n r = \frac{1}{n!} \sum_{\sigma \in S_n} \prod_{k=1}^{n} (x - a_k - b_{\sigma(k)})$$

- $\mathrm{Symb}_{BB}(T_p) = \prod_k (x + z - a_k)$ is real-stable.
- BB: T_p preserves real-rootedness.

Another Example

Recall: $\text{Symb}_{BB}(T)(z,x) := T[(z+x)^\lambda] = \sum_{0 \le \mu \le \lambda} \binom{\lambda}{\mu} z^{\lambda-\mu} T(x^\mu)$

Consider MAP as discussed above:

$$\begin{aligned}
\text{Symb}_{BB}(MAP) &= MAP[(z+x)^\lambda] \\
&= \prod_k MAP[(z_k + x_k)^{\lambda_k}] \\
&= \prod_k (z_k^{\lambda_k} + \lambda_k z_k^{\lambda_k - 1} x_k) \\
&= z^{\lambda-1} \prod_k (z_k + \lambda_k x_k)
\end{aligned}$$

$\text{Symb}_{BB}(MAP)$ is real stable $\Rightarrow$ $MAP(\prod_{(i,j)\in E}(1 - x_i x_j))$ is real stable.

The Symbol

Where does the symbol come from?

$$\text{bilinear form } \langle \cdot, \cdot \rangle : \mathbb{C}_\lambda[x] \otimes \mathbb{C}_\lambda[x] \to \mathbb{C}$$

$$\iff$$

$$\text{Hom}(\mathbb{C}_\lambda[x], \mathbb{C}_\gamma[x]) \cong \mathbb{C}_\lambda[x]^* \otimes \mathbb{C}_\gamma[x] \cong \mathbb{C}_\lambda[x] \otimes \mathbb{C}_\gamma[x] \cong \mathbb{C}_{(\lambda,\gamma)}[z, x]$$

Definition

The symbol map Symb corresponding to $\langle \cdot, \cdot \rangle$ is given for any p, x as:

$$T[p](x) = \langle \text{Symb}(T)(z, x), p(z) \rangle$$

- $\text{Symb}(T)$ "acts on" p via $\langle \cdot, \cdot \rangle$ to get $T(p)$.
- $\text{Symb}(T)$ encodes all info about what T does.

Which bilinear form?

The Apolarity Form

$\langle p, q \rangle := \sum_{0 \le \mu \le \lambda} \binom{\lambda}{\mu}^{-1} (-1)^\mu p_\mu q_{\lambda-\mu}$ (notice: coeff. $\leftrightarrow$ evaluation)

Remark

This is the unique SL_2^m-invariant (variable-wise Mobius transformations) bilinear form on polynomials (up to scalar).

Lemma

The Symb *map corresponding to the apolarity form is:*

$$\text{Symb}(T)(z, x) = T[(x-z)^\lambda] = \sum_{0 \le \mu \le \lambda} \binom{\lambda}{\mu} (-z)^{\lambda-\mu} T(x^\mu)$$

Properties of the apolarity form (denote $\langle \cdot, \cdot \rangle$):

- Provides stability information (classical Grace's theorem)
- Symmetry properties (avoids Mobius transformations)
- Spaces of polynomials are now SL_2^m-modules

Example Revisited

A quick aside: Why the algebraic mindset?

Consider $\boxplus^n$ as a map with a *single* input in $\mathbb{C}_n[x] \otimes \mathbb{C}_n[x] \cong \mathbb{C}_{(n,n)}[x_1, x_2]$.

