"ν" $\geqslant 2$ for Random k-SAT

$2-\alpha \geqslant 2$ for Random k-SAT

On the critical exponents for random k-SAT

David B. Wilson
Microsoft Research

Random 2-SAT (k-SAT)

n variables

$$F_{n,m} = \underbrace{(x_3 \vee \bar{x}_7) \wedge (x_2 \vee \bar{x}_{11}) \wedge \cdots \wedge (\bar{x}_8 \vee \bar{x}_{13})}_{\substack{m \text{ clauses} \\ \text{chosen at random}}}$$

$$F_{n,p} = (x_3 \vee \bar{x}_7) \wedge (x_2 \vee \bar{x}_{11}) \wedge \cdots$$

each possible clause appears with probability p

$$2^2 \binom{n}{2} p \approx m$$

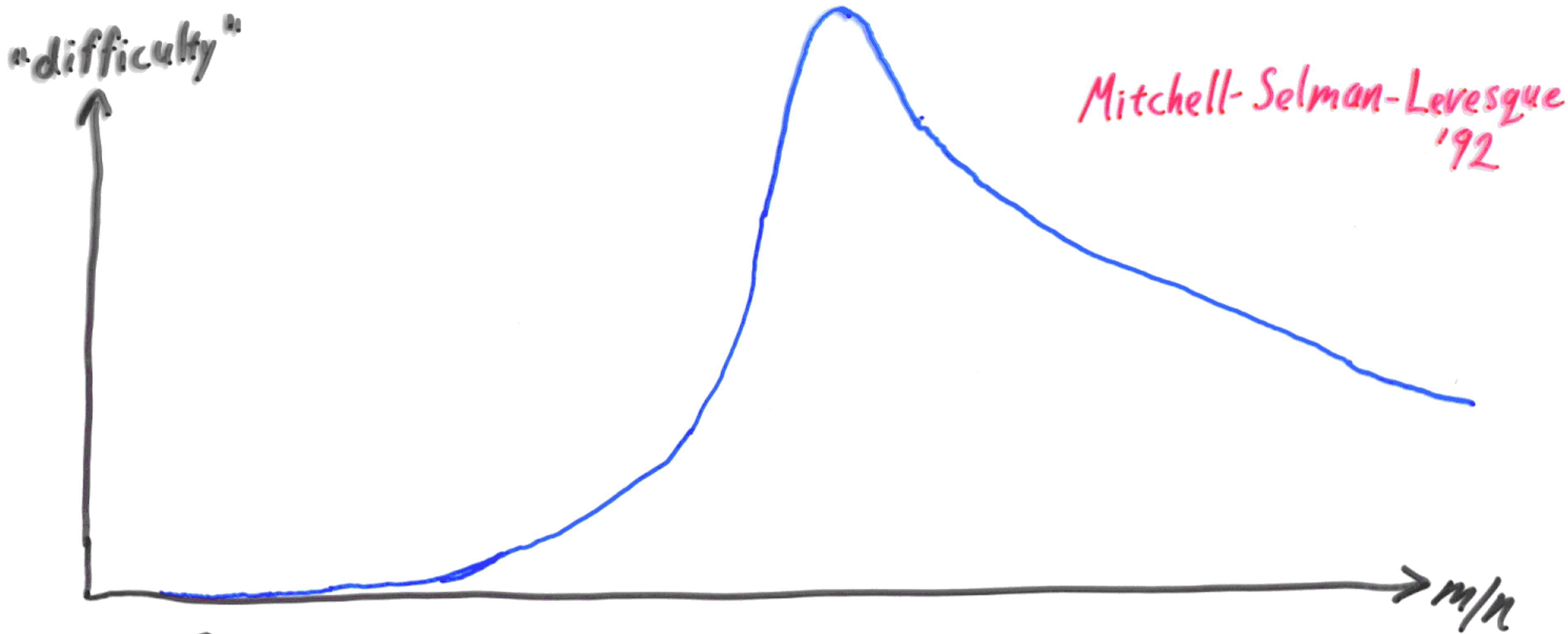
Satisfying Assignments

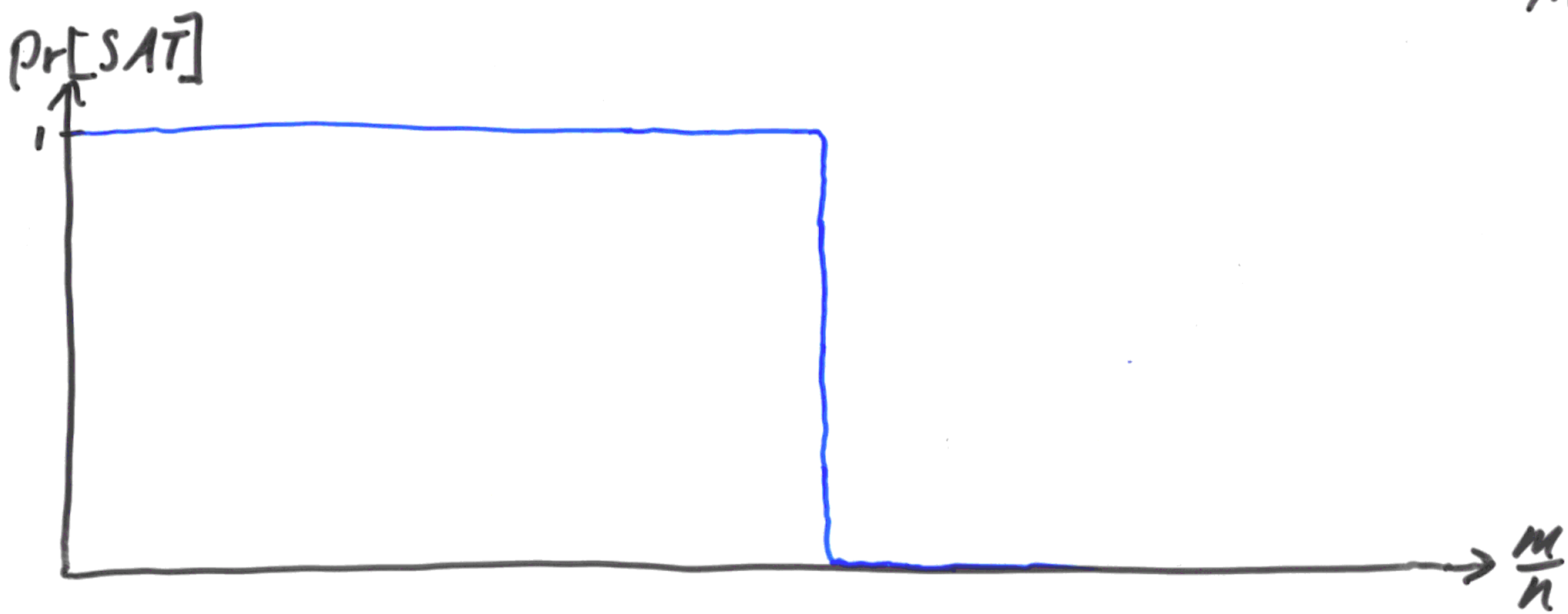
$$(\bar{X}_2 \vee \bar{X}_3) \wedge (X_1 \vee X_3) \wedge (X_1 \vee \bar{X}_2) \wedge (\bar{X}_1 \vee X_3)$$

If $X_1 = True$, $X_2 = False$, $X_3 = True$
Then all clauses are true
making the formula true.

