Performance enhancement of quantum annealing by non-traditional quantum driving

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Outline

1. Introduction to quantum annealing

2. Three methods to enhance performance of quantum annealing
   a. Non-stoquastic Hamiltonians
   b. Inhomogeneous field-driving
   c. Reverse annealing

Introduction to quantum annealing

Goal : To solve ombinatorial optimization problems

→ Ground-state search of the Ising model

Given $\{J_{ij}\}$ and $\{h_i\}$, find the values of variables $\{\sigma_i^z\}$ to minimize H_0

$$H_0 = -\sum_{i,j} J_{ij}\sigma_i^z\sigma_j^z - \sum_{i=1}^{N} h_i\sigma_i^z, \quad (\sigma_i^z = \pm 1)$$

2^Npossibilities

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Uparrow$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Downarrow$$

Use quantum fluctuations to search for the solution.

$$H = sH_0 - (1-s)\underline{\sum_{i=1}^{N}}\sigma_i^x \quad (s: 0 \to 1)$$

$$H = H_0 - \Gamma(t)\sum_i \sigma_i^x$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Uparrow \text{ to } \Downarrow$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Downarrow \text{ to } \Uparrow$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

PHYSICAL REVIEW E VOLUME 58, NUMBER 5 NOVEMBER 1998

Quantum annealing in the transverse Ising model

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(Received 30 April 1998)

We introduce quantum fluctuations into the simulated annealing process of optimization problems, aiming at faster convergence to the optimal state. Quantum fluctuations cause transitions between states and thus play the same role as thermal fluctuations in the conventional approach. The idea is tested by the transverse Ising model, in which the transverse field is a function of time similar to the temperature in the conventional method. The goal is to find the ground state of the diagonal part of the Hamiltonian with high accuracy as quickly as possible. We have solved the time-dependent Schrödinger equation numerically for small size systems with various exchange interactions. Comparison with the results of the corresponding classical (thermal) method reveals that the quantum annealing leads to the ground state with much larger probability in almost all cases if we use the same annealing schedule. [S1063-651X(98)02910-9]

PACS number(s): 05.30.−d, 75.10.Nr, 89.70.+c

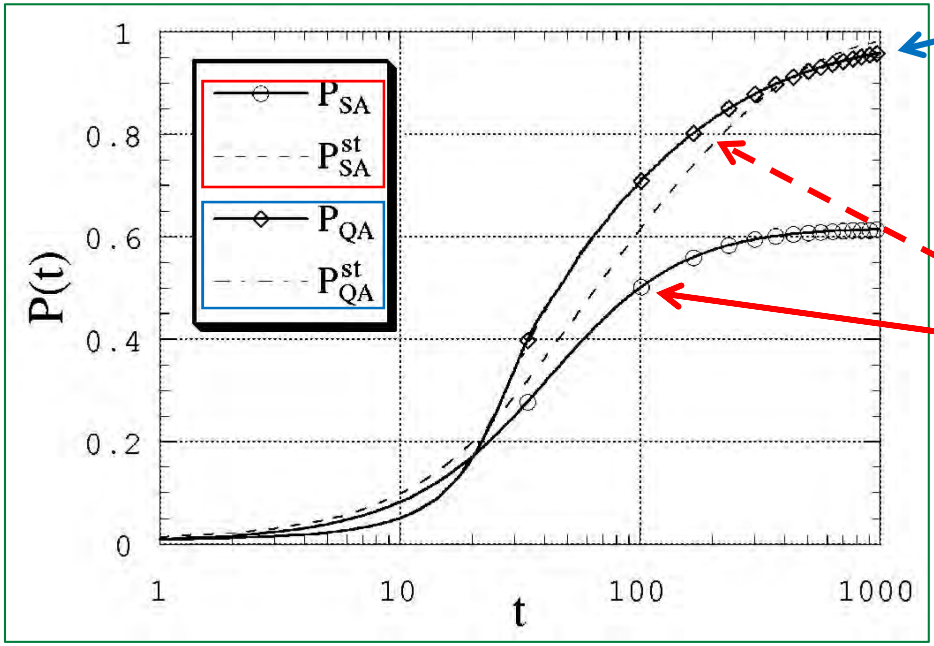
I. INTRODUCTION

The technique of simulated annealing (SA) was first proposed by Kirkpatrick *et al.* [1] as a general method to solve optimization problems. The idea is to use thermal fluctuations to allow the system to escape from local minima of the cost function so that the system reaches the global minimum under an appropriate annealing schedule (the rate of decrease of temperature). If the temperature is decreased too quickly, the system may become trapped in a local minimum. Too slow annealing, on the other hand, is practically useless although such a process would certainly bring the system to the global minimum. Geman and Geman proved a theorem

