Convolutions and fluctuations: free, finite, quantized.

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Hermitian matrix operations

$$A = \begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & a_N \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & & \\ 0 & b_2 & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & b_N \end{pmatrix}$$

$U, V -$ Haar–random in $\mathit{Unitary}(N; \mathbb{R} \,/\, \mathbb{C} \,/\, \mathbb{H})$

$$\boxed{C = UAU^* + VBV^*}$$

$\nwarrow \quad \nearrow$

uniformly random eigenvectors

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or $\boxed{C = (UAU^*) \cdot (VBV^*)}$ or $\boxed{C = P_k(UAU^*)P_k}$

Question: What can you say about **eigenvalues** of C?

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Question: What can you say about **eigenvalues** of C?

I) As $\beta (= 1, 2, 4) \to \infty$ II) As $N \to \infty$ III) In discretization

As $\beta(= 1, 2, 4) \to \infty$

Theorem. (Gorin–Marcus–17) Eigenvalues of C **crystallize** (=they become deterministic) as $\beta \to \infty$:

$$\lim_{\beta\to\infty} \boxed{C = UAU^* + VBV^*}$$

$$\prod_{i=1}^{N}(z - c_i) = \frac{1}{N!} \sum_{\sigma\in S(N)} \prod_{i=1}^{N}(z - a_i - b_{\sigma(i)})$$

$$\lim_{\beta\to\infty} \boxed{C = (UAU^*)\cdot(VBV^*)}$$

$$\prod_{i=1}^{N}(z - c_i) = \frac{1}{N!} \sum_{\sigma\in S(N)} \prod_{i=1}^{N}(z - a_i b_{\sigma(i)})$$

$$\lim_{\beta\to\infty} \boxed{C = P_k(UAU^*)P_k}$$

$$\prod_{i=1}^{k}(z - c_i) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{i=1}^{N}(z - a_i)$$

Finite free convolutions and projection

As $N \to \infty$

Theorem. (Voiculescu, 80s) At $\beta = 1, 2$ empirical measure of eigenvalues of C becomes deterministic as $N \to \infty$.

$$\mu_A = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i} \quad \mu_B = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{b_i} \quad \mu_C = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{c_i}$$

$$G_\mu(z) = \int \frac{\mu(dx)}{z - x}$$

$$R_\mu(z) = \Big(G_\mu(z)\Big)^{-1} - \frac{1}{z}, \quad S_\mu(z) = \frac{z}{1+z}\Big(1 - zG_\mu(z)\Big)^{-1}$$

$$\lim_{N\to\infty} \boxed{C = UAU^* + VBV^*} \qquad R_{\mu_C}(z) = R_{\mu_A}(z) + R_{\mu_B}(z)$$

$$\lim_{N\to\infty} \boxed{C = (UAU^*)\cdot(VBV^*)} \qquad S_{\mu_C}(z) = S_{\mu_A}(z)\cdot S_{\mu_B}(z)$$

$$\lim_{N\to\infty} \boxed{C = P_k(UAU^*)P_k} \qquad R_{\mu_C}(z) = \tfrac{N}{k} R_{\mu_A}(z)$$

Free convolutions and projection

As $N \to \infty$

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Free convolutions and projection

Conjecture. Same is true for any fixed $\beta > 0$.

Discretization

T_λ irreducible (linear) representations of $U(N;\mathbb{C})$

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N, \quad \lambda_i \in \mathbb{Z}.$$

$$T_\lambda \otimes T_\nu = \bigoplus_\kappa c_{\lambda,\nu}^\kappa T_\kappa$$

Littlewood–Richardson coefficients $c_{\lambda,\nu}^\kappa$ intractable as $N \to \infty$.

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Littlewood–Richardson coefficients $c^\kappa_{\lambda,\nu}$ intractable as $N \to \infty$.

Random κ through $P(\kappa) = \dfrac{\dim(T_\kappa) c^\kappa_{\lambda,\nu}}{\dim T_\lambda \cdot \dim T_\nu}$.

