Performance enhancement of quantum annealing by non-traditional quantum driving

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1. Introduction to quantum annealing

2. Three methods to enhance performance of quantum annealing
   a. Non-stoquastic Hamiltonians
   b. Inhomogeneous field-driving
   c. Reverse annealing
Introduction to quantum annealing

Goal: To solve combinatorial optimization problems

Ground-state search of the Ising model

Given \( \{J_{ij}\} \) and \( \{h_i\} \), find the values of variables \( \{\sigma_i^z\} \) to minimize \( H_0 \)

\[
H_0 = - \sum_{i,j} J_{ij} \sigma_i^z \sigma_j^z - \sum_{i=1}^{N} h_i \sigma_i^z, \quad (\sigma_i^z = \pm 1)
\]

Use quantum fluctuations to search for the solution.

\[
H = sH_0 - (1 - s) \sum_{i=1}^{N} \sigma_i^x \quad (s : 0 \rightarrow 1)
\]

\[
H = H_0 - \Gamma(t) \sum_i \sigma_i^x
\]
Quantum annealing in the transverse Ising model

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We introduce quantum fluctuations into the simulated annealing process of optimization problems, aiming at faster convergence to the optimal state. Quantum fluctuations cause transitions between states and thus play the same role as thermal fluctuations in the conventional approach. The idea is tested by the transverse Ising model, in which the transverse field is a function of time similar to the temperature in the conventional method. The goal is to find the ground state of the diagonal part of the Hamiltonian with high accuracy as quickly as possible. We have solved the time-dependent Schrödinger equation numerically for small size systems with various exchange interactions. Comparison with the results of the corresponding classical (thermal) method reveals that the quantum annealing leads to the ground state with much larger probability in almost all cases if we use the same annealing schedule. [S1063-651X(98)02910-9]

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I. INTRODUCTION

The technique of simulated annealing (SA) was first proposed by Kirkpatrick et al. [1] as a general method to solve optimization problems. The idea is to use thermal fluctuations to allow the system to escape from local minima of the cost function so that the system reaches the global minimum under an appropriate annealing schedule (the rate of decrease of temperature). If the temperature is decreased too quickly, the system may become trapped in a local minimum. Too slow annealing, on the other hand, is practically useless although such a process would certainly bring the system to the global minimum. Ceperley and Lieb proved a theorem specific model system, rather than to develop a general argument, to gain insight into the role of quantum fluctuations in the situation of optimization problem. Quantum effects have been found to play a very similar role to thermal fluctuations in the Hopfield model in a transverse field in thermal equilibrium [5]. This observation motivates us to investigate dynamical properties of the Ising model under quantum fluctuations in the form of a transverse field. We therefore discuss in this paper the transverse Ising model with a variety of exchange interactions. The transverse field controls the rate of transition between states and thus plays the same role as the temperature does in SA. We assume that the system has no thermal fluctuations in the QA context and the term
First example of performance advantage over classical simulated annealing

Spin-glass problem (8 spins)

\[ H = H_0 - \Gamma(t) \sum_i \sigma_i^x \]


\[ \Gamma(t) = \frac{3}{\sqrt{t}} \]

Schrödinger equation

\[ T(t) = \frac{3}{\sqrt{t}} \]

Master equation/thermal equilibrium
Recent benchmark
Spin-glass problem of size 16x16

Chimera embedding (8 physical qubits = single logical qubit)

Direct embedding (a physical qubit = a logical qubit)

Convergence conditions

Sufficient condition for convergence in the infinite-time limit

\[ H = H_0 + H_{\text{quantum}} = -\sum J_{ij} \sigma_i^z \sigma_j^z - \Gamma(t) \sum \sigma_i^x \]

\[ T(t) = \frac{cN}{\log t} \]

\( \Gamma(t) = t^{-e'^{1/N}} \)

(Geman-Geman 1984 for SA)

Control parameter

Morita & Nishimori (2007)
Time complexity to reach a fixed amount of error in energy

Stop the annealing process at a finite but large time and measure the residual energy (difference between the true ground-state energy and the actually reached energy)