specific model system, rather than to develop a general argument, to gain insight into the role of quantum fluctuations in the situation of optimization problem. Quantum effects have been found to play a very similar role to thermal fluctuations in the Hopfield model in a transverse field in thermal equilibrium [5]. This observation motivates us to investigate dynamical properties of the Ising model under quantum fluctuations in the form of a transverse field. We therefore discuss in this paper the transverse Ising model with a variety of exchange interactions. The transverse field controls the rate of transition between states and thus plays the same role as the temperature does in SA. We assume that the system has no thermal fluctuations in the QA context and the term

First example of performance advantage over classical simulated annealing

Spin-glass problem (8 spins)



$$\Gamma(t) = \frac{3}{\sqrt{t}}$$

Schrödinger equation

$$T(t) = \frac{3}{\sqrt{t}}$$

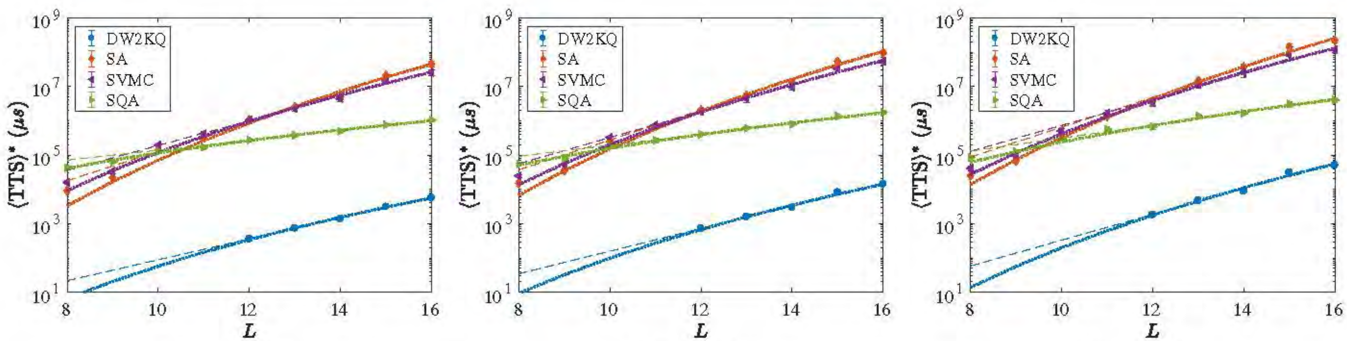
Master equation/
thermal equilibrim

$$H = H_0 - \Gamma(t) \sum_i \sigma_i^x$$

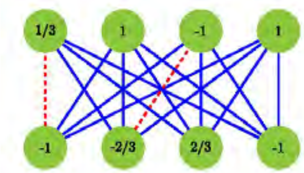
Kadowaki and Nishimori, Phys. Rev. E (1998)

Recent benchmark
Spin-glass problem of size 16x16

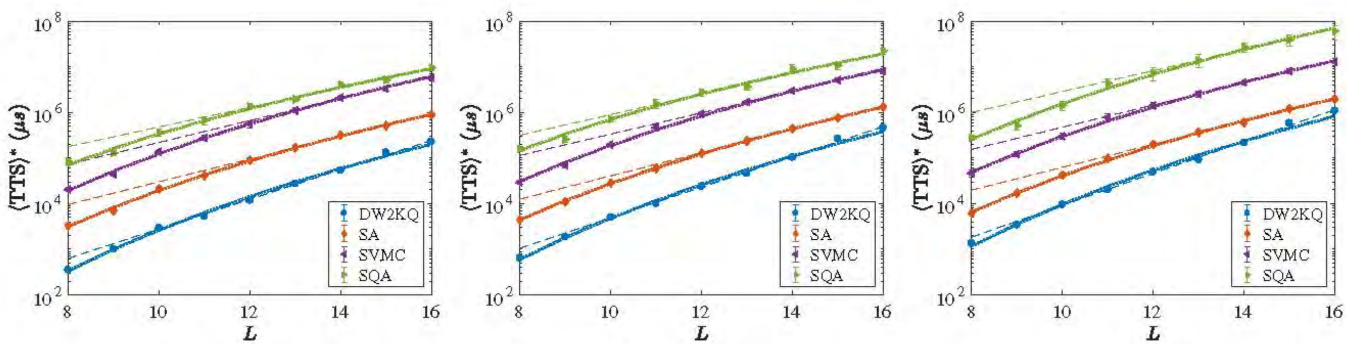
Chimera embedding (8 physical qubits = single logical qubit)



