# Mick Gets What He Needs

*— or —*

# Why $2^k$?

Dimitris Achlioptas, Microsoft Research

Cristopher Moore, University of New Mexico
and the Santa Fe Institute

# The $Y2^k$ Problem

- [Franco & Paull] The First Moment Method gives $r_k < 2^k \ln 2$
- [Chao & Franco, Frieze & Suen] Unit Clause Resolution (UC) gives $r_k > 2^k/k$
- [Monasson & Zecchina] The Replica Method predicts that for large $k$ the *annealed approximation* is tight, and so

$$r_k \approx 2^k \ln 2$$

How can we close the gap between $\Theta(2^k/k)$ and $\Theta(2^k)$?

## We Close This Asymptotic Gap

**Theorem.** The $k$-SAT threshold $r_k$ is bounded below by

$$r_k \geq 2^{k-1} \ln 2 - c$$

for some constant $c > 0$, for all $k$.

**Theorem.** For NAE-$k$-SAT and $k$-uniform Hypergraph 2-Coloring,

$$r_k = 2^{k-1} \ln 2 - c$$

for some constant $c > 0$, for all $k$.

(In fact $c < 1$.)

# The Second Moment Method

Non-Constructive, unlike algorithms

- $\Pr[X > 0] \leq \mathbf{E}[X]$
- $\Pr[X > 0] \geq \mathbf{E}[X]^2/\mathbf{E}[X^2]$

If the mean is large but the variance is small, we're in business . . .

But this doesn't work for 3-SAT!

## How To Use The Second Moment

Let $s$ be an assignment. A random $k$-SAT clause is satisfied with probability

$$p_1 = 1 - 2^{-k}$$

$\mathbf{E}[X^2]$ is the expected number of pairs of satisfying assignments.

Let $s$ and $t$ be two assignments with overlap $q = \alpha \cdot n$. A random $k$-SAT clause is satisfied by *both* of them with probability

$$p_2(\alpha) = 1 - \alpha^k \, 2^{-k} - (1 - \alpha^k) \, 2^{1-k}$$

# How To Use The Second Moment, cont'd

Since the $m = rn$ clauses are independent,

$$\mathbf{E}[X] = 2^n \, p_1^{rn} = (2p_1^r)^n$$

Since there are $2^n \binom{n}{q}$ pairs of assignments with overlap $q = \alpha n$,

$$\mathbf{E}[X^2] = 2^n \sum_{q=0}^{n} \binom{n}{q} \, p_2(q/n)^{rn}$$

so

$$\Pr[X > 0] \geq \frac{\mathbf{E}[X]^2}{\mathbf{E}[X^2]} = 1 \Big/ \sum_{q=0}^{n} \binom{n}{q} \left( \frac{p_2(q/n)^r}{2p_1^{2r}} \right)^n$$

# A Little Asymptotic Combinatorics

So we need an upper bound on sums of the form

$$S = \sum_{q=0}^{n} \binom{n}{q} f(q/n)^n$$

**Lemma.** Let

$$g(\alpha) = \frac{f(\alpha)}{\alpha^\alpha (1-\alpha)^{1-\alpha}}$$

Suppose $g$ has a unique $g_{\max}$ where $g'' < 0$. Then for some $C$,

$$S \leq C \times g_{\max}^n$$

*Proof.* Stirling $\binom{n}{\alpha n} \approx \frac{1}{\sqrt{n}} \left( \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \right)^n$ (entropy) $+$ Laplace's method