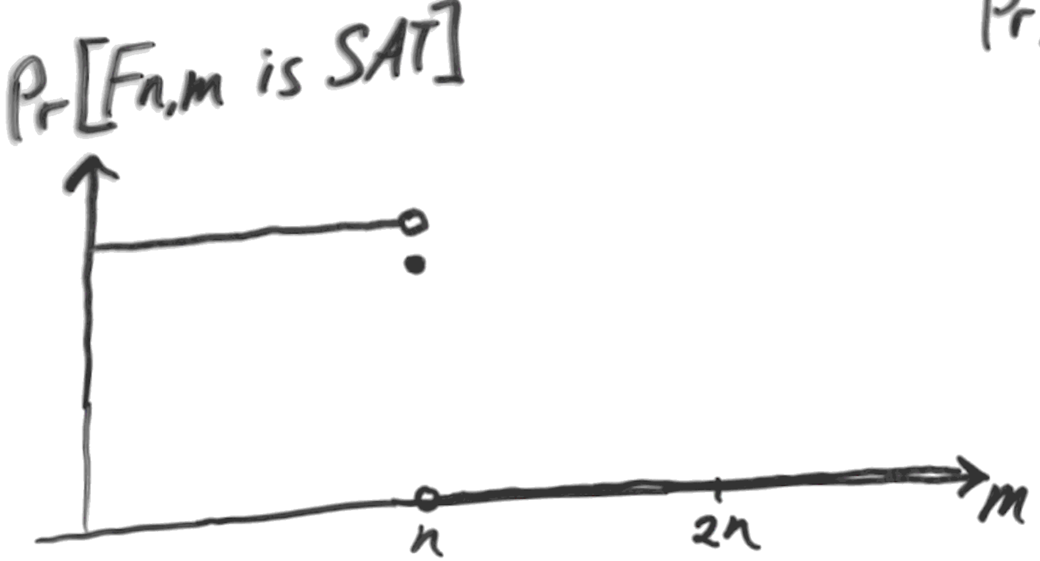
The formula is "satisfiable" if there is an assignment of the Boolean variables which makes it true.

