On Akemann-Weaver Conjecture

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Expected Characteristic Polynomial Techniques and Applications

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Abstract

Akemann and Weaver (2014) have shown an interesting variant of classical Lyapunov's theorem on ranges of non-atomic vector-valued measures in the setting of discrete frames. Their result is an extension of Weaver's conjecture (2004), which was proved by Marcus, Spielman, and Srivastava (2015) in their breakthrough solution of the Kadison-Singer problem. Akemann and Weaver have conjectured a generalization of their Lyapunov-type theorem for higher rank operators. We show the validity of this conjecture for trace class operators by generalizing the main result of Marcus-Spielman-Srivastava to positive definite matrices of higher ranks. The method of the proof requires developing some new properties of mixed characteristic polynomials.

Marcus-Spielman-Srivastava result

Theorem (MSS)

Let $\epsilon > 0$. Suppose that $v_1, \dots, v_m$ are jointly independent random vectors in $\mathbb{C}^d$, which take finitely many values and satisfy

$$\sum_{i=1}^{m} \mathbb{E}[v_i v_i^*] = \mathbf{I} \qquad \text{and} \qquad \mathbb{E}[\|v_i\|^2] \le \epsilon \quad \text{for all } i.$$

Then,

$$\mathbb{P}\left(\left\| \sum_{i=1}^{m} v_i v_i^* \right\| \le (1 + \sqrt{\epsilon})^2 \right) > 0.$$

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$X_i = v_i v_i^*$ is $d \times d$ positive semidefinite random matrices of rank 1

$$\operatorname{tr}(X_i) = \|v_i\|^2$$

Improvement of Marcus-Spielman-Srivastava result

Theorem (M. Cohen, MB)

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$X_1, \dots, X_m$ have rank 1 $\implies$

$$\mathbb{E}\left[\det\left(z\mathbf{I} - \sum_{i=1}^{m} X_i \right) \right] = \mu[X_1, \dots, X_m](z)$$

Mixed characteristic polynomial

Definition

Let $A_1, \ldots, A_m$ be $d \times d$ matrices. The mixed characteristic polynomial is defined as for $z \in \mathbb{C}$ by

$$\mu[A_1, \ldots, A_m](z) = \left(\prod_{i=1}^{m} (1 - \partial_{z_i}) \right) \det \left(z\mathbf{I} + \sum_{i=1}^{m} z_i A_i \right) \Bigg|_{z_1 = \ldots = z_m = 0}$$

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$\mu[A_1, \dots, A_m](z)$ is a monic polynomial in $\mathbb{C}[z]$ of degree d.

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$$\mu[\underbrace{A, \ldots, A}_{d}][z] = \det(z\mathbf{I} - A)$$

Multi-affinity of MCP

Lemma

For a fixed $z \in \mathbb{C}$, the mixed characteristic polynomial mapping

$$\mu : M_{d\times d}(\mathbb{C}) \times \ldots \times M_{d\times d}(\mathbb{C}) \to \mathbb{C}$$

is multi-affine and symmetric. That is, μ affine in each variable and its value is the same for any permutation of its arguments $A_1, \ldots, A_m$.

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Corollary

Let $X_1, \ldots, X_m$ be $d \times d$ jointly independent random matrices, which take finitely many values. Then,

$$\mathbb{E}\left[\mu[X_1, \ldots, X_m](z)\right] = \mu[\mathbb{E}\left[X_1\right], \ldots, \mathbb{E}\left[X_m\right]](z) \qquad z \in \mathbb{C}.$$

Real stable polynomials

Definition

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the upper half plane. We say that a polynomial $p \in \mathbb{C}[z_1, \ldots, z_m]$ is *stable* if $p(z_1, \ldots, z_m) \neq 0$ for every $(z_1, \ldots, z_m) \in \mathbb{C}_+^m$. A polynomial is called *real stable* if it is stable and all of its coefficients are real.

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Lemma (MSS)

If $A_1, \ldots, A_m$ are positive semidefinite hermitian $d \times d$ matrices, then $\det\left(z\mathbf{I} + \sum_{i=1}^m z_i A_i\right)$ is a real stable polynomial.