- $\text{Symb}_{BB}(\boxplus^n) = (x+z)^n \boxplus^n (x+t)^n = (x+z+t)^n$
- $\text{Symb}(\boxplus^n) = (x-z)^n \boxplus^n (x-t)^n = (x-z-t)^n$

Why not multivariate? $p \boxplus^\lambda q := \frac{1}{\lambda!} \sum_\mu \partial^\mu p(0) \partial^{\lambda-\mu} q(x)$

- $\text{Symb}_{BB}(\boxplus^\lambda) = (x+z)^\lambda \boxplus^\lambda (x+t)^\lambda = (x+z+t)^\lambda$
- $\text{Symb}(\boxplus^\lambda) = (x-z)^\lambda \boxplus^\lambda (x-t)^\lambda = (x-z-t)^\lambda$

Notice: $\boxplus^\lambda$ preserves real stability by the BB characterization.

Grace's Theorem

Theorem

If p is $(\mathcal{H}_+ \cup \overline{\mathbb{R}_+})^m$*-stable and q is* $(\mathcal{H}_- \cup \overline{\mathbb{R}_-})^m$*-stable then* $\langle p, q \rangle \neq 0$.

Corollaries:

- SL_2^m-invariance $\Rightarrow$ any circular regions with portion of boundary
- compactness of $\mathbb{CP}^1$ $\Rightarrow$ closed and open (classical) circular regions

Corollary (Grace, Borcea-Brändén)

For any closed circular regions C_i*, if p is* $(C_1 \times \cdots \times C_m)$*-stable and q is* $(C_1^c \times \cdots \times C_m^c)$*-stable, then* $\langle p, q \rangle \neq 0$.

For input polynomials with given stability properties, the form is nonzero. This says that certain evaluations of $T(p)$ are nonzero.

The Main Characterization

Theorem

T maps $(C_1^c \times \cdots \times C_m^c)$-stable polynomials to S-stable polynomials iff Symb(T) *is $(C_1 \times \cdots \times C_m) \times S$-stable. Here S can be any set.*

Proof.

($\Leftarrow$) Recall that $T[p](x) = \langle \text{Symb}(T)(z,x), p(z) \rangle$.
($\Rightarrow$) Apolarity form dual of C_i is C_i^c. ($\mathbb{CP}^1 \sim \{\text{linear polys}\}$) □

- Output stability region is completely free.
- No need for Mobius transformations: all stability info included.
- Unifies the many BB characterization results.

Corollary (Walsh)

Let $a_k \in \mathbb{C}$ denote the roots of p. If r has all its roots in C, then $p \boxplus^n r$ has all its roots in $\cup_k (C + a_k)$.

More Results

Possible to use other stability regions besides circular regions?

Theorem

T preserves the set of polynomials with roots in $C^\circ \cup \gamma$ iff $\mathrm{Symb}(T)$ is $(C^\circ \cup \gamma) \times (C^\circ \cup \gamma)^c$-stable, for γ connected portion of the boundary of C.

What about real intervals and rays?

Corollary

Let $T : \mathbb{R}_n[x] \to \mathbb{R}_m[x]$ be a linear operator with $\dim(\mathrm{Im}(T)) > 2$, and let I, J be real intervals. Then T preserves real-rootedness and maps I-rooted polynomials to J-rooted polynomials iff $\mathrm{Symb}(T)$ is either $(\mathcal{H}_- \cup I) \times (\overline{\mathcal{H}_+} \setminus J)$-stable or $(\mathcal{H}_- \cup I) \times (\overline{\mathcal{H}_-} \setminus J)$-stable.

Relation to BB Characterization

How does this mesh with BB characterization (e.g. set complements)?

Recall: $\text{Symb}(T)(z,x) = T[(x-z)^\lambda] = \sum_{0 \leq \mu \leq \lambda} \binom{\lambda}{\mu} (-z)^{\lambda-\mu} T(x^\mu)$

Negate z to obtain *BB* symbol. $(\text{Symb}_{BB}(T)(z,x) = T[(x+z)^\lambda])$

Corollary

T preserves $\mathcal{H}_+^m$-stable polynomials iff $\text{Symb}_{BB}(T)$ is $(\overline{\mathcal{H}_+}^m \times \mathcal{H}_+^m)$-stable.

