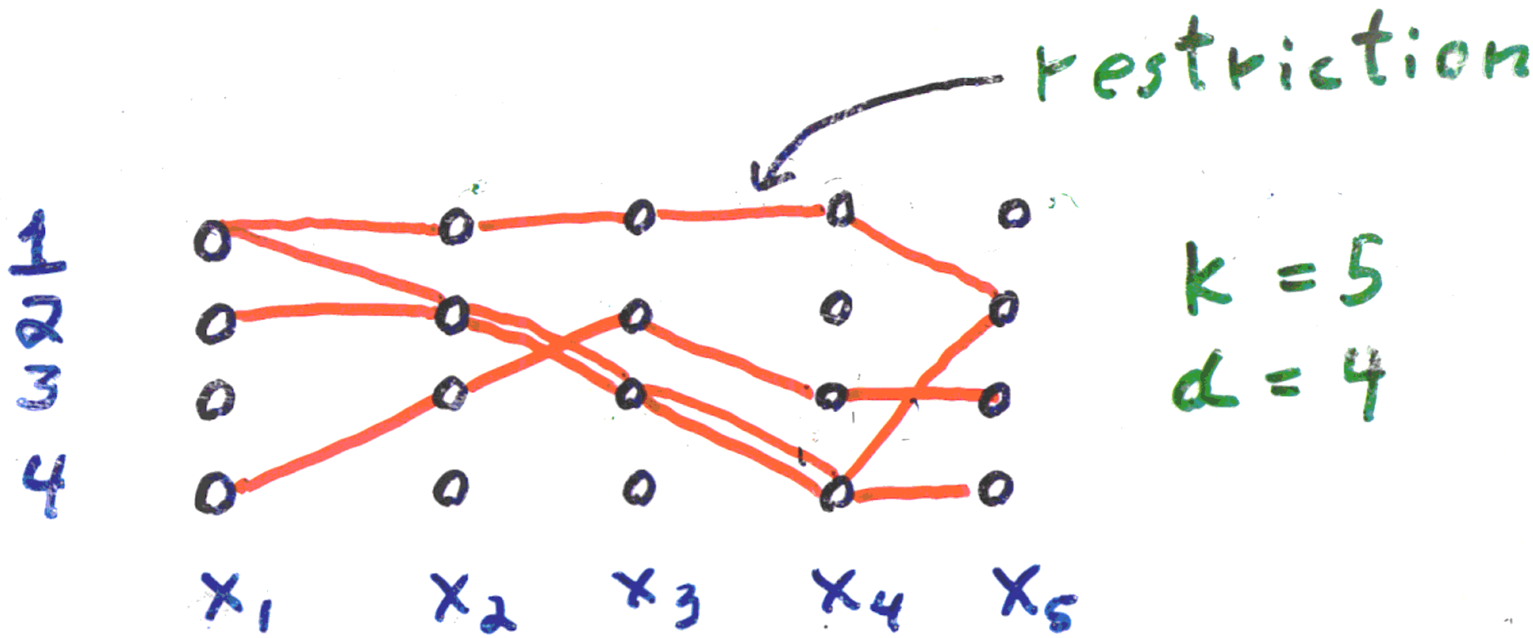
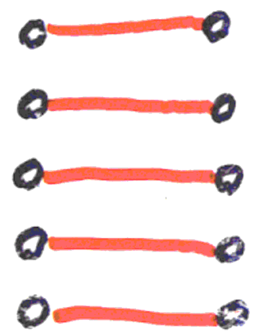
CONSTRAINT:



$k = 5$

$d = 4$

Eg.