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The mixed characteristic polynomial

$$\mu[A_1, \ldots, A_m](z) = \left(\prod_{i=1}^m (1 - \partial_{z_i})\right) \det\left(z\mathbf{I} + \sum_{i=1}^m z_i A_i\right)\Bigg|_{z_1 = \ldots = z_m = 0}$$

is real stable and monic of degree d.

Multivariate barrier of Marcus, Spielman, and Srivastava

Theorem (MSS)

Let $\epsilon > 0$. Suppose $A_1, \ldots, A_m$ are $d \times d$ positive semidefinite matrices satisfying

$$\sum_{i=1}^{m} A_i = \mathbf{I} \qquad \text{and} \qquad \mathrm{Tr}(A_i) \le \epsilon \quad \text{for all } i.$$

Then, all roots of the mixed characteristic polynomial $\mu[A_1, \ldots, A_m]$ are real and the largest root is at most $(1 + \sqrt{\epsilon})^2$.

Interlacing polynomials

Lemma

Let $p_1, \dots, p_n \in \mathbb{R}[z]$ be polynomials of the same degree. Suppose that every convex combination $\sum_{i=1}^n t_i p_i$, $\sum_{i=1}^n t_i = 1$, $t_i \geq 0$, is a real rooted polynomial. Then, there exists $1 \leq i_0 \leq n$ such that

$$\operatorname{maxroot}(p_{i_0}) \leq \operatorname{maxroot}\left(\sum_{i=1}^n t_i p_i\right).$$

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Lemma

Suppose that $X_1, \ldots, X_m$ are jointly independent random positive semidefinite $d \times d$ matrices which take finitely many values. Then with positive probability

$$\operatorname{maxroot}(\mu[X_1, \ldots X_m]) \leq \operatorname{maxroot}(\mu[\mathbb{E}[X_1], \ldots, \mathbb{E}[X_m]]).$$

Monotonicty of maximal root

Lemma

Suppose that $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are positive semidefinite hermitian $d \times d$ matrices satisfying

$$A_i \le B_i \qquad \text{for all } i = 1, \dots, m.$$

Then,

$$\operatorname{maxroot}(\mu[A_1, \dots, A_m]) \le \operatorname{maxroot}(\mu[B_1, \dots, B_m]).$$

Proof.

It suffices to consider the special case when $A_i = B_i$ for all $i = 2, \ldots, m$. For any $0 < p < 1$, consider a random matrix

$$X_p = \begin{cases} \frac{1}{p} A_1 & \text{with probability } p \\ \frac{1}{p-1}(B_1 - A_1) & \text{with probability } 1-p \end{cases}$$

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$\mathbb{E}[X_p] = B_1 \implies$ with positive probability

$$\operatorname{maxroot} \mu[X_p, A_2, \ldots, A_m] \le \operatorname{maxroot} \mu[B_1, A_2, \ldots, A_m]$$

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$$\begin{aligned} (1-p) \operatorname{maxroot} \mu\left[\frac{1}{1-p}(B_1 - A_1), A_2, \dots, A_m\right] \\ = \operatorname{maxroot} \mu[B_1 - A_1, (1-p)A_2, \dots, (1-p)A_m] \\ \to \operatorname{maxroot} \mu[B_1 - A_1, 0, \dots, 0] = \operatorname{tr}(B_1 - A_1) \qquad \text{as } p \to 1. \end{aligned}$$

Hence, if $A_1 \le B_1$ and $A_1 \ne B_1$, then

$$\operatorname{maxroot} \mu\left[\tfrac{1}{1-p}(B_1 - A_1), A_2, \dots, A_m\right] \to \infty \qquad \text{as } p \to 1.$$

$$\operatorname{maxroot} \mu[\tfrac{1}{p} A_1, A_2, \dots, A_m] \le \operatorname{maxroot} \mu[B_1, A_2, \dots, A_m]. \qquad \square$$

Lemma

$B_1, \dots, B_m$ positive semidefinite hermitian $d \times d$ matrices $\implies$

$$\left\| \sum_{i=1}^{m} B_i \right\| \le \operatorname{maxroot} \mu[B_1, \dots, B_m].$$

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Proof.