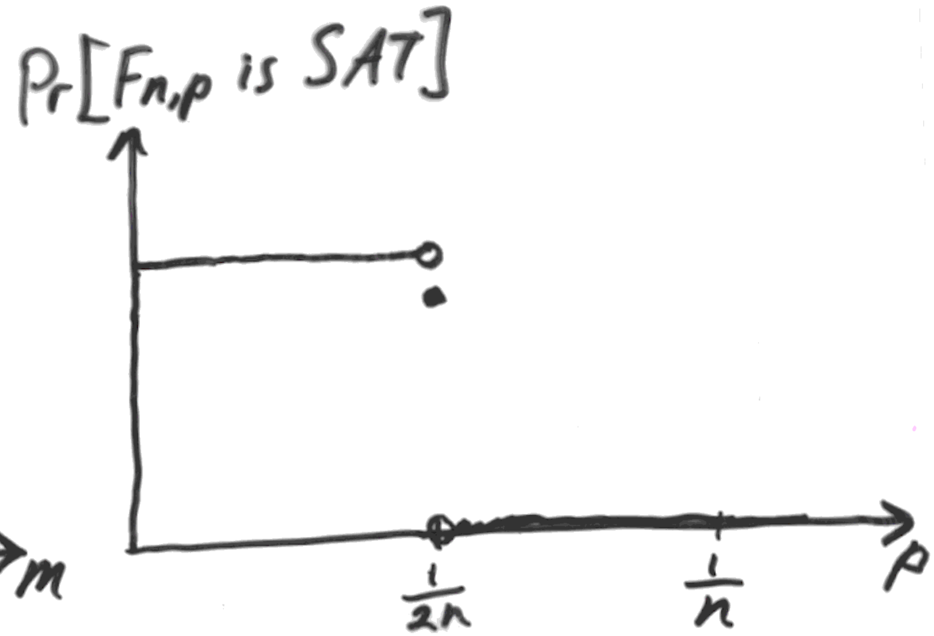
The k-SAT Threshold





Sharp Threshold for 2-SAT





$\frac{m}{n} = \alpha$

If $\alpha < 1$, $\Pr[SAT] \to 1$

If $\alpha > 1$, $\Pr[SAT] \to 0$

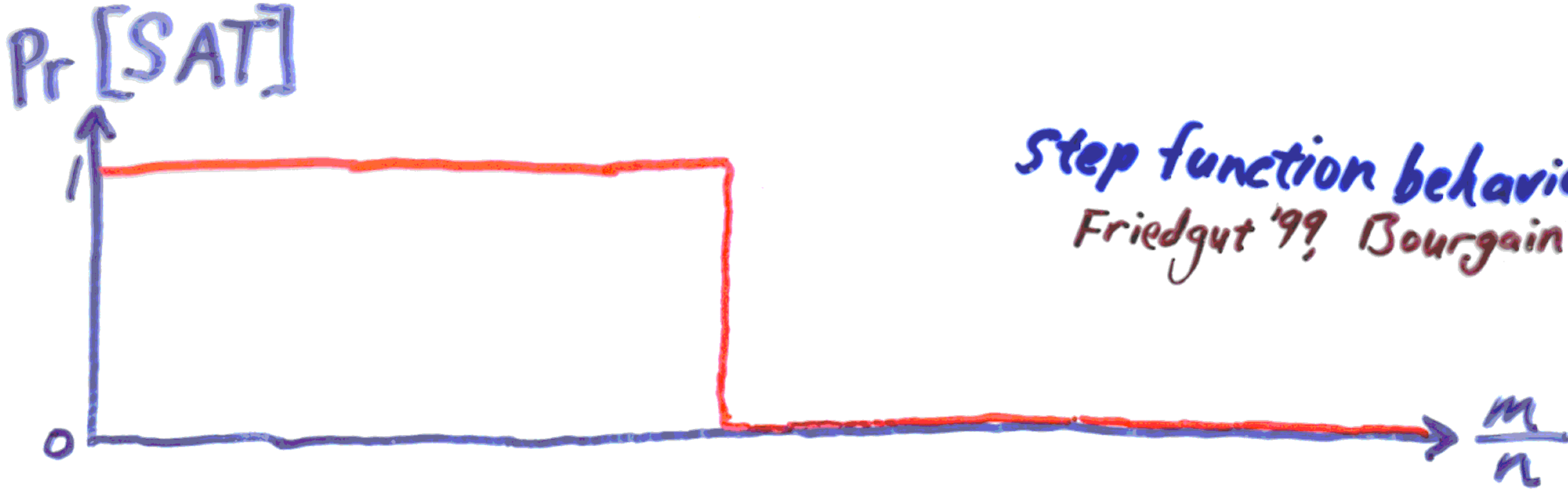
$p2n = \alpha$

Chvátal and Reed '92

Fernandez de la Vega '92

Goerdt '92

Threshold Behavior of 3-SAT



Step function behavior:
Friedgut '99, Bourgain '99

Monte Carlo estimates
4.2 - 4.3

lower bounds

1	Chao-Franco '86, '90
1.63	Broder-Frieze-Upfal '93
3.003	Frieze-Suen '96
3.145	Achlioptas '00
3.26	Achlioptas-Sorkin '00
3.42	Kaporis-Kirousis-Lalas '02

upper bounds

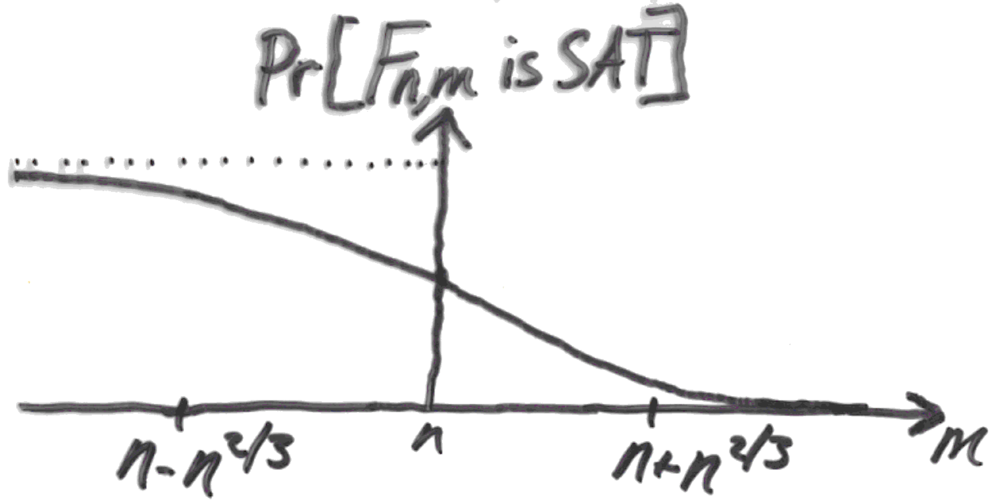
5.191	Franco-Paul '83
5.08	El Maftouhi - Fernandez de la Vega '95
4.758	Kamath-Motwani-Palem-Spirakis '95
4.643	Dubois-Boufkhad '97
4.602	Kirousis-Kranakis-Krizanc '96
4.596	Janson-Stamatiou-Vamvakari '00
(4.506)	Dubois-Boufkhad-Mandler '99
4.571	Kirousis-Stamatiou-Vamvakari-Zito '9[illegible]