(a) $q = 0.25$ (b) $q = 0.50$ (c) $q = 0.75$



Direct embedding (a physical qubit = a logical qubit)

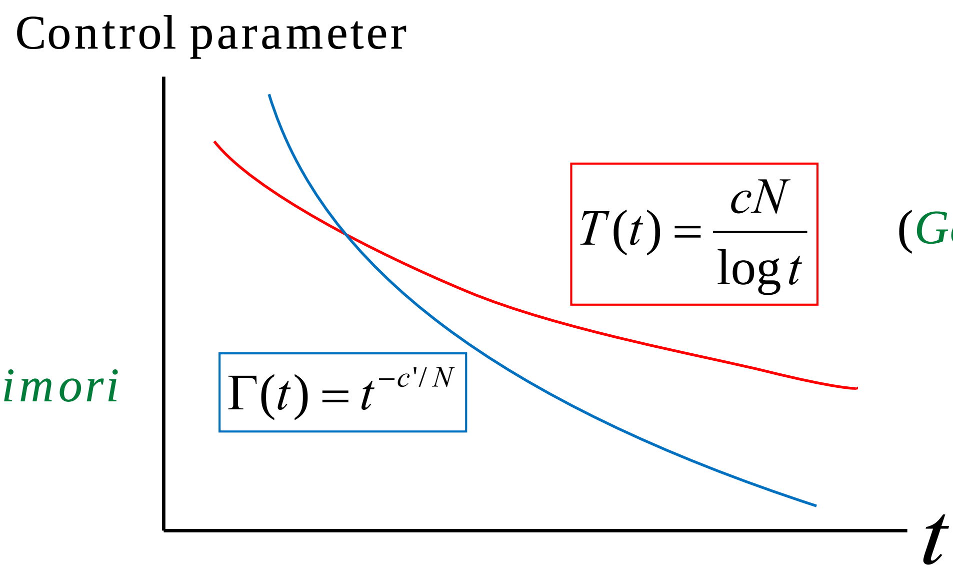


(a) $q = 0.25$ (b) $q = 0.50$ (c) $q = 0.75$

Albash and Lidar, Phys. Rev. X (2018)

Convergence conditions

Sufficient condition for convergence in the infinite-time limit



(*Geman-Geman 1984* for SA)

Morita & Nishimori (2007)

$$H = H_0 + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z\sigma_j^z - \Gamma(t)\sum \sigma_i^x$$

Time complexity to reach a fixed amount of error in energy

Stop the annealing process at a finite but large time and measure the residual energy
(difference between the true ground-state energy and the actually reached energy)

Simulated annealing

$$\Delta E(t) \approx T(t) = \frac{cN}{\ln t} = \delta \quad \Rightarrow \quad t = e^{\frac{cN}{\delta}}$$

$$\frac{1}{\delta} >> \ln \delta \quad (\text{for small } \delta)$$

Quantum annealing: perturbation with respect to Γ

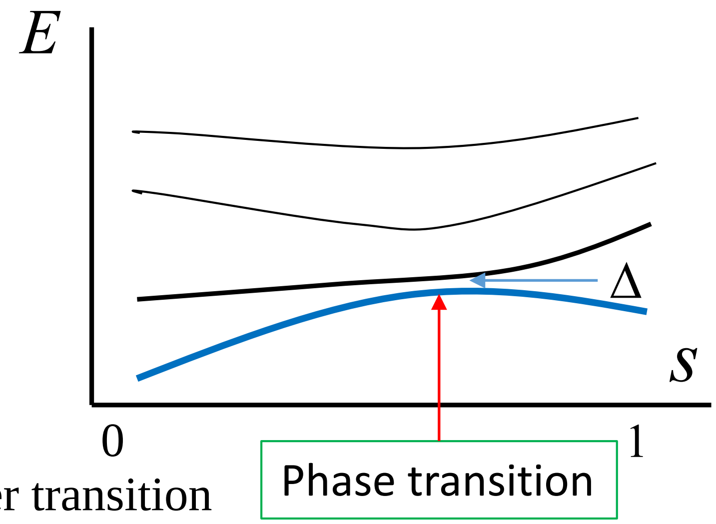
$$\Delta E(t) \approx \Gamma(t)^2 = t^{-2c'/N} = \delta \quad \Rightarrow \quad t = e^{\frac{N|\ln \delta|}{2c'}}$$

