

Quantum polar decomposition algorithm

arXiv:2006.08841

polar decomposition: $\alpha = e^{i\varphi} b$

↑ ↑
complex real

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matrix polar decomposition:

$$A = U B$$

unitary / isometry Hermitian

$$U = A(A^+A)^{-1/2}, \quad B = (A^+A)^{1/2}$$

Problem: Given ability to apply A ,
apply U , and e^{-iBt} deterministically

Applications: Quantum Procrustes problem, pretty good measurements, ...

Method: Hamiltonian quantum singular value transformation: arXiv: 2104.01416

Write $A = \sum_j \sigma_j l_j r_j^\dagger$ $\leftarrow$ right singular vectors

$\uparrow$ singular values $\uparrow$ left singular vectors

$$\Rightarrow A = UB, \quad U = \sum_j l_j r_j^\dagger \quad B = \sum_j \sigma_j r_j r_j^\dagger$$

isometry Hermitian

Assume ability to apply $H = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}$, Hamiltonian block encoding, $Z = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$

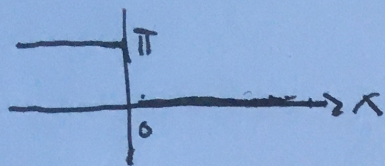
note $H \begin{pmatrix} r_j \\ \pm l_j \end{pmatrix} = \pm \sigma_j \begin{pmatrix} r_j \\ l_j \end{pmatrix}$

eigenvectors/values

$$\Rightarrow \text{can apply } e^{-i f(H)} \begin{pmatrix} r_j \\ \pm l_j \end{pmatrix} = e^{-i f(\pm \sigma_j)} \begin{pmatrix} r_j \\ \pm l_j \end{pmatrix}$$

using either quantum phase estimation (HtL)
or QSVT

choose $f(x) = \pi/2 (1 - \text{sign}(x))$



$$\Rightarrow e^{-i f(H)} = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \Rightarrow \text{deterministically perform } U$$

choose $f(x) = |x|t$



$\Rightarrow$

$$e^{-i f(H)} = \begin{pmatrix} e^{-i \sqrt{A^\dagger A} t} & 0 \\ 0 & e^{-i \sqrt{A A^\dagger} t} \end{pmatrix}$$

$\Rightarrow$ deterministically exponentiate B



takes time $O(k)$, $k = \text{condition \# of } A$

Quantum Procrustes problem

Input/output pairs $|\varphi_j\rangle \rightarrow \boxed{} \rightarrow |\psi_j\rangle$

Find unitary that best approximates $\rightarrow \boxed{} \rightarrow$ in ℓ_2 (Frobenius)

- U is polar decomposition unitary for $A = \sum_j |\psi_j\rangle\langle\varphi_j|$

$\Rightarrow$ perform U deterministically given oracle access to $|\varphi_j\rangle, |\psi_j\rangle$

Pretty good (square root) measurements for $\{|\varphi_j\rangle\}$

- project onto rows of polar decomposition U for $A = \sum_j |\psi_j\rangle\langle\varphi_j|$

$\Rightarrow$ perform pretty good measurement deterministically given oracle access to $|\varphi_j\rangle$

+ much more!