Semi-classical limit degenerates representations of a Lie group into orbital measures on its Lie algebra

$$T_\lambda \otimes T_\nu \longrightarrow UAU^* + VBV^*$$

Discretization

Theorem. (Gorin–Bufetov–13; following Biane in 90s)
Empirical measure of ν becomes deterministic as $N \to \infty$.

$$T_\lambda \otimes T_\nu = \bigoplus_\kappa c^\kappa_{\lambda,\nu} T_\kappa, \qquad P(\kappa) = \frac{\dim(T_\kappa) c^\kappa_{\lambda,\nu}}{\dim T_\lambda \cdot \dim T_\nu}$$

$$\mu_\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i}{N}\right), \qquad \mu_\kappa = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\kappa_i}{N}\right)$$

Scaling is important!

$$G_\mu(z) = \int \frac{\mu(dx)}{z-x}, \quad R^{\text{quant}}_\mu(z) = \Big(G_\mu(z)\Big)^{-1} - \frac{1}{1-\exp(-z)}$$

$$R^{\text{quant}}_{\mu_\kappa}(z) = R^{\text{quant}}_{\mu_\lambda}(z) + R^{\text{quant}}_{\mu_\nu}(z)$$

Quantized free convolution

Discretization

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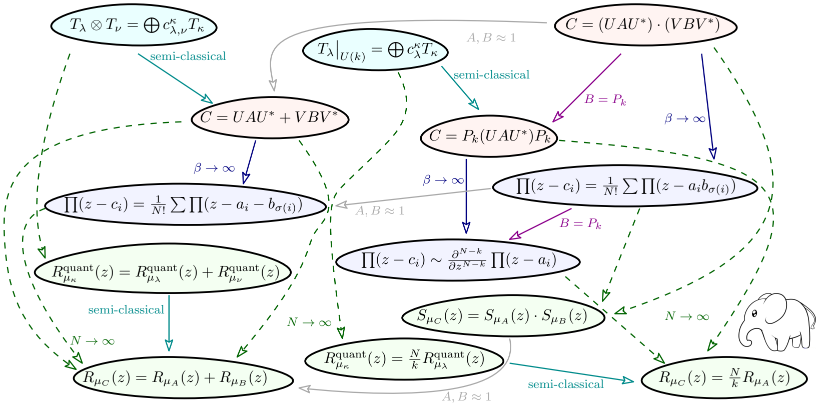
Quantized free convolution

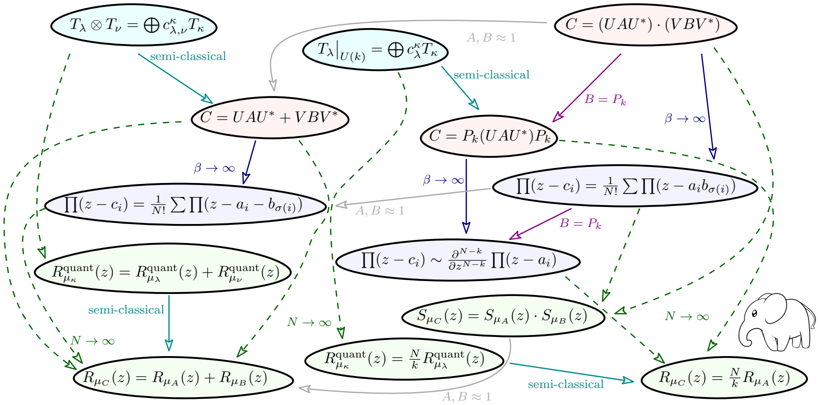
Restrictions to smaller subgroups lead to projection.

Zoo of operations



Operations on matrices and representations lead to a variety of **Laws of Large Numbers**, resulting in convolutions.
They are all cross-related by limit transitions.

Zoo of operations



Operations on matrices and representations lead to a variety of **Laws of Large Numbers**, resulting in **convolutions**.
They are all cross-related by limit transitions.

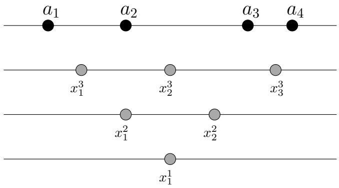
We present a **unifying framework** for the operations.