$$H = H_0 + H_{\text{quantum}} = -\sum J_{ij}\sigma_i^z \sigma_j^z - \Gamma(t)\sum \sigma_i^x$$

Computational complexity
From the perspective of phase transitions

Adiabatic theorem $\tau \propto \Delta^{-2}$

$$H = sH_0 - (1-s)\sum_{i=1}^{N}\sigma_i^x \quad (s: 0 \to 1) \qquad s = \frac{t}{\tau}$$



Gap scaling

$$\Delta \propto \begin{cases} e^{-aN} & \text{1st order transition} \\ N^{-b} & \text{2nd order transition} \end{cases}$$

Complexity

$$\tau \propto \begin{cases} e^{2aN} & \text{(hard)} \\ N^{2b} & \text{(easy)} \end{cases}$$

Very important to avoid 1st order transition

Performance enhancement by non-traditional quantum driving

a. Non-stoquastic Hamiltonian
b. Inhomogeneous field-driving
c. Reverse annealing

Non-"stoquasticity"

Stoquastic=stochastic + quantum

Bravyi and Terhal 2009

Quantum but can be simulated efficiently by a classical stochastic process

$$\exp(-\beta H) \qquad\qquad H = a\,\sigma_1^x \sigma_2^x \quad (a>0)$$

$$\begin{aligned}
&\langle \uparrow\uparrow | e^{-\beta a \sigma_1^x \sigma_2^x} | \downarrow\downarrow \rangle \\
=&\langle \uparrow\uparrow | (\cosh \beta a) - (\sinh \beta a)\, \sigma_1^x \sigma_2^x | \downarrow\downarrow \rangle \\
=&-\sinh \beta a
\end{aligned}$$

Negative probability shows up if a>0: Non-stoquastic

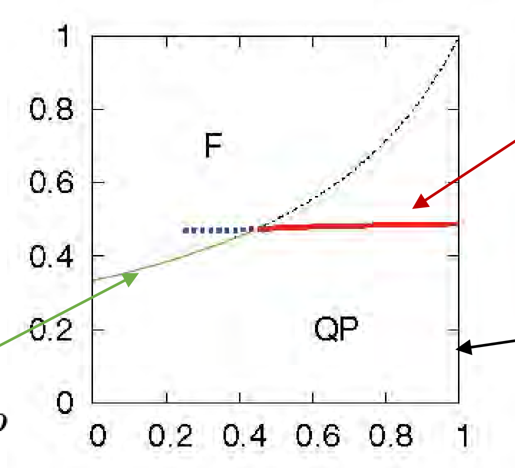
Non-stoquastic Hamiltonian

$$H = s\lambda H_0 - (1-s)\sum_{i=1}^{N}\sigma_i^x + s(1-\lambda)N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$$

$$\exp(-\beta H)$$

Impossible to simulate classically (sign problem). “Strong” quantum effects.

$$H_0 = -N\left(\frac{1}{N}\sum_i \sigma_i^z\right)^p$$



1st order transition
(Exponential time) $\tau \propto e^{aN}$

2nd order transition
Polynomial time
$\tau \propto N^b$

Conventional method

Non-stoquasticity leads to an exponential speedup (not just impossibility of simulation).