## Sitting On A Fence

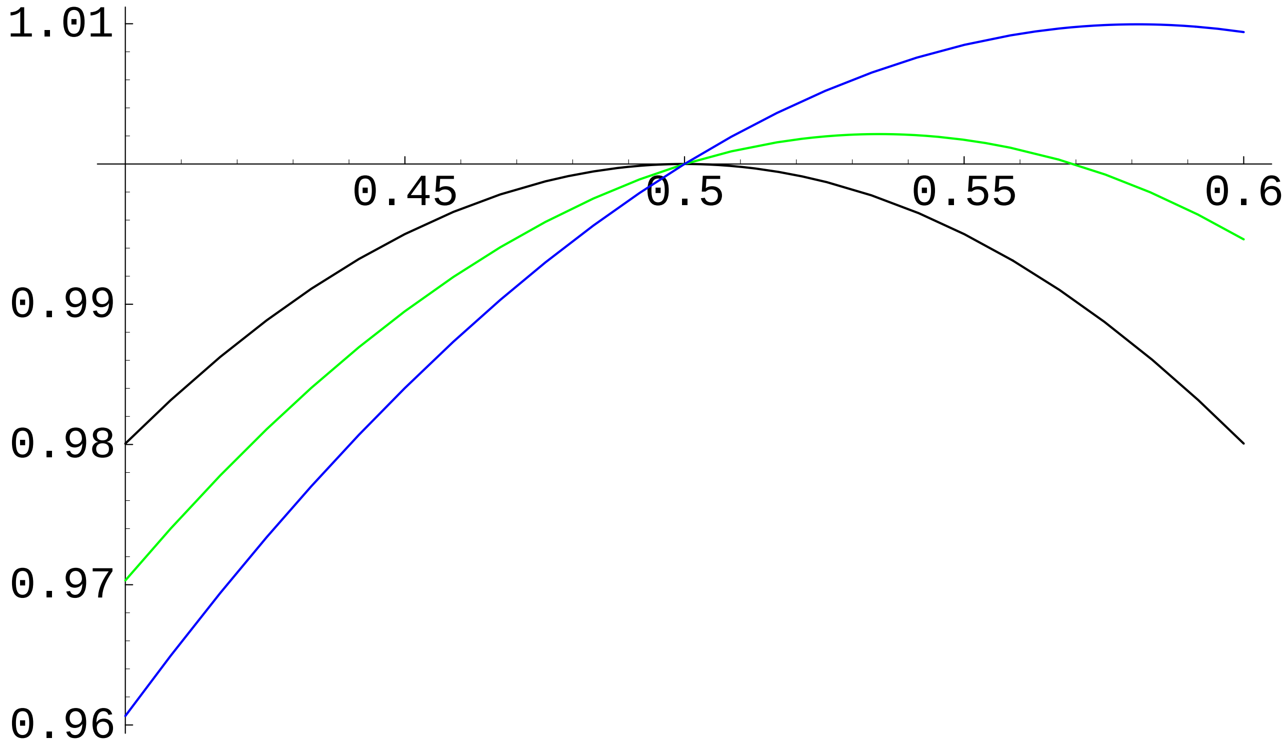
So we need to maximize

$$g(\alpha) = \frac{1}{2\,\alpha^{\alpha}\,(1-\alpha)^{1-\alpha}} \left(\frac{p_2(\alpha)}{p_1^2}\right)^r$$

Note: $g(1/2) = 1$ since $p_2(1/2) = p_1^2$ (since if $\alpha = 1/2$, $s$ and $t$ are effectively independent)

Is this the global maximum?

# I Can't Get No . . .




For 3-SAT, any $r > 0$ moves $g_{\max}$ away from $\alpha = 1/2$, and $g_{\max} > 1$.

So the Second Moment Method fails to give a nontrivial bound.

# Symmetrize!

Let's consider only those assignments $s$ such that both $s$ and $\overline{s}$ are satisfying.

In other words, let's think about NAE $k$-SAT instead of $k$-SAT.