$d = 2$

k-SAT

d-COL

Two Natural Models

- Choose cn random k-tuples of variables

For each k-tuple:

- randomly choose t of the d^k possible restrictions

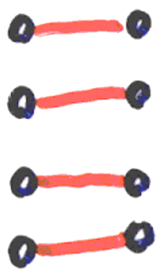
OR

- include each possible restriction with probability

$$p = t/d^k$$

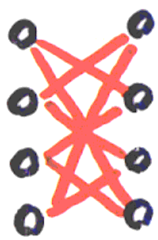
$k = 2 \qquad d = 4$

"colour"



$\Pr = \frac{1}{3}$

"same parity"



$\Pr = \frac{2}{3}$

NON-SHARP FULL THRESHOLD

$c_1 = \frac{3}{4}, \quad c_2 > \frac{3}{2}$

AKKKMS

For any $c > 0$

2^{nd} model is a.s.

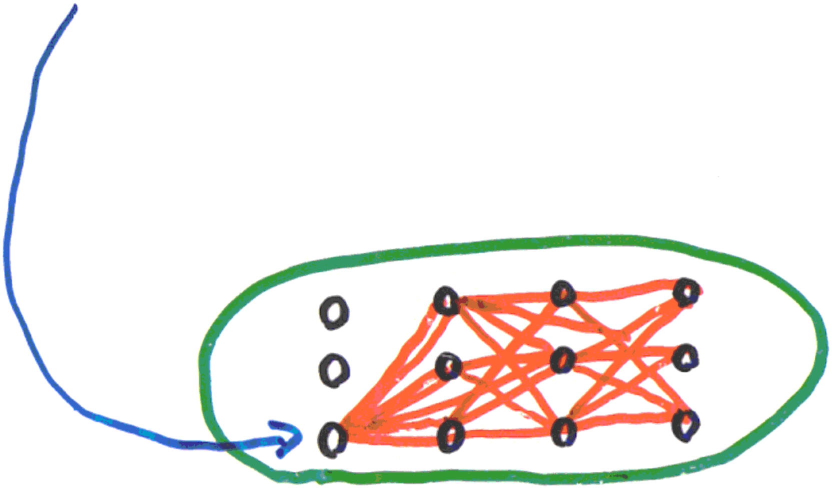
UNSATISFIABLE

1^{st} model is a.s.

UNSATISFIABLE

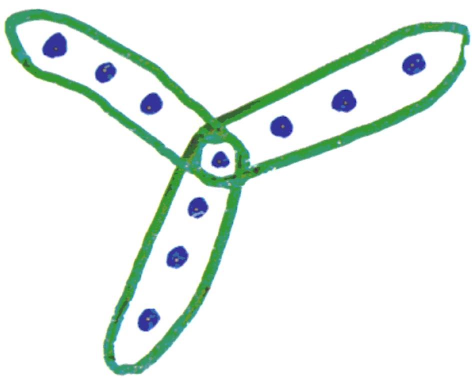
if $t \geq d^{k-1}$

Forbidden Value



$K = 4$

$d = 3$



A General Model

- Choose cn random k-tuples

For each k-tuple:

- Choose a random constraint from some fixed distribution

Generalizes

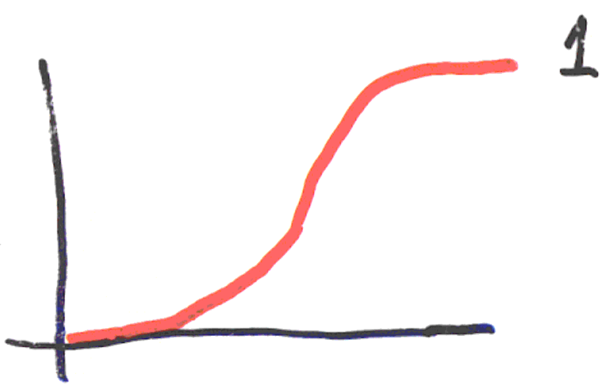
k-SAT
k-COL
Gent et. al
K-XOR
NAE-k-SAT
$(2+p)$-SAT
⋮

A model has a threshold if

$\exists\, c_1, c_2$ s.t.