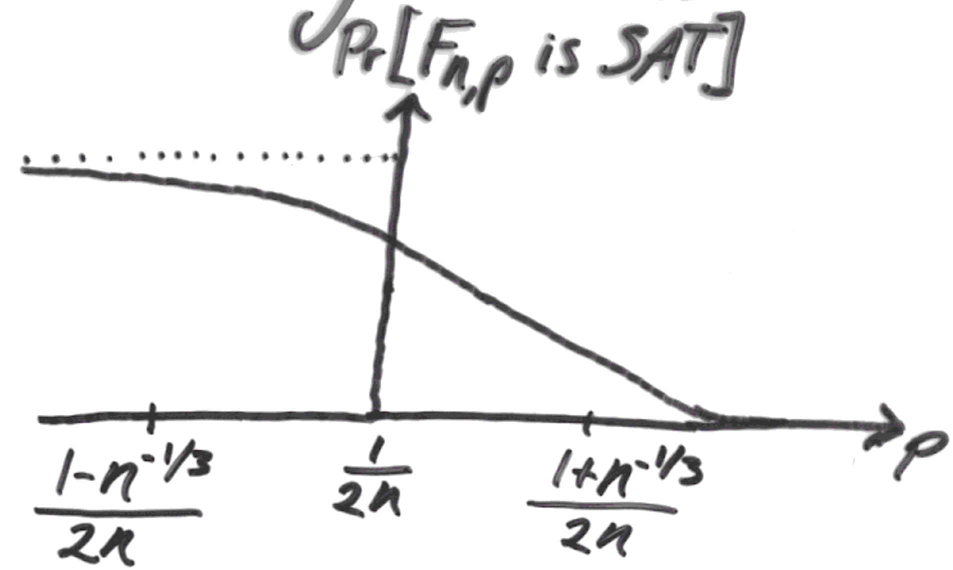
"It could be that $\alpha_3(n) = 4.1$ for odd n and $\alpha_3(n) = 4.3$ for even n."

No it couldn't.

If $n_1 < n_2$, $\alpha_3(n_1) = 4.1$ and $\alpha_3(n_2) = 4.3$, then $n_2 \geqslant n_1 + \Theta(n_1^{1/2})$.

Width of the 2-SAT Scaling Window





$m = n \pm \Theta(n^{2/3})$

$p = \frac{1}{2n} \pm \Theta(n^{-4/3})$

Exponents predicted by BBCK '98
Monasson & Zecchina '98

Exponents proved by BBCKW '99
(RS&A 2001)