What about the set closure? And the lack of dimension condition?

- Zero polynomial does not count as stable here.
- Hurwitz' theorem and extra arguments yield the BB results.
- Excluding root "edge cases" gives the clean characterization.

Analytic Information

Next question: what about analytic information?

Recall $T[p](x) = \langle \text{Symb}(T)(z,x), p(z) \rangle$. So:

- Bounds on $\langle \cdot, \cdot \rangle$ transfer to bounds on $T[p](x)$.
- Analytic notion needs to relate to polynomial evaluation.

The point: the above expression equates *evaluation* of $T(p)$ to a bilinear form which explicitly refers to *coefficients* of p and $\text{Symb}(T)$.

What analytic notion relates evaluation and coefficients?

Capacity

Definition (Gurvits)

Given a polynomial $p \in \mathbb{R}^+_\lambda[x]$ and $\alpha \in \text{Newt}(p)$, define $\text{Cap}_\alpha(p) := \inf_{x>0} \frac{p(x)}{x^\alpha}$ (where $x^\alpha := \prod_k x_k^{\alpha_k}$ as usual).

Theorem (Gurvits)

Let $p \in \mathbb{R}^+_\lambda[x]$ be m-homogeneous and real stable. For $c_k := \min(k, \lambda_k)$:

$$p_{(1^m)} = \partial_{x_1} \cdots \partial_{x_m} p(0) \geq \text{Cap}_{(1^m)}(p) \prod_{k=2}^{m} \left(\frac{c_k - 1}{c_k} \right)^{c_k - 1}$$

- Simple proof of permanent inequality for doubly stochastic matrices and related results (e.g. Schrijver bound, mixed discriminants).
- Progress measure for matrix/operator scaling algorithms: non-commutative symbolic matrix singularity problem $\in \mathcal{P}$

A Recent Relevant Result

Definition

For $p \in \mathbb{C}_{(\lambda,\lambda)}[z,x]$, define $\langle p \rangle_{SO_2} := \sum_{\mu \le \lambda} \binom{\lambda}{\mu}^{-1} p_{\mu,\mu}$.

Properties of $\langle \cdot \rangle_{SO_2}$:

- $\langle p, q \rangle_{SO_2} := \langle p(z) q(x) \rangle_{SO_2} = \langle z^\lambda p(-1/z), q(z) \rangle$ is a bilinear form.
- If p is $(\mathcal{H}_- \cup \overline{\mathbb{R}_+})^m \times (\mathcal{H}_+ \cup \overline{\mathbb{R}_+})^m$-stable, then $\langle p \rangle_{SO_2} \neq 0$.
- $\langle \cdot, \cdot \rangle_{SO_2}$ is SO_2^m-invariant.

Theorem (Anari-Gharan 2017)

If $p \in \mathbb{R}^+_{(1^m,1^m)}[z,x]$ is $(\mathcal{H}_-^m \times \mathcal{H}_+^m)$-stable ("bistable"), then:

$$\langle p \rangle_{SO_2} \ge \alpha^\alpha (1-\alpha)^{1-\alpha} \operatorname{Cap}_{(\alpha,\alpha)}(p)$$

Proof uses strongly Rayleigh inequalities for real stable $p \in \mathbb{R}^+_{(1^m)}[x]$.

Using the Framework

Corollary

If $p \in \mathbb{R}^+_{(\lambda,\lambda)}[z,x]$ is $(\mathcal{H}^m_- \times \mathcal{H}^m_+)$-stable ("bistable"), then:

$$\langle p \rangle_{SO_2} \geq \frac{\alpha^\alpha (\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \operatorname{Cap}_{(\alpha,\alpha)}(p)$$

For T and p with desired stability properties and any $x > 0$:

$$\begin{aligned} T[p](x) &= \langle \operatorname{Symb}_{SO_2}(T)(z,x), p(z) \rangle_{SO_2} \\ &\geq \frac{\alpha^\alpha (\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_\alpha(\operatorname{Symb}_{SO_2}(T)(\cdot,x)) \end{aligned}$$