$c < c_1 \Rightarrow$ a.s. SATISFIABLE

$c > c_2 \Rightarrow$ a.s. UNSATISFIABLE



Pr(UNSAT)

SYMMETRIC CASE

THEOREM

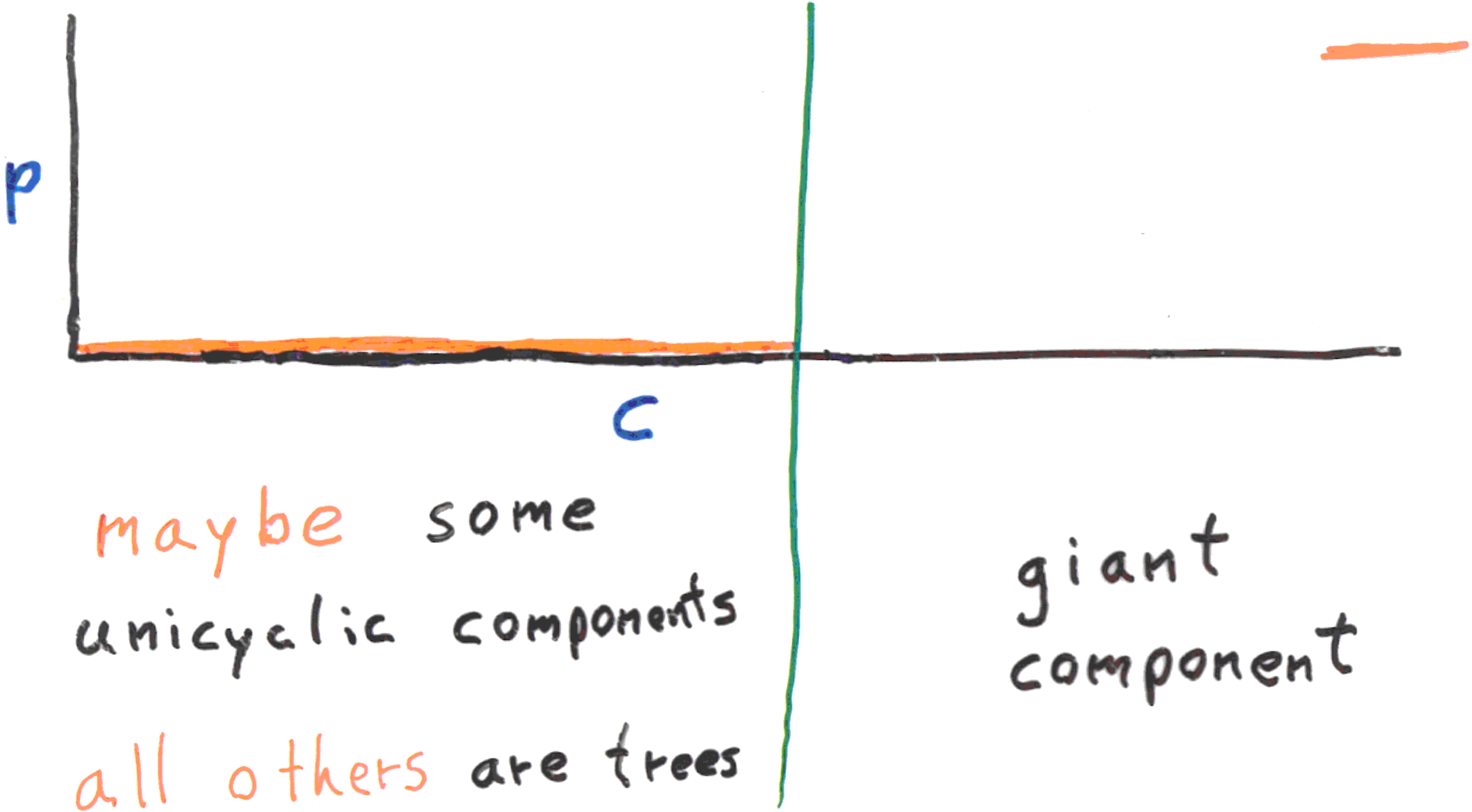
There is a threshold IFF the constraint distribution satisfies:

(i) $Pr\left(X_1 = X_2 = \ldots = X_k = 1 \text{ is acceptable} \right) < 1$

(ii) $Pr\left(X_i = 1 \text{ is forbidden} \right) = 0 \quad \forall i$

(iii) for any cycle C
$Pr\left(C \text{ is satisfiable} \right) = 1$

(i), (ii) $\Longrightarrow$ half-threshold


p
c
maybe some
unicyclic components
all others are trees
giant
component

$k = 2$

$\mathcal{C} = \{ c : \Pr(c) > 0 \}$

$G(\mathcal{C})$:

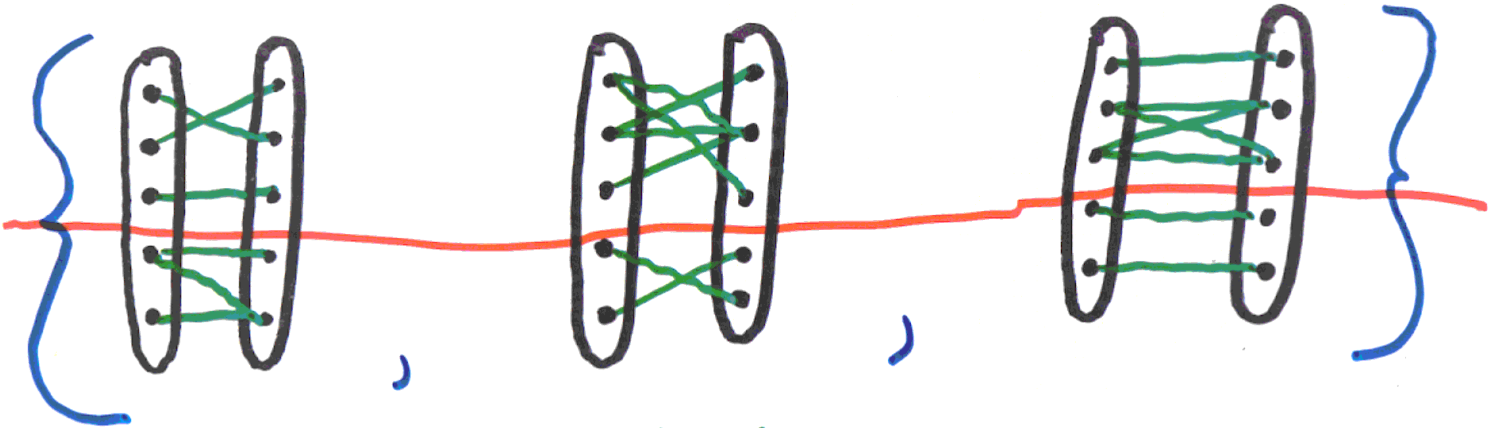
vertices $\{1, \ldots, d\}$

edges $\{ xy : xy$ is permitted by at least one $c \in \mathcal{C} \}$

"$\mathcal{C}$ is connected"

