# The Principle of Deferred Decisions

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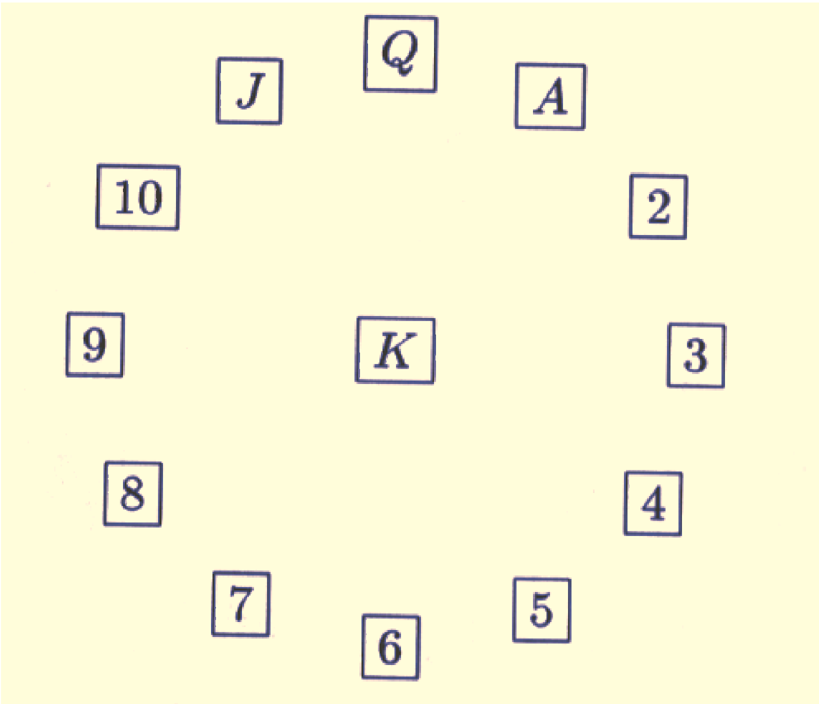
D.E. Knuth
Mariages stables et leurs relations avec d'autres problèmes combinatoires.
Presses de l'Université de Montréal, 1976.

- A uniformly random structure $\mathcal{S}$ subject to certain constraints (*conditionals*).
- Uniformly random: all such structures are equiprobable.
- Apply an algorithmic step to $\mathcal{S}$ that operates on certain elements of the structure.
- Is the part of the structure comprised of the untouched elements uniformly random subject to the same conditionals?

Basic idea:

- Assume that the elements of the structure are stored in registers whose contents are hidden (unexposed).
- To apply the algorithmic step, expose only the registers where this step operates.
- The remaining registers are still unexposed and therefore remain uniformly random.

# The clock solitaire




- The game ends when you arrive at an empty stack. Equivalently, when the fourth king is turned over.

- What is the probability that all cards are turned over? Equivalently, what is the probability that the last card to be turned over is a king?

- The sequence that the cards will be exposed is uniformly random.
- This is still true after the first card is exposed, conditional that the remaining cards belong to the unexposed part of the deck.
- Therefore the probability of success is:

$$4 \times \frac{51}{52} \times \frac{50}{51} \times \cdots \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{13}.$$

# A graph game

Let $G$ be a uniformly random graph, conditional on the:

- The number of vertices, $n$,
- the number of edges, $m$,
- the number of vertices of degree 0, $X_0$ (isolated vertices),
- the number of vertices of degree 1, $X_1$.

Claim: If we arbitrarily delete a vertex of degree 1 and its incident edge, the remaining graph is still random conditional on the new values of the parameters $n, m, X_0, X_1$.

Store the information about the graph on unexposed:

- $n$ vertex-registers,
- $m$ edge-registers.

Additionally, to each vertex-register assign an *exposed* degree-register with information about the degree of the vertex: 0,1 or $h$.

# Another graph game

Let $G$ be a graph whose edges are distinguished into black and white (a *B&W-graph*).

A vertex is called *all-white* if all its incident edges are white.

The *w-degree* of a vertex $v$ is the number of all-white vertices connected with $v$.

Assume that the B&W graph $G$ is uniformly random conditional on the:

- The number of vertices, $n$,
- the number of edges, $m$,
- the number of vertices of w-degree 0, $X_0$,
- the number of vertices of w-degree 1, $X_1$,

Delete an arbitrary vertex of w-degree 1 and all edges (black and white) incident on it.

Is the remaining graph random conditional on the new values of the parameters $n, m, X_0, X_1$?