Let P be a rank 1 orthogonal projection such that

$$\left\| \sum_{i=1}^{m} B_i \right\| = \left\| \sum_{i=1}^{m} P B_i P \right\|.$$

$A_i = P B_i P$ rank 1 $\implies$ $\det(z\mathbf{I} - \sum_{i=1}^m A_i) = \mu[A_1, \ldots, A_m](z)$

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$$\left\| \sum_{i=1}^{m} A_i \right\| = \operatorname{maxroot} \det \left(z\mathbf{I} - \sum_{i=1}^{m} A_i \right)$$
$$= \operatorname{maxroot} \mu[A_1, \dots, A_m] \leq \operatorname{maxroot} \mu[B_1, \dots, B_m].$$

Theorem (M. Cohen, MB)

Let $\epsilon > 0$. Suppose that $X_1, \ldots, X_m$ are jointly independent $d \times d$ positive semidefinite random matrices satisfy

$$\sum_{i=1}^{m} \mathbb{E}[X_i] = \mathbf{I} \qquad \textit{and} \qquad \mathbb{E}[\operatorname{tr} X_i] \le \epsilon \quad \textit{for all } i.$$

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Proof.

Let $A_i = \mathbb{E}[X_i]$. Applying MSS multivariate barrier estimate

$$\left\| \sum_{i=1}^{m} X_i \right\| \le \operatorname{maxroot}(\mu[X_1, \ldots X_m]) \le \operatorname{maxroot}(\mu[A_1, \ldots A_m])$$

$$\le (1+\sqrt{\epsilon})^2.$$

□

KS_r conjecture of Weaver

Rank 1 random matrix MSS result $\implies$

Theorem (Akemann-Weaver)

Let $\{u_i\}_{i\in[m]}$ in $\mathbb{C}^d$ be a Parseval frame

$$\sum_{i=1}^{m} u_i u_i^* = \mathbf{I} \qquad \text{and} \qquad \|u_i\|^2 \le \delta \quad \text{for all } i. \tag{1}$$

Let $r \in \mathbb{N}$ and $t_1, \dots, t_r > 0$ satisfy $\sum_{k=1}^{r} t_k = 1$. Then, there exists a partition $\{I_1, \dots, I_r\}$ of $[m]$ such that each $\{u_i\}_{i\in I_k}$, $k = 1, \dots, r$, is a Bessel sequence with the bounds

$$\left\| \sum_{i\in I_k} u_i u_i^* \right\| \le t_k(1 + \sqrt{r\delta})^2. \tag{2}$$

Akemann-Weaver conjecture

Higher rank random matrix MSS result $\implies$

Theorem (MB)

Let $\epsilon > 0$. Suppose $A_1, \ldots, A_m$ are $d \times d$ positive semidefinite matrices satisfying

$$\sum_{i=1}^{m} A_i = \mathbf{I} \qquad \text{and} \qquad \mathrm{Tr}(A_i) \le \epsilon \quad \text{for all } i.$$

Let $t_k > 0$, $k = 1, \ldots, r$, where $r \in \mathbb{N}$, be such that $\sum_{k=1}^{r} t_k = 1$. Then, there exists a partition $\{I_1, \ldots, I_r\}$ of $[m]$ such that

$$\left\| \sum_{i \in I_k} A_i \right\| \le t_k \left(1 + \sqrt{r\epsilon}\right)^2 \qquad \text{for all } k = 1, \ldots, r.$$

Proof.