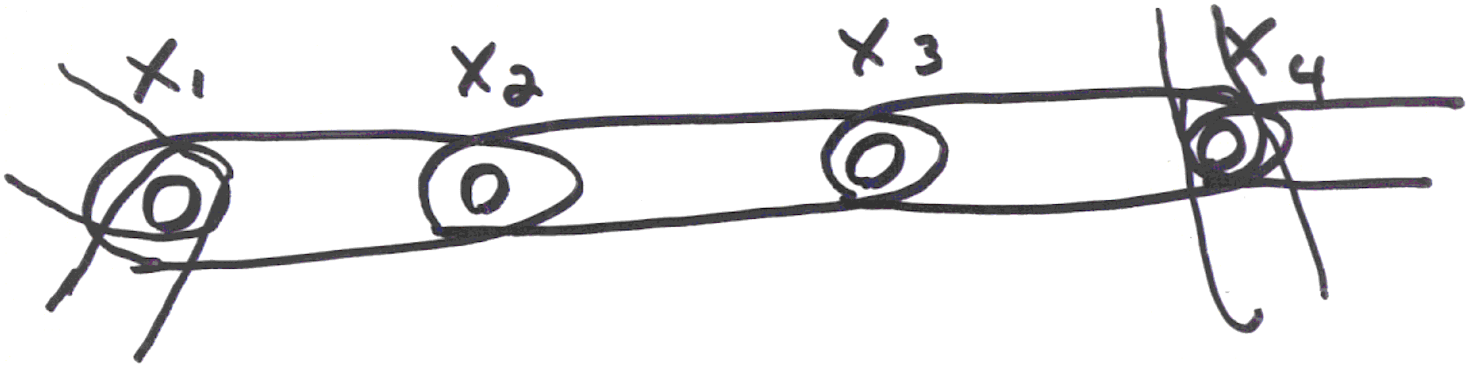
means

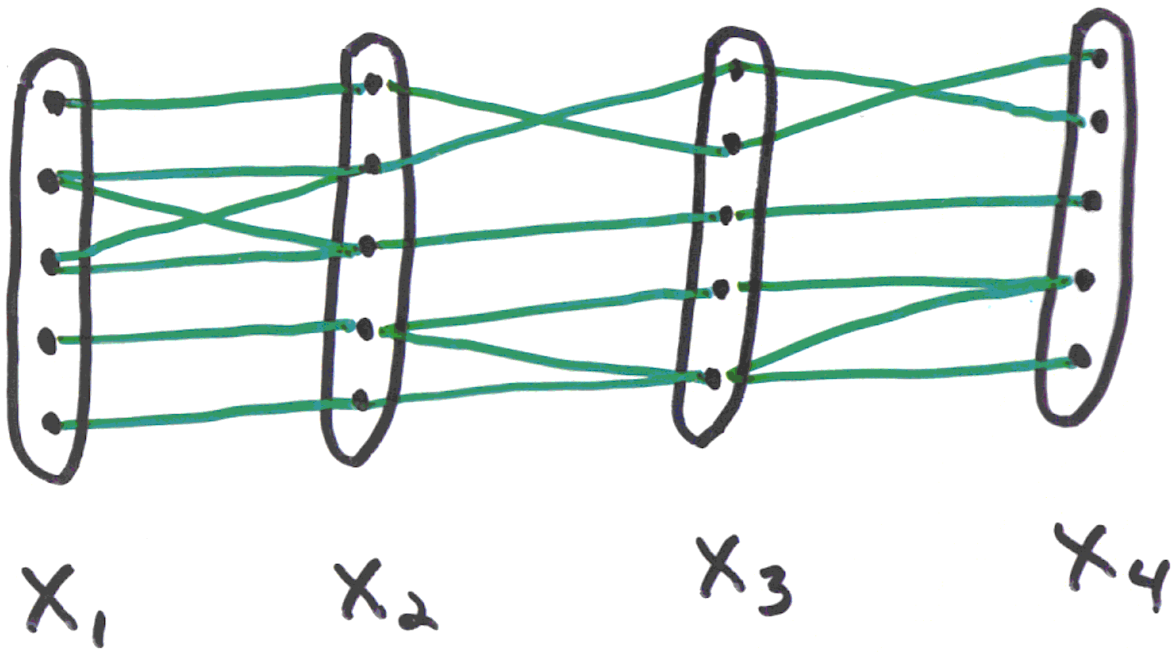
"$G(\mathcal{C})$ is connected"