Matrix corners

The most explicit case.

$N \times N$ matrix UAU^*

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



Theorem. (Gelfand–Naimark–50s; Baryshnikov, Neretin – 00s) With $(x_1^N, \dots, x_N^N) = (a_1, \dots, a_N)$, the joint law of particles is

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

- A basis of extension from $\beta = 1, 2, 4$ to **general** $\beta > 0$.
- Consistent with (Hermite/Laguerre/Jacobi) β log–gases

Matrix corners

$N \times N$ matrix UAU^*

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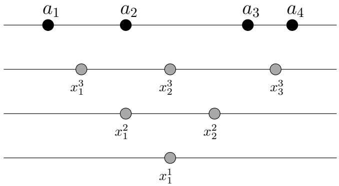

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

Multivariate Bessel Function

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \mathbb{E} \exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1} \right) \right)$$

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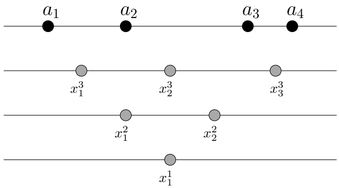

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

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Diagonal matrix elements at $\beta = 1, 2, 4$

Proposition. $\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N)$ is symmetric in $z_1,\ldots,z_N$.

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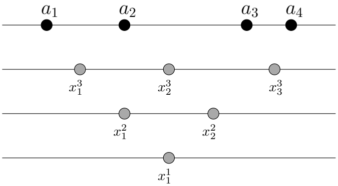

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

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Laplace transform for full matrix at $\beta = 1, 2, 4$.

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \mathbb{E} \exp \left[\mathrm{Trace}(UAU^* Z) \right], \quad Z \sim \{z_i\} \text{ eigenvalues}$$

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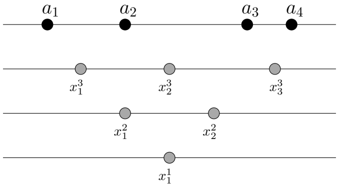

Multivariate Bessel Function

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \mathbb{E}\exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1}\right)\right)$$

(Harish Chandra; Itzykson–Zuber) **At $\beta = 2$:**

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) \sim \frac{\det\left[\exp(a_i z_j)\right]_{i,j=1}^{N}}{\prod_{i<j}(a_i - a_j)(z_i - z_j)}$$

Matrix corners

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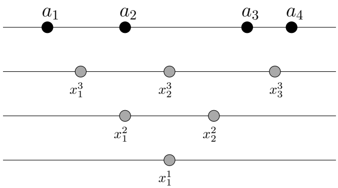

Multivariate Bessel Function

$$\mathcal{B}_{a_1,\dots,a_N}(z_1,\dots,z_N) = \mathbb{E} \exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1}\right)\right)$$

$\left[\mathcal{B}_{a_1,\dots,a_N}(z_1,\dots,z_N) \cdot \prod_{i<j}(z_i - z_j)^{\beta/2}\right]$ is an eigenfunction of rational Calogero–Sutherland Hamiltonian for any $\beta > 0$

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + \frac{\beta(2-\beta)}{2} \sum_{i<j} \frac{1}{(z_i - z_j)^2}$$

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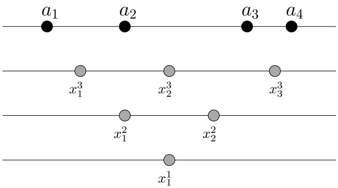

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \mathbb{E}\exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1}\right)\right)$$

The probability law of interest is readily reconstructed:

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_k,0,\ldots,0) = \iiint \mathcal{B}_{x_1^k,x_2^k,\ldots,x_k^k}(z_1,\ldots,z_k)\mathbb{P}(dx_1^k, dx_2^k,\ldots,dx_k^k)$$

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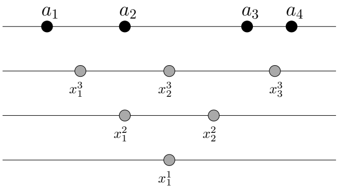