Frontiers in ICT, **4**, 2 (2017), *PRE* **85**, 051112 (2012), *J. Phys. A* **48**, 335301 (2015)

Hopfield model: random interaction

$$H_0 = \sum_{i_1 < \cdots < i_k} J_{i_1 \cdots i_k} \sigma^z_{i_1} \cdots \sigma^z_{i_k}$$

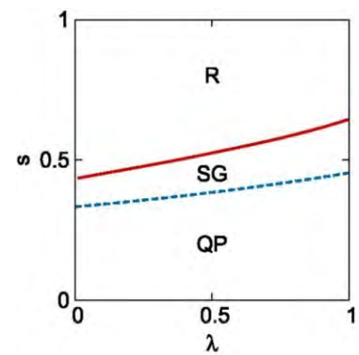
$$J_{i_1 \cdots i_k} = \frac{1}{N^{k-1}} \sum_{\mu=1}^{p} \xi^{\mu}_{i_1} \cdots \xi^{\mu}_{i_k} \quad (\xi^{\mu}_{i_l} = \pm 1)$$

$$H = s\lambda H_0 - (1-s)\sum_{i=1}^{N} \sigma_i^x + s(1-\lambda)N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^x\right)^2$$

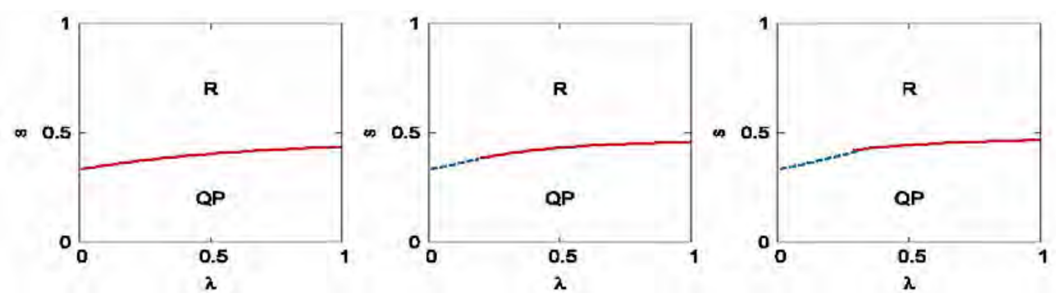
k-body interaction. Randomness in interactions.

$k = 2, \quad p = 0.04N$

$$\boxed{-\sum_{i<j} J_{ij}\sigma_i^z \sigma_j^z}$$



1st order reduced to 2nd !

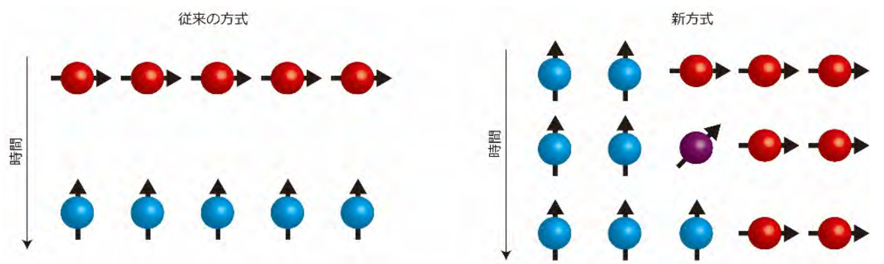


$p = 0.04N^{k-1} \quad k = 3, 4, 5$

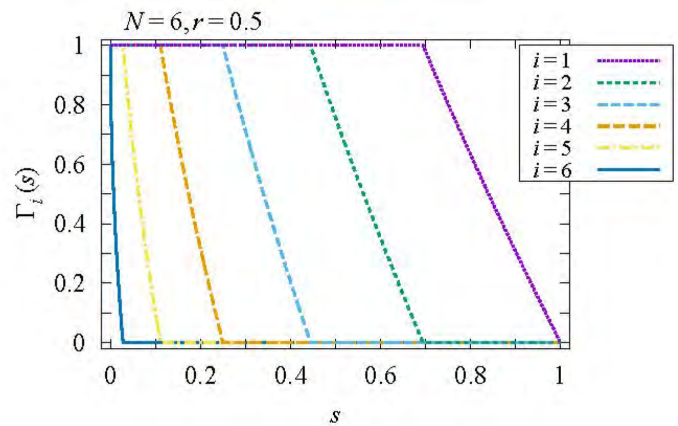
Non-stoquastic Hamiltonian is effective to speedup QA even for a problem with randomness.