Simulated annealing

\[ \Delta E(t) \approx T(t) = \frac{cN}{\ln t} = \delta \quad \Rightarrow \quad t = e^{\frac{cN}{\delta}} \]

Quantum annealing: perturbation with respect to \( \Gamma \)

\[ \Delta E(t) \approx \Gamma(t)^2 = t^{-2c'/N} = \delta \quad \Rightarrow \quad t = e^{\frac{N|\ln \delta|}{2c'}} \]

\[ H = H_0 + H_{\text{quantum}} = -\sum J_{ij} \sigma_i^z \sigma_j^z - \Gamma(t) \sum \sigma_i^x \]
Computational complexity
From the perspective of phase transitions

Adiabatic theorem
\[ \tau \propto \Delta^{-2} \]
\[ H = sH_0 - (1 - s) \sum_{i=1}^{N} \sigma_i^x \quad (s : 0 \rightarrow 1) \quad s = \frac{t}{\tau} \]

Gap scaling
\[ \Delta \propto \begin{cases} e^{-aN} & \text{1st order transition} \\ N^{-b} & \text{2nd order transition} \end{cases} \]

Complexity
\[ \tau \propto \begin{cases} e^{2aN} & \text{(hard)} \\ N^{2b} & \text{(easy)} \end{cases} \]

Very important to avoid 1st order transition
Performance enhancement by non-traditional quantum driving

a. Non-stoquastic Hamiltonian
b. Inhomogeneous field-driving
c. Reverse annealing
Stoquastic = stochastic + quantum

Quantum but can be simulated efficiently by a classical stochastic process

\[ \exp(-\beta H), \quad H = a \sigma_1^x \sigma_2^x \quad (a > 0) \]

\[ \langle \uparrow\uparrow | e^{-\beta a \sigma_1^x \sigma_2^x} | \downarrow\downarrow \rangle = \langle \uparrow\uparrow | (\cosh \beta a) - (\sinh \beta a) \sigma_1^x \sigma_2^x | \downarrow\downarrow \rangle = -\sinh \beta a \]

Negative probability shows up if a>0: Non-stoquastic
Non-stoquastic Hamiltonian

\[
H = s \lambda H_0 = \left(1 - s\right) \sum_{i=1}^{N} \sigma_{i}^{x} + \left(1 - \lambda \right) \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_{i}^{x} \right)^{2}
\]

\[\exp(-\beta H)\]

Impossible to simulate classically (sign problem). “Strong” quantum effects.

\[H_0 = -N \left( \frac{1}{N} \sum_{i} \sigma_{i}^{z} \right)^{p}\]

1st order transition (Exponential time) \(\tau \propto e^{aN}\)

2nd order transition (Polynomial time) \(\tau \propto N^{b}\)

Conventional method

Non-stoquasticity leads to an exponential speedup (not just impossibility of simulation).

Hopfield model: random interaction

\[ H_0 = \sum_{i_1 < \ldots < i_k} J_{i_1 \ldots i_k} \sigma_{i_1}^z \cdots \sigma_{i_k}^z \]

\[ J_{i_1 \ldots i_k} = \frac{1}{N^{k-1}} \sum_{\mu=1}^{p} \xi_{i_1}^\mu \cdots \xi_{i_k}^\mu \quad (\xi_{i_1}^\mu = \pm 1) \]

\[ H = s\lambda H_0 - (1 - s) \sum_{i=1}^{N} \sigma_i^x + s(1 - \lambda)N \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_i^x \right)^2 \]

\[ -\sum_{i<j} J_{ij} \sigma_i^z \sigma_j^z \]

\[ k = 2, \quad p = 0.04N \]

1st order reduced to 2nd!

\[ p = 0.04N^{k-1} \quad k = 3, 4, 5 \]

Non-stoquastic Hamiltonian is effective to speedup QA even for a problem with randomness.
Inhomogeneous driving of the transverse field

\[ H = sH_0 - (1 - s) \sum_{i=1}^{N} \sigma_i^x \]

\[ H = sH_0 - (1 - s) \sum_{i=1}^{N(1-\tau)} \sigma_i^x - 0 \sum_{i=N(1-\tau)+1}^{N} \sigma_i^x \]

\[ H_0 = -N \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_i^z \right)^p - \sum_{i=1}^{N} h_i \sigma_i^z \]

Random-longitudinal-field Ising model, for which non-stoquastic method doesn’t work
Result

Initial

Final

1\textsuperscript{st} order phase transition disappears.