Let $X_1, \ldots, X_m$ be jointly independent $dr \times dr$ random matrices such that each matrix X_i is block diagonal taking values

$$\begin{bmatrix} t_1^{-1}A_i & & & \\ & \mathbf{0}_d & & \\ & & \ddots & \\ & & & \mathbf{0}_d \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_d & & & \\ & \ddots & & \\ & & \mathbf{0}_d & \\ & & & t_r^{-1}A_i \end{bmatrix}$$

with probability t_k, $k = 1, \ldots, r$, resp. Then,

$$\sum_{i=1}^{m} \mathbb{E}\left[X_i\right] = \mathbf{I}_{dr} \quad \text{and} \quad \mathbb{E}\left[\operatorname{tr}(X_i)\right] = \sum_{k=1}^{r} t_k (t_k)^{-1} \operatorname{tr}(A_i) \le r\epsilon$$

Improved MSS result with $r\epsilon$ in place of ϵ implies an outcome which yields the required partition

$$I_k = \{i \in [m] : X_i \text{ is non-zero in } k^{\text{th}} \text{ entry}\}, \qquad k = 1, \ldots, r.$$

Lyapunov's theorem

Theorem (Lyapunov)

The range of a non-atomic vector-valued measure with values in $\mathbb{R}^n$ is a convex and compact subset of $\mathbb{R}^n$.

Approximate Lyapunov's theorem

Theorem (Akemann-Weaver)

There exists a universal constant $C_0 > 0$ such that the following holds. Suppose $\{\phi_i\}_{i\in I}$ is a Bessel family with bound 1 in a separable Hilbert space $\mathcal{H}$

$$\sum_{i\in I} |\langle f, \phi_i\rangle|^2 \le 1 \qquad \text{for all } f \in \mathcal{H},$$

and $\|\phi_i\|^2 \le \varepsilon$, where $\varepsilon > 0$. Suppose that $0 \le t_i \le 1$ for all $i \in I$. Then, there exists a subset $I_0 \subset I$ such that

$$\left\| \sum_{i\in I_0} \phi_i \otimes \phi_i - \sum_{i\in I} t_i \phi_i \otimes \phi_i \right\| \le C_0 \varepsilon^{1/8}.$$

$$\phi_i \otimes \phi_i : \mathcal{H} \to \mathcal{H}, \qquad \phi_i \otimes \phi_i(f) = \langle f, \phi_i\rangle \phi_i$$

Theorem (Akemann-Weaver)

If $\{\phi_i\}_{i \in I}$ is a Bessel family with bound 1 and $\|\phi_i\|^2 \le \varepsilon$, then the set of all partial frame operators

$$\mathcal{S} = \left\{ \sum_{i \in I'} \phi_i \otimes \phi_i : I' \subset I \right\}$$

is an approximately convex subset of $\mathcal{B}(\mathcal{H})$. That is, for every T in the convex hull of $\mathcal{S}$, there exists $S \in \mathcal{S}$ s. t. $||S - T|| \le C_0 \varepsilon^{1/8}$.

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Theorem (MB)

Let $\{\phi_t\}_{t\in X}$ be a continuous Bessel family on a **non-atomic** *measure space (X,μ). Let $\mathcal{S}$ be the set of all partial frame operators*

$$S_{\phi,E} f = \int_E \langle f, \phi_t\rangle \phi_t d\mu(t) \qquad E \subset X \text{ is measurable}$$

Then, the operator norm closure $\overline{\mathcal{S}} \subset \mathcal{B}(\mathcal{H})$ is convex.

Approximate Lyapunov for trace class operators

Theorem (MB)

Let I be countable and $\epsilon > 0$. Suppose that $\{T_i\}_{i \in I}$ is a family of positive trace class operators in a separable Hilbert space $\mathcal{H}$ such that

$$\sum_{i \in I} T_i \le \mathbf{I} \quad \text{and} \quad \operatorname{tr}(T_i) \le \varepsilon \qquad \text{for all } i \in I.$$

Suppose that $0 \le t_i \le 1$ for all $i \in I$. Then, there exists a subset of indices $I_0 \subset I$ such that

$$\left\| \sum_{i \in I_0} T_i - \sum_{i \in I} t_i T_i \right\| \le C_0 \epsilon^{1/8},$$

where $C_0 > 0$ is a universal constant.

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The proof follows closely the strategy Akemann-Weaver theorem for rank one case.

THE END