Inhomogeneous driving of the transverse field

$$H = sH_0 - (1-s)\sum_{i=1}^{N}\sigma_i^x$$



$$H = sH_0 - (1-s)\sum_{i=1}^{N(1-\tau)}\sigma_i^x - 0\sum_{i=N(1-\tau)+1}^{N}\sigma_i^x$$



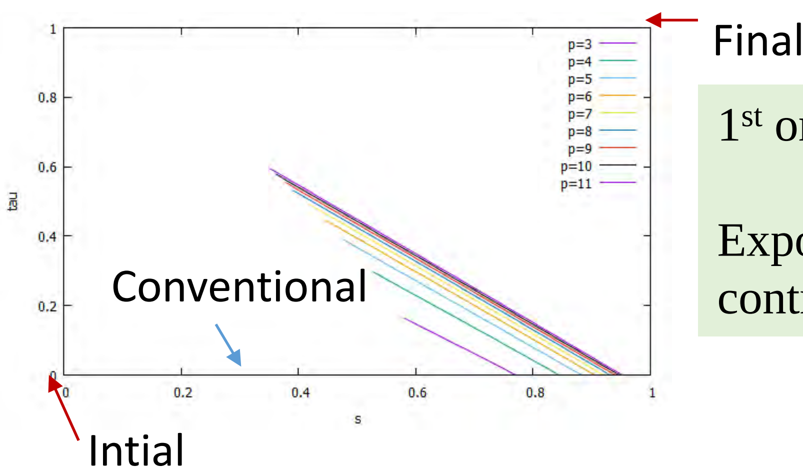
$$\boxed{H_0 = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - \sum_{i=1}^{N}h_i\sigma_i^z}$$

Random-longitudinal-field Ising model, for which non-stoquastic method doesn't work

Result

$$H = sH_0 - (1-s)\sum_{i=1}^{N(1-\tau)} \sigma_i^x - 0 \sum_{i=N(1-\tau)+1}^{N} \sigma_i^x$$

$$H_0 = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - \sum_{i=1}^{N} h_i\sigma_i^z$$



1st order phase transition disappears.

Exponential speedup by a simple inhomogeneous control of the transverse field.

J. Phys. Soc. Jpn. ***87****, 023002 (2018), arXiv:1808.01582*

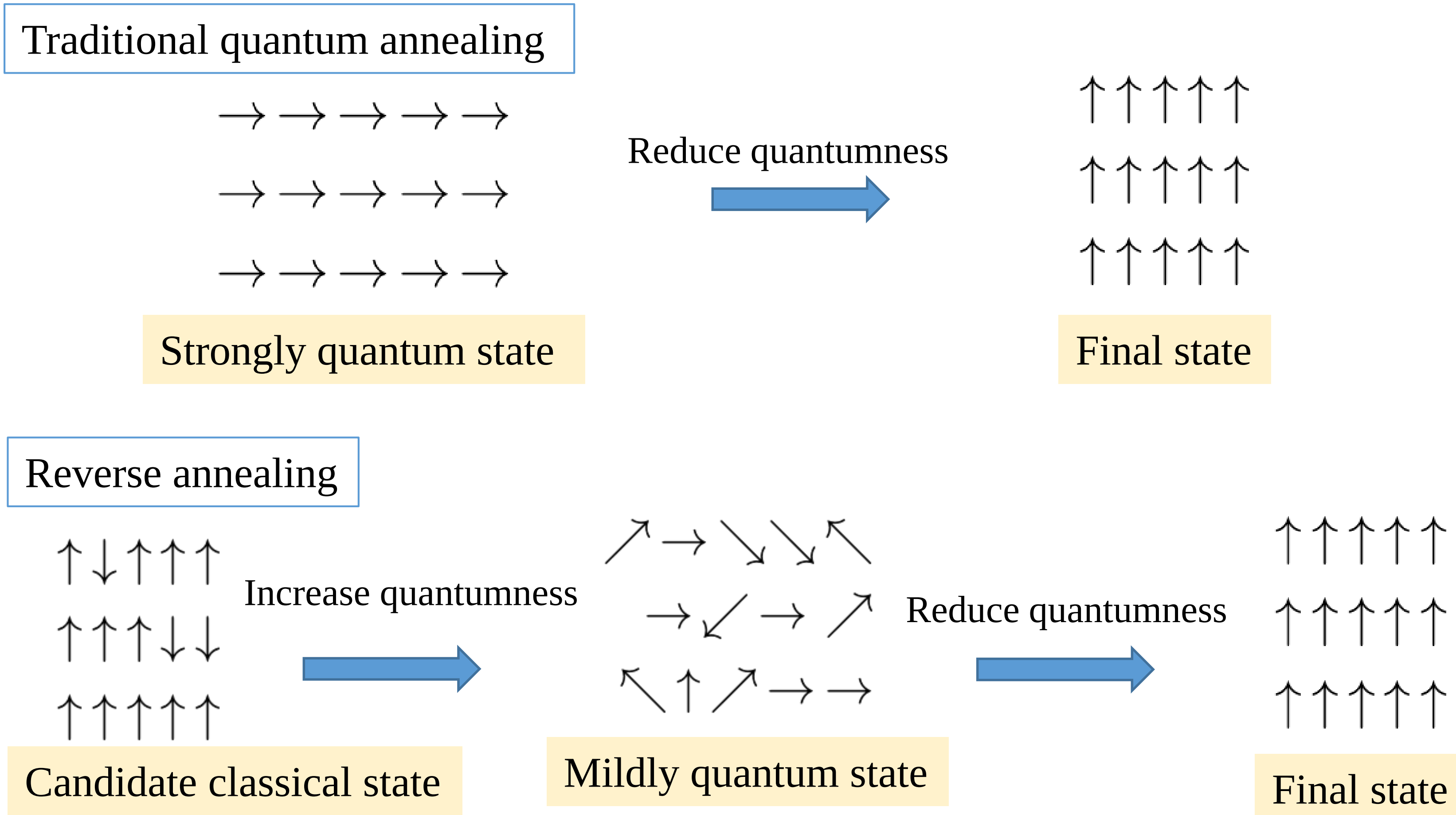
Classical simulated annealing with inhomogeneous temperature drive