Then

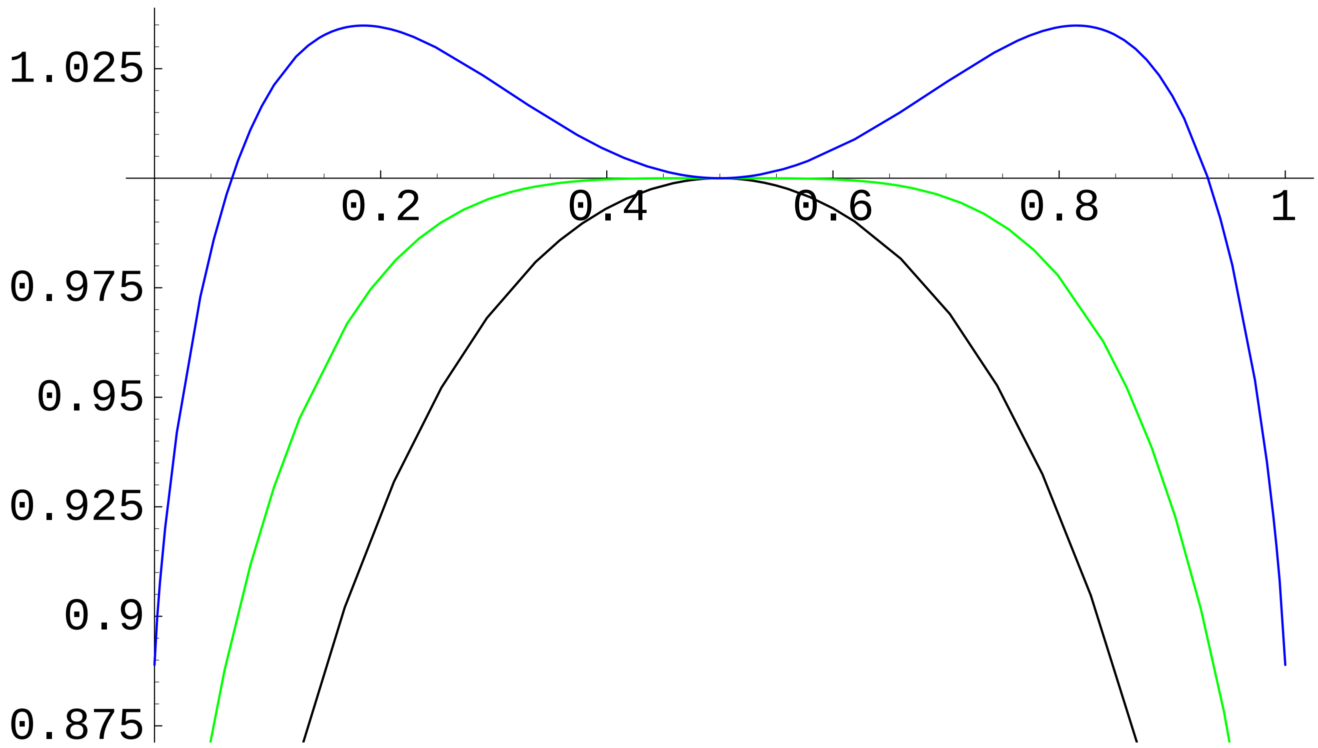
$$p_1 = 1 - 2^{1-k}$$

and

$$p_2(\alpha) = 1 - \left(\alpha^k + (1-\alpha)^k\right) 2^{1-k} - \left(1 - \alpha^k - (1-\alpha)^k\right) 2^{2-k}$$

Now $g(\alpha)$ is symmetric around $\alpha = 1/2$!