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The probability law of interest is readily **reconstructed**:

$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_k,0,\ldots,0) = \iiint \mathcal{B}_{x_1^k,x_2^k,\ldots,x_k^k}(z_1,\ldots,z_k)\mathbb{P}(dx_1^k,dx_2^k,\ldots,dx_k^k)$$

Equivalently, we have a family of **observables**

$$\boxed{\mathbb{E}\mathcal{B}_{x_1^k,x_2^k,\ldots,x_k^k}(z_1,\ldots,z_k) = \mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_k,0,\ldots,0)}$$

Matrix addition

$N \times N$ matrix UAU^* $\qquad$ $N \times N$ matrix VBV^*

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$$\mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) = \mathbb{E}\exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1}\right)\right)$$

$$\mathcal{B}_{b_1,\ldots,b_N}(z_1,\ldots,z_N) = \mathbb{E}\exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} y_i^k - \sum_{j=1}^{k-1} y_j^{k-1}\right)\right)$$

$$\boxed{C = UAU^* + VBV*}$$

Question: $\mathbb{E}\mathcal{B}_{c_1,\ldots,c_N}(z_1,\ldots,z_N) = ?$

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$$C = UAU^* + VBV*$$

$$\mathbb{E}\mathcal{B}_{c_1,\ldots,c_N}(z_1,\ldots,z_N) = \mathcal{B}_{a_1,\ldots,a_N}(z_1,\ldots,z_N) \cdot \mathcal{B}_{b_1,\ldots,b_N}(z_1,\ldots,z_N)$$

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Any $\beta > 0$! **Caveat:** positivity in $\mathbb{E}$ is **open** outside $\beta = 1, 2, 4$.

Operations through symmetric functions

$C = UAU^* + VBV^*$	Multivariate Bessel $\mathbb{E}\mathcal{B}_C(\vec{z}) = \mathcal{B}_A(\vec{z}) \cdot \mathcal{B}_B(\vec{z})$
$C = P_k(UAU^*)P_k$	Multivariate Bessel $\mathbb{E}\mathcal{B}_C(\vec{z}) = \mathcal{B}_A(\vec{z}, 0^{N-k})$
$C = (UAU^*) \cdot (VBV^*)$	Heckman–Opdam hypergeometric $\mathbb{E}\mathcal{HO}_C(\vec{z}) = \mathcal{HO}_A(\vec{z}) \cdot \mathcal{HO}_B(\vec{z})$
$T_\lambda \otimes T_\nu = \bigoplus_\kappa c^\kappa_{\lambda,\nu} T_\kappa$	Schur polynomials $\mathbb{E}\left[\frac{s_\kappa(\vec{z})}{s_\kappa(1^N)}\right] = \frac{s_\lambda(\vec{z})}{s_\lambda(1^N)} \cdot \frac{s_\nu(\vec{z})}{s_\nu(1^N)}$
$T_\lambda\Big\|_{U(k)} = \bigoplus_\kappa c^\kappa_\lambda T_\kappa$	Schur polynomials $\mathbb{E}\left[\frac{s_\kappa(\vec{z})}{s_\kappa(1^k)}\right] = \frac{s_\lambda(\vec{z}, 1^{N-k})}{s_\lambda(1^N)}$

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These all are degenerations of Macdonald polynomials.

Macdonald polynomials

$P_\lambda(x_1, \ldots, x_N; q, t)$: homogenous, symmetric, leading term $\prod_{i=1}^{N} x_i^{\lambda_i}$

$$[T_{i;q} f](x_1, \ldots, x_N) = f(x_1, \ldots, x_{i-1}, q x_i, x_{i+1}, \ldots, x_N)$$

$$\mathcal{D} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{i;q}$$

$$\mathcal{D} P_\lambda(x_1, \ldots, x_N; q, t) = \left[\sum_{i=1}^{N} q^{\lambda_i} t^{N-i} \right] P_\lambda(x_1, \ldots, x_N; q, t)$$