Assign local (inverse) temperature to each site and increase each of them one by one.

$$H = -N\left(\frac{1}{N}\sum_{i=1}^{N}\beta_i\sigma_i\right)^p - \sum_{i=1}^{N}h_i\sigma_i \qquad \beta = 1/T$$

- First-order transition persists.
- To be contrasted with the quantum case: inhomogeneous transverse field erased the first order transition.
- Quantum approach is better than the corresponding classical approach. “Limited quantum speedup”

Reverse annealing



Perdomo-Ortiz et al (2010)

Static properties

-- Analytical solution by the mean-field theory (1) --

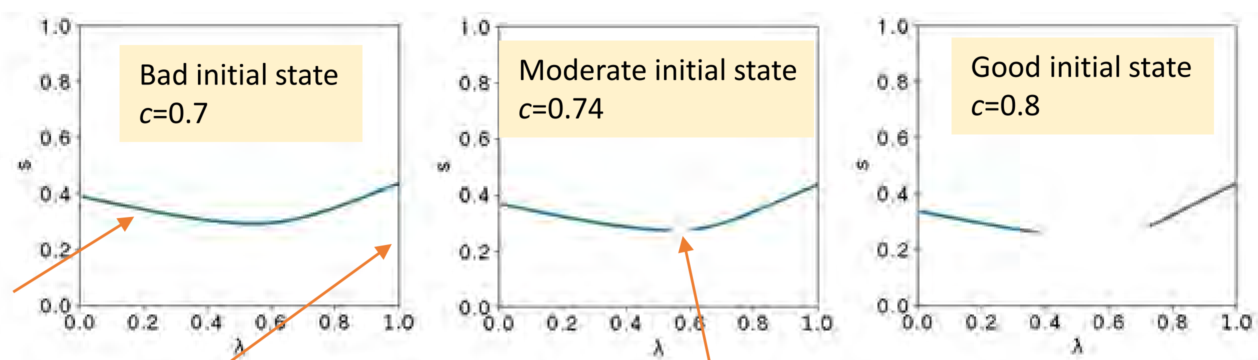
$$H = sH_0 + (1-\lambda)(1-s)H_{\text{init}} + (1-s)\lambda H_{\text{TF}}$$

$$\boxed{H_{\text{int}} = -\sum_{i=1}^{N} \epsilon_i \sigma_i^z \quad (\epsilon_i = 1\ (\text{prob } c), \text{or} -1\ (\text{prob } 1-c))}$$

$$\boxed{H_0 = -N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^z\right)^p - \sum_{i=1}^{N} h_i \sigma_i^z}$$