Increase $\frac{m}{n}$ by $\Theta(n^{-\frac{1}{3}})$

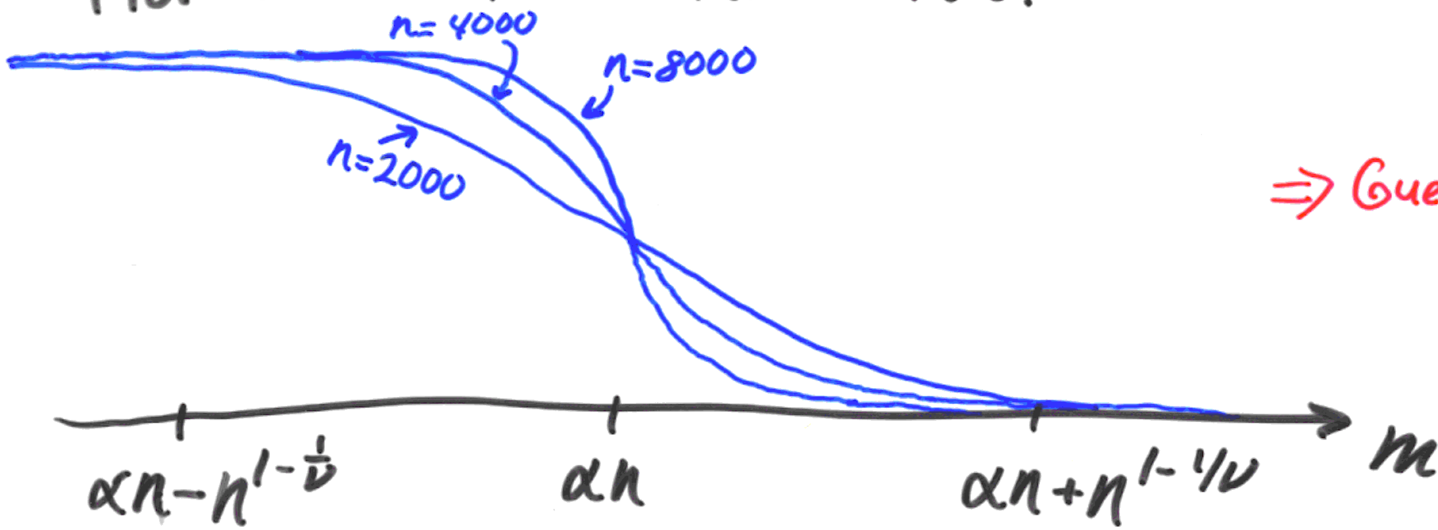
"ν" or $2-\alpha$

Measuring "ν" by experiment

Guess α_k $\quad(\alpha_2 = 1, \; \alpha_3 \approx 4.2)$

Guess ν_k

Plot curves for different n's.



$\Rightarrow$ Guess for ν too big.

If α_k and ν_k are correct, curves should line up.

For 2-SAT, when $n \geq 10{,}000$ evident that $\nu = 3$.
when $n < 1{,}000$ not so obvious.

Values of "ν" for k-SAT

k	Experimental	Heuristic	Rigorous
2	$2.6 \pm .2$, 2.8, 3	3	3
3	$1.5 \pm .1$		≥ 2
4	$1.25 \pm .1$		≥ 2
5	$1.10 \pm .1$		≥ 2
6	$1.05 \pm .05$		≥ 2
$k \to \infty$		1, ∞	≥ 2

$F_{n,m}$, $F_{n,p}$, and ν

Consider the property "Has $\geq 4.2n$ clauses"

In $F_{n,m}$ very sharp transition, $\nu = 1$

In $F_{n,p}$, there are $\sqrt{}$ fluctuations in #clauses, $\nu = 2$

For the property "Formula is satisfiable"

Already known that $\nu \geq 2$ in $F_{n,p}$

A priori it could be that $\nu < 2$ in $F_{n,m}$

Nonetheless, we show $\nu \geq 2$ even in $F_{n,m}$

Bystander Rule

Rule for partitioning a set of clauses into relevant clauses and bystander clauses.

Bystander clauses "don't influence" satisfiability.

For any subset of original set of clauses,
subset is SAT iff (subset \ bystander clauses) is SAT

Isolated-literal Bystander Rule

Given a set of clauses, declare a clause to be a bystander if it contains a variable which is not in any other clause in the set; otherwise declare the clause to be relevant.

For any subset of clauses drawn from the set, adding or removing bystander clauses does not affect satisfiability.

Isolated-literal Bystander Rule

Given a set of clauses, declare a clause to be a bystander if it contains a variable which is not in any other clause in the set; otherwise declare the clause to be relevant.

For any subset of clauses drawn from the set, adding or removing bystander clauses does not affect satisfiability.

If k is fixed, and $m = O(n)$, w.h.p. a constant fraction of set of clauses will be bystanders.

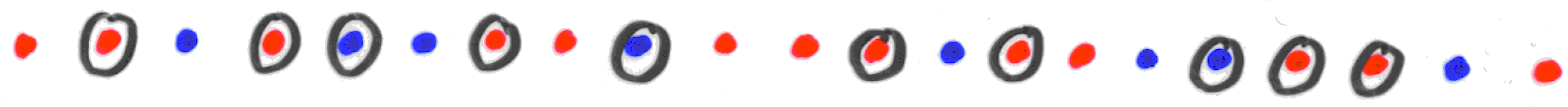
$$1 - (1 - e^{-km/n})^k$$

m clauses, r relevant, b bystanders

$m = r + b$ (in UNSAT phase)

m clauses, r relevant, b bystanders

$m = r + b$ (in UNSAT phase)



Alice picks j of these clauses uniformly at random.

$L = 7$ relevant items in $f_{m,j}$

$j - L = 3$ bystanders

m clauses, r relevant, b bystanders

$m = r + b$ (in UNSAT phase)

Bob: "Alice has a fair formula.
In particular L has the right distribution.
But the m items are already in a random order.
So I'll take the first L relevant clauses,
and the first $j-L$ bystanders."