Divide by x^β and take $\inf_{x>0}$ on both sides (recall $\operatorname{Cap}_\beta(p) := \inf_{x>0} \frac{p(x)}{x^\beta}$):

$$\operatorname{Cap}_\beta(T[p]) \geq \frac{\alpha^\alpha (\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \operatorname{Cap}_\alpha(p) \operatorname{Cap}_{(\alpha,\beta)}(\operatorname{Symb}_{SO_2}(T))$$

Capacity Transferring Operators

Theorem

Let $T : \mathbb{R}_\lambda^+[z] \to \mathbb{R}_\gamma^+[z]$ *be such that* $\mathrm{Symb}_{SO_2}(T)(z,x)$ *is real stable in* z *for every* $x > 0$ *("semistable"). For any real stable* $p \in \mathbb{R}_\lambda^+[x]$*:*

$$\frac{\mathrm{Cap}_\beta(T[p])}{\mathrm{Cap}_\alpha(p)} \geq \frac{\alpha^\alpha(\lambda-\alpha)^{\lambda-\alpha}}{\lambda^\lambda} \mathrm{Cap}_{(\alpha,\beta)}(\mathrm{Symb}_{SO_2}(T))$$

Moreover, this bound is tight for any fixed α, β, *and* T.

Lemma

Given $T : \mathbb{R}_\lambda^+[z] \to \mathbb{R}_\gamma^+[z]$, *we have* $\mathrm{Symb}_{SO_2}(T) = T[(xz+1)^\lambda]$.

Tightness is demonstrated by considering $p(z) = (xz+1)^\lambda$ for fixed $x > 0$.

Gurvits' Theorem

Let real stable $p \in \mathbb{R}_\lambda^+[x]$ be m-homogeneous. Recall ($c_k := \min(k, \lambda_k)$):

$$p_{(1^m)} = \partial_{x_1} \cdots \partial_{x_m} p(0) \geq \mathrm{Cap}_{(1^m)}(p) \prod_{k=2}^{m} \left(\frac{c_k - 1}{c_k} \right)^{c_k - 1}$$

Consider $T = \partial_{x_m}|_{x_m=0}$ and let γ denote λ without the m^{th} coordinate.

- $\mathrm{Symb}_{SO_2}(T) = \partial_{x_m}(xz+1)^\lambda\big|_{x_m=0} = \lambda_m z_m (xz+1)^\gamma$
- $\mathrm{Cap}_{(1^m,1^{m-1})}(\mathrm{Symb}_{SO_2}(T)) = \lambda_m \frac{\gamma^\gamma}{(\gamma-1)^{\gamma-1}}$
- $\mathrm{Cap}_{(1^{m-1})}(\partial_{x_m} p|_{x_m=0}) \geq \left(\frac{\lambda_m - 1}{\lambda_m}\right)^{\lambda_m - 1} \mathrm{Cap}_{(1^m)}(p)$

$\partial_{x_m} p|_{x_m=0}$ is homogeneous of degree $m-1 \Rightarrow$ induction

Further Questions

Applications of capacity-preservers, beyond differential operators?

Can we get similar bounds based only on the *total* degree of a given homogeneous polynomial?

SO_m-invariant form: $\langle p, q \rangle_{SO_m} := \sum_\mu \binom{d}{\mu}^{-1} p_\mu q_\mu$

Conjecture (Gurvits 2009)

For real stable d-homogeneous polynomials $p, q \in \mathbb{R}^+[x]$, we have:

$$\langle p, q \rangle_{SO_m} \geq m^{-d} \, \mathrm{Cap}_\alpha(p) \, \mathrm{Cap}_\alpha(q)$$

What about the matrix capacity case used in operator scaling result?

- Some bound on Frobenius inner product? Some other bilinear form?
- Possibly related to SO_m form above.