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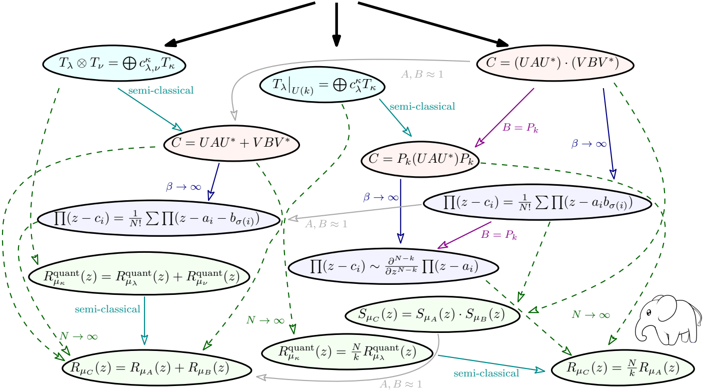
$$\frac{P_\lambda(x_1, \ldots, x_N)}{P_\lambda(1, t, \ldots, t^{N-1})} \frac{P_\nu(x_1, \ldots, x_N)}{P_\nu(1, t, \ldots, t^{N-1})} = \sum_\kappa c_{\lambda,\nu}^{\kappa} \frac{P_\kappa(x_1, \ldots, x_N)}{P_\kappa(1, t, \ldots, t^{N-1})}$$

Conjecture. $c_{\lambda,\nu}^{\kappa} \geq 0$ whenever $0 < q, t < 1$.

Known cases: $q = t$; $q = 0$; $t = 0$; $q \to 1, t = q^{\beta/2}, \beta = 1, 2, 4$.

Macdonald polynomials

$$\frac{P_\lambda(x_1,\ldots,x_N)}{P_\lambda(1,t,\ldots,t^{N-1})}\,\frac{P_\nu(x_1,\ldots,x_N)}{P_\nu(1,t,\ldots,t^{N-1})}=\sum_\kappa c^\kappa_{\lambda,\nu}\,\frac{P_\kappa(x_1,\ldots,x_N)}{P_\kappa(1,t,\ldots,t^{N-1})}$$



$\sum_\kappa c^\kappa_{\lambda,\nu} = 1$ and they define a distribution on κ's.

Everything we saw so far is a limit of this distribution.

Macdonald polynomials

$$\boxed{\frac{P_\lambda(x_1,\dots,x_N)}{P_\lambda(1,t,\dots,t^{N-1})}\frac{P_\nu(x_1,\dots,x_N)}{P_\nu(1,t,\dots,t^{N-1})} = \sum_\kappa c^\kappa_{\lambda,\nu}\frac{P_\kappa(x_1,\dots,x_N)}{P_\kappa(1,t,\dots,t^{N-1})}}$$

Our analysis of $c^\kappa_{\lambda,\nu}$–random κ:

1. Direct combinatorial limits of Macdonald polynomials.
2. Applying difference/differential operators in $x_1,\dots,x_N$ to the defining identity.
3. Contour integral expressions for $\frac{P_\lambda(x_1,1,t,\dots,t^{N-2})}{P_\lambda(1,t,\dots,t^{N-1})}$.

Then $N\to\infty$ analysis of the most general Macdonald case is **open.**

Beyond the Law of Large Numbers

$$\boxed{\frac{P_\lambda(x_1,\dots,x_N)}{P_\lambda(1,t,\dots,t^{N-1})}\frac{P_\nu(x_1,\dots,x_N)}{P_\nu(1,t,\dots,t^{N-1})}=\sum_\kappa c^\kappa_{\lambda,\nu}\frac{P_\kappa(x_1,\dots,x_N)}{P_\kappa(1,t,\dots,t^{N-1})}}$$

Information is lost by considering deterministic limits of **random** κ.

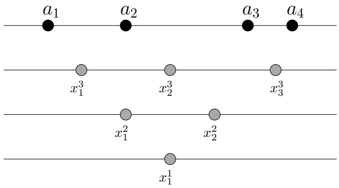
What can be recovered?

Matrix corners again

Back to the most explicit case.