## Oh Baby (We've Got A Good Thing Goin')



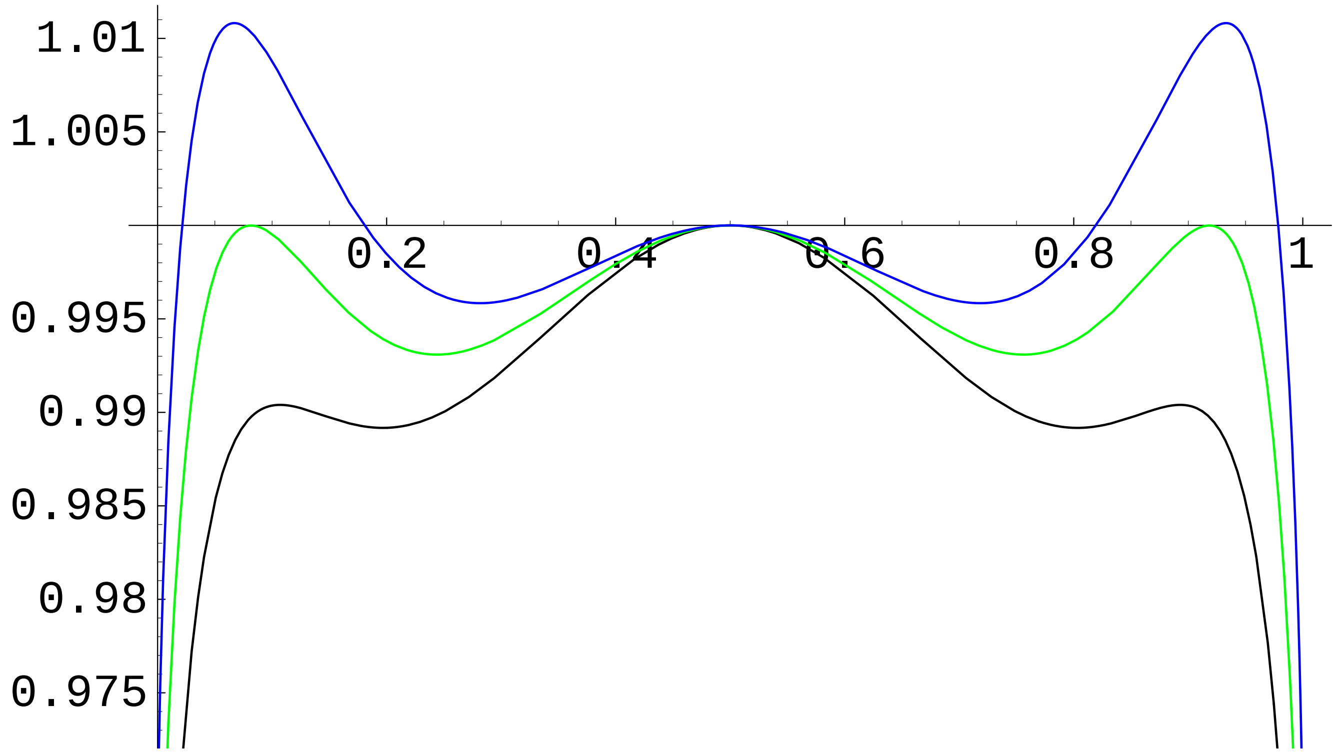

For NAE 3-SAT, $g_{\max} = 1$ for $r < 3/2$. So $r_k \geq 3/2$.

This is better than UC which gives $4/3$,

and almost as good as the best-known algorithm which gives $1.514$.

Of course, a lower bound for NAE $k$-SAT is also a lower bound for $k$-SAT.

# What Happens For Larger $k$?




For $k \geq 5$, two new maxima appear near $\alpha = 0, 1$

When do the new maxima exceed $1$?

## Maximizing $g$

**Lemma.** For every $\epsilon > 0$, there exists $k_0 = k_0(\epsilon)$ such that for all $k \geq k_0$, if

$$r \leq 2^{k-1} \ln 2 - \frac{\ln 2}{2} - \frac{1}{2} - \epsilon$$

then $g_{\max} = g(1/2) = 1$ is the unique global maximum and $g''(1/2) < 0$.

*Proof.* For $k > 11$ and $r < 2^k$, we have:

Step 1: $g''(1/2) < 0$

Step 2: $g(\alpha) \leq 1$ if $|\alpha - 1/2| \leq 1/10$ (Taylor series)

Step 3: We have to locate and bound the maxima on the sides . . .

## Finding the Side Maxima

Turn things around: for a given $\alpha$, define $r^*(\alpha)$ as the $r$ that makes $g(\alpha) = 1$.

Solving $g(\alpha) = 1$ for $r$ and applying some inequalities gives

$$
\begin{aligned}
r^*(\alpha) &= \frac{\ln 2 - h(\alpha)}{\ln p_2(\alpha) - 2\,\ln p_1} \\
&> \left(\frac{2^k}{\alpha^k + (1-\alpha)^k} - 1\right) \frac{\ln 2 - h(\alpha)}{2} - 2 \times \left(\frac{25}{32}\right)^k
\end{aligned}
$$

where $h(\alpha) = -\alpha \ln \alpha - (1-\alpha)\ln(1-\alpha)$ is the entropy function.