$\hat{f}_{m,j}$ fair formula.

m clauses, r relevant, b bystanders

$m = r + b$ (in UNSAT phase)

$\hat{f}_{m,j}$

Satisfiability of $\hat{f}_{m,j}$ depends only on L.

Critical L_0, $\hat{f}_{m,j}$ is SAT iff $L < L_0$.

If Alice picks clauses one at a time, there is a first j_0 such that Bob's $\hat{f}_{m,j_0}$ is UNSAT.

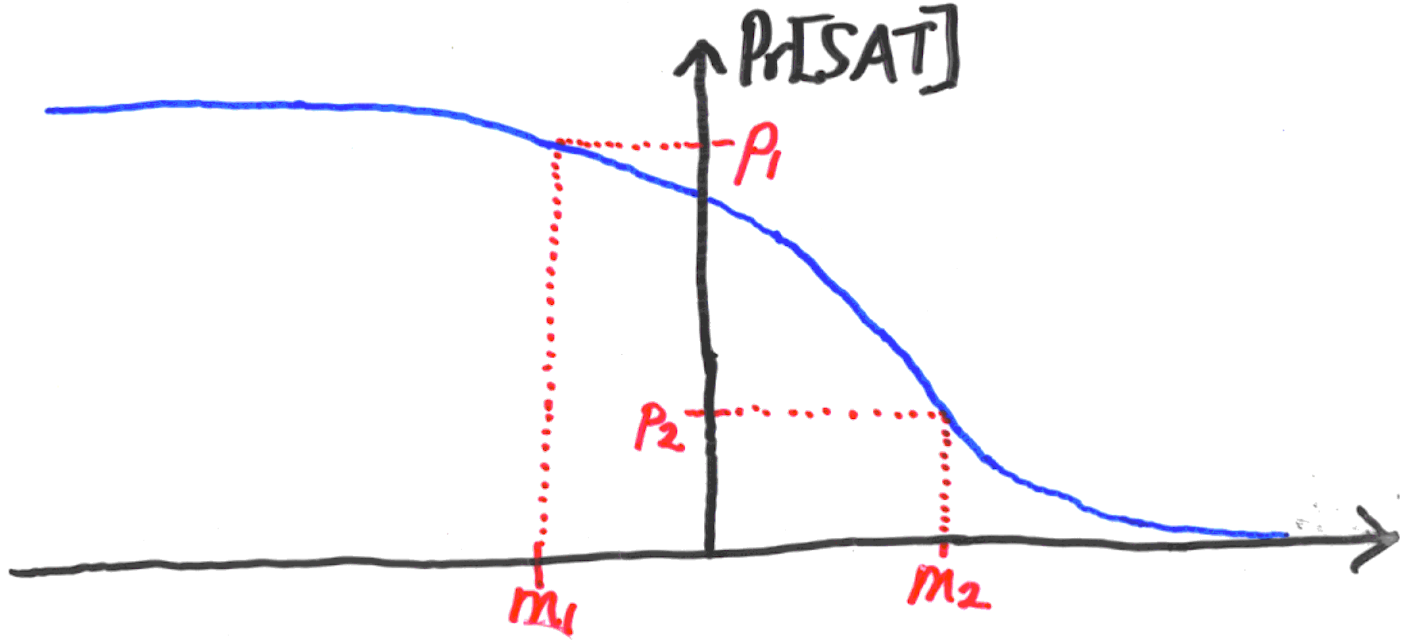
When Alice picks $f_{m,j}$, $\sqrt{}$ fluctuations in L

$\Downarrow$

$\sqrt{}$ fluctuations in j_0.