$N \times N$ matrix UAU^*

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



With $(x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N)$, the joint law of particles is

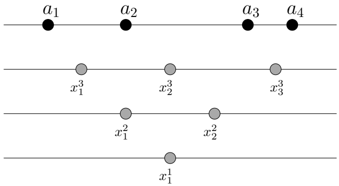
$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

What is happening as $\beta \to \infty$?

Matrix corners again

$N \times N$ matrix UAU^*

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



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As $\beta \to \infty$, particles maximize

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{-2} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}| \to \max$$

Matrix corners again

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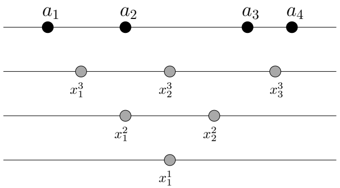

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Proposition. The optimal configuration is given by

$$\prod_{i=1}^{k} (z - \bar{x}_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N} (z - a_j), \qquad k = 1, 2, \ldots, N.$$

Matrix corners again

With $(x_1^N, \dots, x_N^N) = (a_1, \dots, a_N)$, the joint law of particles is

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

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Proposition. $\xi_j^i := \lim_{\beta \to \infty} \sqrt{\beta}(x_j^i - \bar{x}_j^i)$ has Gaussian density

$$\exp\left(\sum_{k=1}^{N-1} \left[\sum_{1 \le i < j \le k} \frac{(\xi_i^k - \xi_j^k)^2}{2(x_i^k - x_j^k)^2} - \sum_{a=1}^{k} \sum_{b=1}^{k+1} \frac{(\xi_a^k - \xi_b^{k+1})^2}{4(x_a^k - x_b^{k+1})^2} \right] \right).$$

A version of discrete Gaussian Free Field.

Central Limit Theorems for fluctuations

1. $\lim_{\beta\to\infty} [C = P_k(UAU^*)P_k]$. (Gorin–Marcus–17): **discrete GFF**
2. $\lim_{\beta\to\infty} [C = (UAU^*)(+ \text{ or } \cdot)(VBV^*)]$. **Open problem**
3. $\lim_{N\to\infty} \left[T_\lambda \Big|_{U(k)} = \bigoplus_\kappa c_\lambda^\kappa T_\kappa \right]$. (Petrov–12, Bufetov–Gorin–15): Fluctuations are given by **Gaussian Free Field** (= covariance given by the Green function of the Laplace operator in suitable complex structure); bijection with random lozenge tilings.
4. $\lim_{N\to\infty} [C = P_k(UAU^*)P_k]$. Same methods and results apply.
5. $\lim_{N\to\infty} [C = UAU^* + VBV^*]$. (Collins–Mingo–Sniady–Speicher–04) **Second order freeness**. Covariance similar to GFF **(??)**.
6. $\lim_{N\to\infty} [C = (UAU^*) \cdot (VBV^*)]$. (Vasilchuk–16) Similar **(??)**.
7. **Conjecture:** The last 3 answers extend to general $\beta > 0$.
8. $\lim_{N\to\infty} \left[T_\lambda \otimes T_\nu = \bigoplus_\kappa c_{\lambda,\nu}^\kappa T_\kappa \right]$. (Bufetov–Gorin–15) CLT with covariance similar to GFF. **Conceptual explanation?**

The simplest open problem

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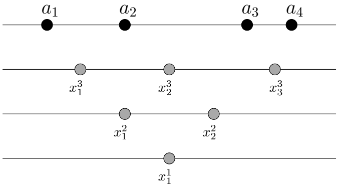

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Fix **arbitrary** $\beta > 0$ and send $N \to \infty$

Conjecture. The Law of Large Numbers is β–independent.

Conjecture. The Central Limit Theorem as $N \to \infty$ is β–independent after rescaling by $\sqrt{\beta}$.

Difficulty: Negative powers in the interaction.

Summary

1. Operations on matrices and representations possess Laws of Large Numbers as $N \to \infty$ or $\beta \to \infty$ leading to various **convolutions.**
2. Fluctuations are described by Gaussian fields. Whenever formula–less identification is made, it is **GFF**.
3. **Macdonald polynomials multiplication** is the mother of all.
4. Many **open** questions remain!