Start from the classical state ε_i and then increase quantum fluctuations by H_{TF}

$$s = \lambda = 0 \longrightarrow s = \lambda = 1$$



1[st] order

Conventional method

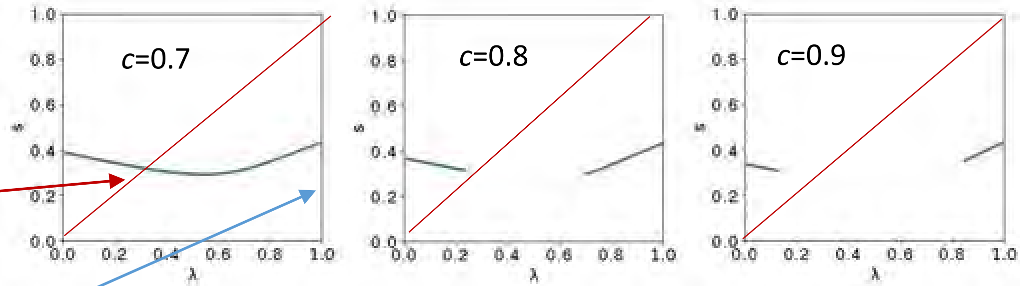
1[st] order disappears

Phys. Rev. A 98, 022314 (2018)

Dynamic properties

-- Direct solution of the Schrodinger equation --

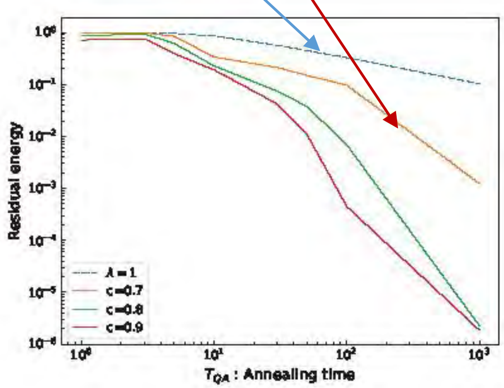
$$H = sH_0 + (1-\lambda)(1-s)H_{\mathrm{init}} + \Gamma \cdot (1-s)\lambda H_{\mathrm{TF}}, \quad s = \lambda = \frac{t}{T_{\mathrm{QA}}}$$



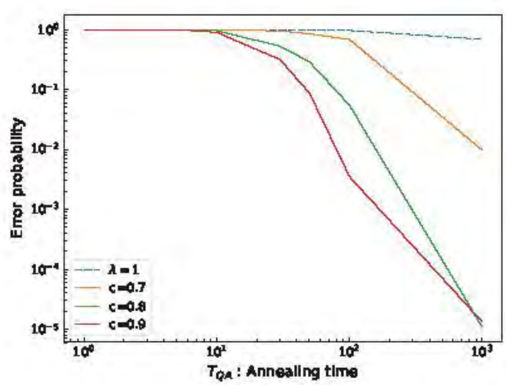
s=λ

Conventional QA

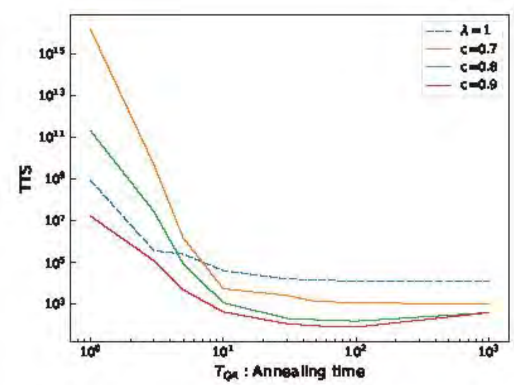
Results for N=45, p=3, Γ=2



Residual energy vs. T_{QA}



Error prob. vs. T_{QA}



TTS vs. T_{QA}

Summary

Quantum annealing with non-stoquastic Hamiltonian
Quantum annealing with inhomogeneous field driving
Quantum annealing with reverse annealing

- Exponential speedup *in comparison with the conventional quantum annealing.* 1^{st} order → 2^{nd} order or no transition.
- Also *in comparison with the corresponding classical simulated annealing* for the inhomogeneous protocol
- Inhomogeneous driving and reverse annealing are realized (at least partially) on the latest D-Wave machine.
- Efforts exist toward hardware implementation of non-stoquastic Hamiltonians.