Answer: **No.**

Because if the w-degree of a vertex $u$ changes from 0 to 1 or to $h$, it is revealed that $u$ is connected to a vertex that has become all-white as a result of the deletion of edges.

What if we assume that more information about the graph is initially exposed.

E.g.:

- Full information on the ordinary degree of each vertex and/or
- full information on the w-degree of each vertex and/or
- information about the color of each edge and/or
- information about a vertex being all-white or not?

Still the answer is No.

Yet intuitively we know that certain characteristics of the new graph retain their uniform randomness.

**Basic question:** How can isolate the characteristics of a B&W-graph that retain their randomness in this game of deletions?

Given a structure $\mathcal{S}$ and an algorithm on $\mathcal{S}$ we want to find what conditionals should be imposed on $\mathcal{S}$, so that the structure retains its randomness under the algorithm, given the updated conditionals.

Rough outline of the method to isolate the characteristics whose randomness is retained:

- Store the information about the structure $\mathcal{S}$ into registers. Some are exposed, some are unexposed.

- The unexposed correspond to non-random characteristics. Characteristics that are to be included in the initial conditional.

- Apply the algorithmic step and update all registers.

- If the contents of an unexposed register $r$ can be inferred from the changes in the exposed registers, expose $r$ as well.

- If at the end of the algorithmic step the same type of registers as in the beginning remain exposed, then the algorithmic step retains randomness conditional on the exposed type of information.

- It pays to store information into "small pieces."

# Formulas

Consider a CNF formula comprised only of 3- and 2-clauses. Apply a Davis-Putman-Logemann-Loveland (DPLL) algorithmic step on it. In other words:

- Select a literal and set it to TRUE. Set its negation to FALSE.
- Reduce the formula, i.e. delete the clauses that contain the TRUE literal and delete appearances of the FALSE literal (thus, certain clauses are either deleted or shrunk).
- Repeatedly set to TRUE literals in unit clauses, set their negation to FALSE and reduce the formula, until no unit clauses remain. If an empty clause appears, stop.

If no empty clause appears, the new formula contains again only 3- and 2-clauses.

*Degree* of a literal: the number of its occurrences in the formula. *Co-degree:* the degree of its negation.

Assume now that we select to make TRUE a literal whose co-degree is 1.

Assume also that the unique occurrence of the negation of this literal appears in 2-clauses (or 3-clauses).

What kind of randomness is retained under such an operation?

# Storing information in registers

- One register for each literal and one register for each clause in the formula. Unexposed.

- Attach to each literal an unexposed negation-register pointing to its negation.

- Attach to each literal an exposed degree-register with information on its degree (0, 1 in 3-clauses, 1 in 2-clauses, or none of these)

- Attach to each clause an exposed register with information on its type (3-clause or 2-clause).

Unfortunately randomness is not retained. Because a change in the type of a clause together with changes in the degrees of literals, reveals information supposed to be unexposed.

One extra ingredient: The number of literals of degree 1 in 2-clauses that pair with same type of literals is assumed to be known.

**Theorem.** *A formula retains its randomness if an algorithmic step as above is applied, given that the following information is contained in the conditional:*

- *The number of 3-clauses and the number of 2-clauses.*
- *The number of literals of degree 0, of degree 1 in 3-clauses and of degree 1 in 2-clauses.*
- *The number of 2-clauses whose both literals have degree 1.*

What if we want to select a literal of degree $i$ in 3-clauses and degree $j$ in 2-clauses?

**Theorem.** *A formula retains its randomness when we set a literal of degree $i$ in 3-clauses and degree $j$ in 3-clauses given that we know:*

- *The number of 3-clauses and the number of 2-clauses.*
- *The number of literals of degree $i$ in 3-clauses and degree $j$ in 2-clauses.*
- *The way literals are paired into 2-clauses.*

Important: The last conditional does not reveal what is the negation of each literal.

# B&W-graphs

$W$, the set of *w-degree witnesses*, is the set of edges incident on all-white vertices.

How to construct a uniformly random graph with a given number of vertices, a given number of edges and a given *set* of w-degree witnesses:

- Uniformly at random ut as many additional edges as required between vertices that are not all-white and then randomly color them so as at least one black edge is incident on a vertex that is not all-white.

**Theorem.** *Let $G$ be a random B&W-graph, conditional on the number of vertices, the number of edges and the set of w-degree witnesses. Arbitrarily delete a vertex of a specified w-degree. Then the graph obtained is random conditional on the same information.*

**Informal Conjecture.** *The conditionals in all theorems above is the weakest possible so that randomness is retained.*