permitted pairs

$\mathcal{C}$ is **NOT** connected



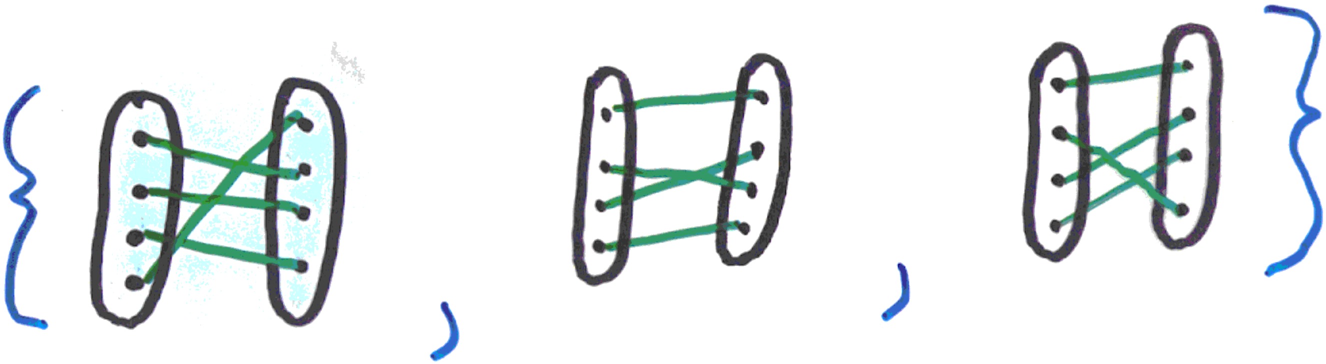


CONSTRAINT PATH

A NULL PATH is a constraint path which permits $X_1 = i, X_{\cancel{2}4} = j$ for every $i, j \in \{1, \ldots, d\}$