Exponential speedup by a simple inhomogeneous control of the transverse field.

\begin{equation}
H = sH_0 - (1 - s) \sum_{i=1}^{N(1-\tau)} \sigma^x_i - 0 \sum_{i=N(1-\tau)+1}^{N} \sigma^x_i \quad H_0 = -N \left( \frac{1}{N} \sum_{i=1}^{N} \sigma^z_i \right)^p - \sum_{i=1}^{N} h_i \sigma^z_i
\end{equation}

Classical simulated annealing with inhomogeneous temperature drive

Assign local (inverse) temperature to each site and increase each of them one by one.

\[ H = -N \left( \frac{1}{N} \sum_{i=1}^{N} \beta_i \sigma_i \right)^p - \sum_{i=1}^{N} h_i \sigma_i \quad \beta = 1/T \]

- First-order transition persists.
- To be contrasted with the quantum case: inhomogeneous transverse field erased the first order transition.
- Quantum approach is better than the corresponding classical approach. “Limited quantum speedup”
Reverse annealing

Traditional quantum annealing

\[
\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow
\]

Strongly quantum state

Reduce quantumness

\[
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow
\]

Final state

Reverse annealing

\[
\uparrow \downarrow \uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \downarrow \downarrow \\
\uparrow \uparrow \uparrow \uparrow \uparrow
\]

Candidate classical state

Increase quantumness

\[
\rightarrow \rightarrow \rightarrow \rightarrow \\
\rightarrow \leftarrow \rightarrow \rightarrow \\
\leftarrow \uparrow \rightarrow \rightarrow
\]

Mildly quantum state

Reduce quantumness

\[
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow \\
\uparrow \uparrow \uparrow \uparrow
\]

Final state

Perdomo-Ortiz et al (2010)
Static properties

--- Analytical solution by the mean-field theory (1) ---

\[ H = sH_0 + (1 - \lambda)(1 - s)H_{\text{init}} + (1 - s)\lambda H_{\text{TF}} \]

\[ H_{\text{int}} = -\sum_{i=1}^{N} \epsilon_i \sigma_i^z \quad (\epsilon_i = 1 \text{ (prob } c), \text{ or } -1 \text{ (prob } 1 - c)) \]

\[ H_0 = -N \left( \frac{1}{N} \sum_{i=1}^{N} \sigma_i^z \right)^p - \sum_{i=1}^{N} h_i \sigma_i^z \]

Start from the classical state \( \epsilon_i \) and then increase quantum fluctuations by \( H_{\text{TF}} \)

\[ s = \lambda = 0 \quad \rightarrow \quad s = \lambda = 1 \]

---

**Conventional method**

1st order disappears

**Bad initial state**

\( c = 0.7 \)

**Moderate initial state**

\( c = 0.74 \)

**Good initial state**

\( c = 0.8 \)

---

Dynamic properties

Direct solution of the Schrödinger equation

\[ H = s H_0 + (1 - \lambda)(1 - s) H_{\text{init}} + \Gamma \cdot (1 - s) \lambda H_{\text{TF}}, \quad s = \lambda = \frac{t}{T_{QA}} \]

Results for \( N=45, p=3, \Gamma=2 \)

- Residual energy vs. \( T_{QA} \)
- Error prob. vs. \( T_{QA} \)
- TTS vs. \( T_{QA} \)
Quantum annealing with **non-stoquastic** Hamiltonian
Quantum annealing with **inhomogeneous field driving**
Quantum annealing with **reverse annealing**

- **Exponential speedup** *in comparison with the conventional quantum annealing.* 1\(^{\text{st}}\) order $\rightarrow$ 2\(^{\text{nd}}\) order or no transition.
- **Also in comparison with the corresponding classical simulated annealing** for the inhomogeneous protocol
- Inhomogeneous driving and reverse annealing are realized (at least partially) on the latest D-Wave machine.
- Efforts exist toward hardware implementation of non-stoquastic Hamiltonians.