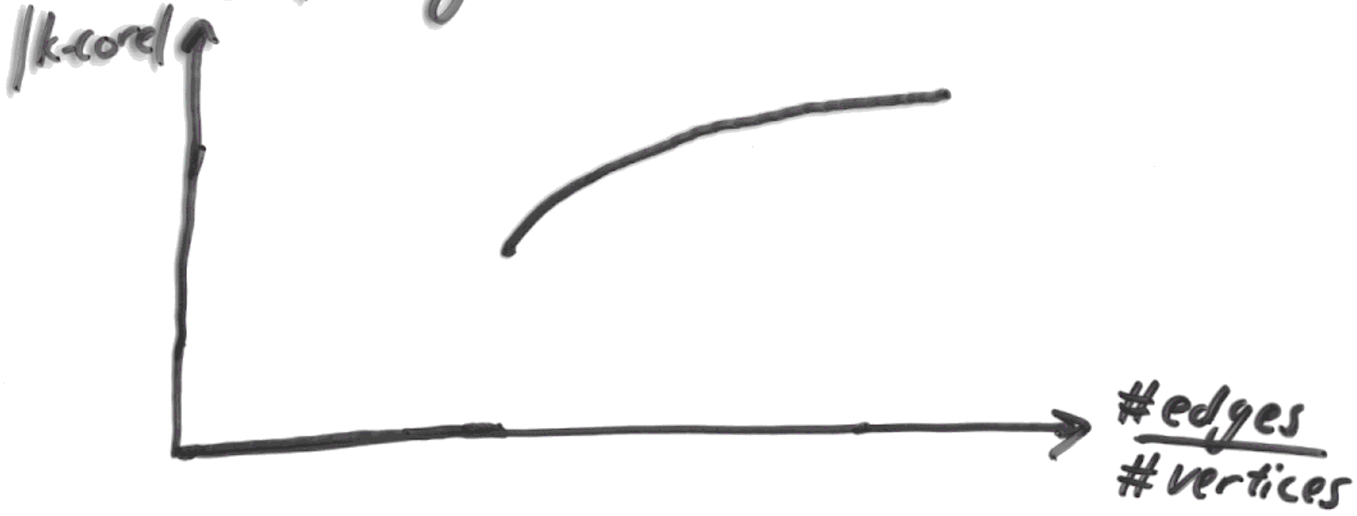
Width of the Scaling Window for 3-SAT

$\nu \geqslant 2$ (even if ν is not well-defined)



$$m_2 - m_1 \geqslant 0.00149 \times (p_1 - p_2 - o(1)) \times \sqrt{n}$$

Emergence of k-core



Add edges one at a time.
Size of k-core suddenly jumps from 0 to constant fraction.
Pittel, Spencer, Wormald '96

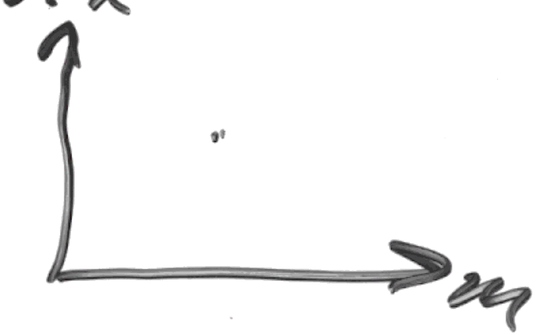
But $\sqrt{}$ fluctuations in location of jump
(same proof as for k-SAT)

Hypothesise exponent ν'; $\nu = \max(\nu', 2)$?

Why can't $\alpha_3(n)$ be 4.3 or 4.1 depending on parity of n?

Alice: Fix m,
pick random N so that $\Pr[N=n] = \frac{6}{\pi^2}\frac{1}{n^2}$.
Pick random formula ϕ from $F_{N,m}$.
Constant fraction of variables unused,
so compactify variable numbers, to get $\tilde{\phi}$.
Give $\tilde{\phi}$ to Bob.

Bob: Count clauses to get m.
Count number of used variables, infer N.
Determine if $\tilde{\phi}$ is SAT.
Collect statistics on $\Pr[\mathrm{SAT}]$.



$\sqrt{}$ fluctuations in inferred value of $N \Rightarrow$

$\Pr[F_{n,m}$ is SAT$]$ can't change too rapidly as function of n

Given $\tilde{\phi}$ with v variables used,

$\binom{n}{v}$ possible formulas on n variables that could have given $\tilde{\phi}$, all equally likely.

$$\Pr[N=n \mid \phi] = \frac{6}{\pi^2}\frac{1}{n^2}\binom{n}{v}\frac{1}{\left(2^k\binom{n}{k}\right)^m}$$

Inference of N only depends on v, m, k;
the satisfiability of $\tilde{\phi}$ does not affect inference of N.