Note: If a CSP is minimally unsatisfiable then it has no null path

- If G only contains permutations then G can't form a null path



- If G is not connected then G can't form a null path

ASSUME: G is connected

no forbidden values

LEMMA 1

If G forms no UNSAT CYCLE

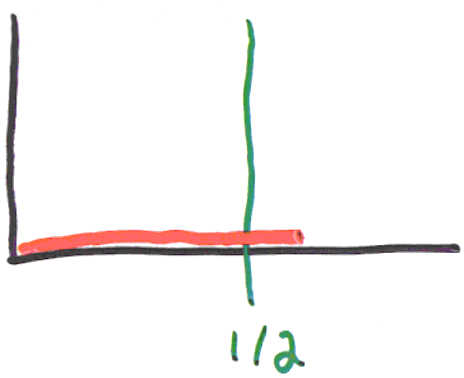
then G forms a null path

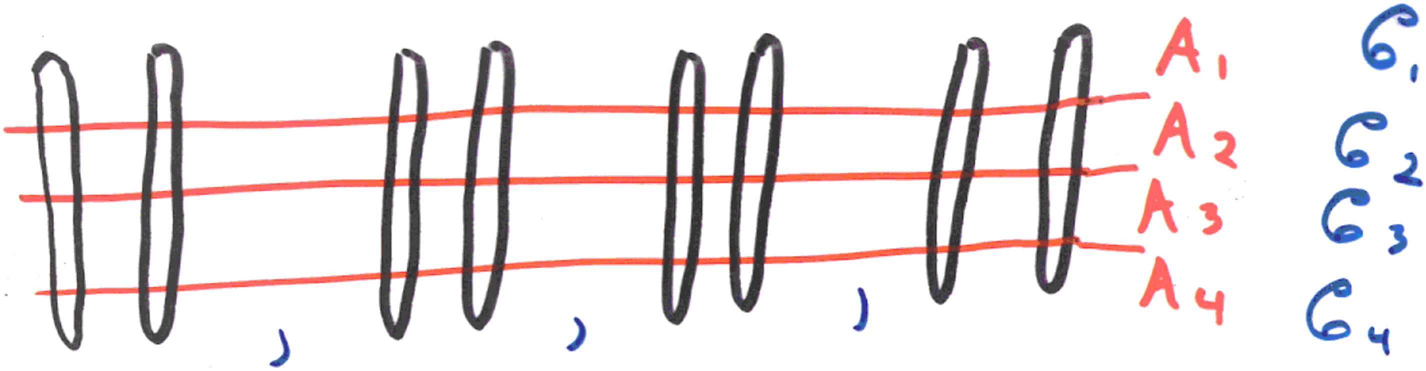
LEMMA 2

If G forms no UNSAT CYCLE

and G forms a null path

then

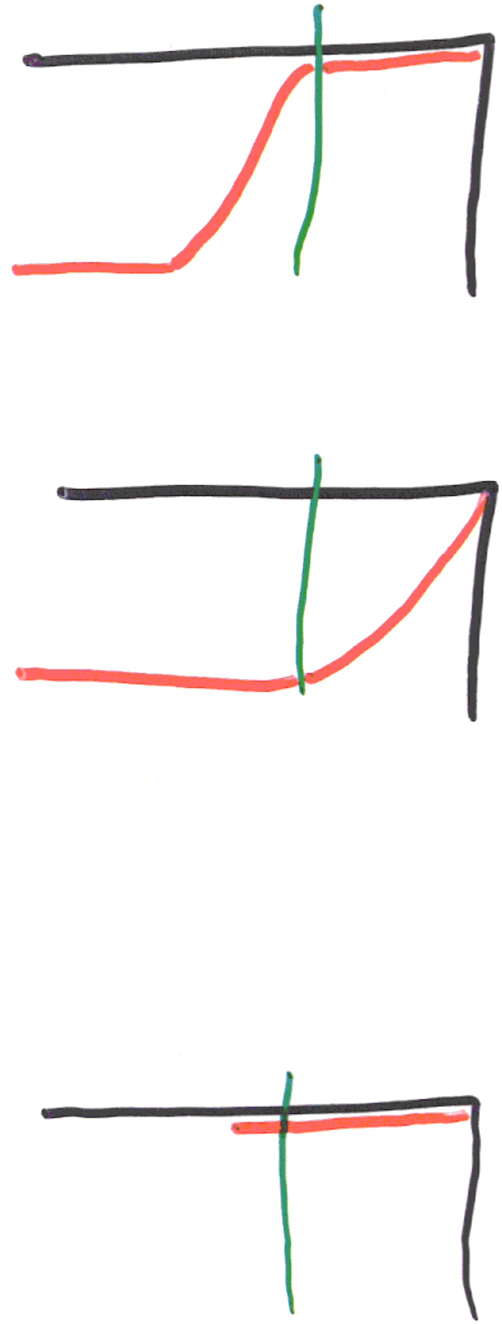




Case 1 Some G_i has no UNSAT CYCLE

Case 2 Some cycle is UNSAT for every G_i

Case 3



$G = \{P_1, P_2, \ldots, P_r\}$

$\mathcal{P} = \{P_1, P_1^{-1}, \ldots, P_r, P_r^{-1}\}^*$

Note: $\mathcal{P}$ is a transitive group

$\mathcal{P}_i = \{P \in \mathcal{P} : i \text{ is a fixed point}\}$

Lemma: $|\mathcal{P}_i| = |\mathcal{P}|/d$

Note: $I \in \mathcal{P}_i \quad \forall i$

$$\Longrightarrow \quad \mathcal{P} - \bigcup_{i=1}^{d} \mathcal{P}_i \neq \emptyset$$

i.e. some $P \in \mathcal{P}$ has no fixed point

THEOREM

(i) full threshold

(ii) uniform constraint distribution

(iii) for every possible constraint NO value lies in ≥ 2 restrictions

$\Longrightarrow$ SHARP THRESHOLD