Now we minimize $r^*(\alpha) \ldots$

## Finding the Side Maxima cont'd

Differentiating $r^*$ with respect to $\alpha$ and setting it to zero gives (roughly)

$$-\ln\alpha = k\left(\ln 2 - h(\alpha)\right) - \ln(1-\alpha)$$

Bootstrapping:

1. Since $\alpha$ is bounded away from $1/2$, $h(\alpha)$ is bounded below $\ln 2$. So $-\ln\alpha = \Theta(k)$ and $\alpha$ is exponentially small.

2. But $h(\alpha) \approx -\alpha\ln\alpha$, so $h$ is exponentially small too . . .

3. So $-\ln\alpha \approx k\ln 2$ and $\alpha \approx 2^{-k}$.

Plugging $\alpha = 2^{-k}$ into $r^*$ completes the proof.

# For NAE $k$-SAT, This Bound Is Tight

For NAE $k$-SAT, the First Moment method gives

$$r_k < 2^{k-1} \ln 2 - \frac{\ln 2}{2}$$

Local maximality [KKKS] tightens this by $-1/4$

| $k$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ |
|---|---|---|---|---|---|---|---|---|
| Our lower bound | $3/2$ | $49/12$ | $9.973$ | $21.190$ | $43.432$ | $87.827$ | $176.570$ | $354.027$ |
| Upper bound | $2.214$ | $4.969$ | $10.505$ | $21.590$ | $43.768$ | $88.128$ | $176.850$ | $354.295$ |

The gap between our lower and upper bounds quickly approaches $1/4$.

## Paint It Black (And White)

When is a random $k$-uniform hypergraph 2-colorable? Erdös' Property B

Let $s$ and $t$ be two assignments, with

- $\alpha \cdot n$ vertices colored black in $s$
- $\beta \cdot n$ colored black in $t$
- $\gamma \cdot n$ colored black in both

The probability a random hyperedge is bichromatic in both $s$ and $t$ is $p_2(\alpha, \beta, \gamma)$, so we have to maximize a three-dimensional function $g(\alpha, \beta, \gamma)$

# Reducing Three Dimensions To One

Like NAESAT without negation . . .

Using convexity arguments we show that $\alpha = \beta = 1/2$ for any maximum of $g(\alpha, \beta, \gamma)$.

Moreover, $g(1/2, 1/2, \gamma) = g_{\mathrm{NAESAT}}(2\gamma)$ so $g_{\max}$ is at $(1/2, 1/2, 1/4)$.

We generalize our Lemma to any number of dimensions (now depends on the determinant of the matrix of second derivatives)

So we get the same lower bound as for NAE $k$-SAT!

Upper bound is the same too.

# Hey You, Get Off Of My Cloud

$g(\alpha) = \text{entropy} \times \text{correlation}$

If I give you a satisfying assignment, and offer you a biased coin . . .

- For $k$-SAT, you should take the offer
- For NAE $k$-SAT, a fair coin is better!

$k$-SAT assignments "attract" and have overlap $> 1/2$

NAE $k$-SAT assignments are independently scattered throughout the hypercube in a uniform cloud.

# The Overlap Distribution

The probability $P(\alpha)$ that two satisfying assignments have overlap $\alpha$ is proportional to $g(\alpha)^n$, in the annealed approximation

When this overlap distribution is tightly peaked (*not* in the annealed approximation) then Replica Symmetry is expected to hold

Under Replica Symmetry Breaking, $P(\alpha)$ develops another peak (one-step) or becomes continuous (Parisi)

Does the appearance of multiple peaks in $g(\alpha)$
have anything to do with Replica Symmetry Breaking?

## Conclusions and Open Questions

- Symmetrizing $k$-SAT lets the Second Moment Method give a lower bound
- Will also work for other Constraint Satisfaction problems with similar symmetry
- How can we close the factor of 2 for $k$-SAT?
- Why is the Second Moment so tight for NAESAT and Hypergraph 2-Coloring?
- What can higher moments tell us?

Connections with Replica Symmetry?