Rigorous Results on Disordered Systems

Lecture notes[1]

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1. Introduction.

Disordered systems are among the most fascinating, and most difficult, problems in statistical physics. They arise immediately whenever the hypothesis of translation invariance is put in question, as happens almost always already in solid state physics (where perfect crystals are expensive and impurities are the rule). More importantly, dropping this hypothesis allows physics to expand well beyond its own field into areas of application where a complex and messy underlying structure of interaction is fundamental, and nobody would dream to use a homogeneous perfect system even as a first approximation.

The reason why physics can still have a say in such situations is by the ingenious idea to bring order back into the chaos by stepping one level up: moving from *disordered systems* to *random systems* allows us to speak instead of some messy hard to describe object of the particular realization of a random object governed by nice and regular probability distribution. This construct does not only offer conceptual advantages, but also makes quantitative computations – at least in principle – feasible.

The statistical mechanics of *random systems* has thus become a huge field with many existing and even more potential applications. Theoretical research by physicist in this field has been and is dominated by what is known as the replica method [MPV]: basically, an ingenious trick that allows to some extend to transform the original random system back into a deterministic system, albeit with same strange properties, which than can be subjected to the usual tools of statistical mechanics such as expansions, renormalization group methods, etc. The power of this method is absolutely stunning, as it allows to solve problems that appear otherwise totally hopeless. This applies in particular to problems of combinatorial optimization that have been heavily tried in vain with conventional methods. The main disadvantage of the method is that from a mathematical point of view, it appears to make no sense whatsoever. In fact, for a mathematician, it consist of a sequence of prescriptions to perform formal manipulations for which there is no justification. However, the results of the replica method, if carefully performed, seem to be, for the most part, perfectly correct. This is, for the mathematician, a tremendous challenge. When does the replica method work, and why does it work?

The other main tool of theoretical physics in the field is numerical simulation, i.e. Monte Carlo methods. However, by and large they do not work terribly well in disordered systems, due to very slow equilibration and strong finite size effects. In a way, the most interesting aspect of Monte Carlo methods in the area of random systems is the analysis of these methods, i.e. Markov chains that are reversible with respect to the Gibbs measures, themselves. After all, they are reasonable models for the dynamical behaviour of such systems, which in many ways are of even greater interest than their equilibrium properties.

Mathematical physics has caught an interest in random systems early on, but compared to the theoretical physics rapid progress, has been slow treading, although not without some remarkable success stories. The most fundamental problems concern the description of the phase structure, i.e. the classification of the Gibbs measures and their description. One can roughly classify the typical problems according to the criteria

- mean field models vs. lattice models
- high temperature vs. low temperature
- weak disorder vs. strong disorder

Let me give a brief survey of the state of the art according to this classification. More details can be found in [N,Fr,B].

1.1. Lattice models.

1.1.1. High temperature.

The high temperature phase of lattice models is probably the best understood part of the entire field. There are reasonable criteria, based on high-temperature expansions or Dobrushin uniqueness arguments augmented with percolation results, when to expect uniqueness of the Gibbs state. This is to a large extend to be expected: high temperature means that the interaction is rather irrelevant, be it random or not. In fact all existing difficulties stem from the fact that in random systems one often cannot require uniform boundedness of potentials. Work has then to be put into showing that, under reasonable assumptions on the distributions of the random parameters, this still does not destroy phase uniqueness[Be,Gi,KM]. A more interesting issue arises if one also drops the assumption of absolute summability of the interaction. This is naturally the case in some models of *spin glasses*. Uniqueness may then only hold in a weak sense [FZ1,FZ2] and even in one dimensions very strange phenomena may take place [GNS].

1.1.2. Low temperature.

1.1.2.1. Weak disorder.

Naturally, the first question one would ask is whether small random perturbation alter the phase structure of the model at low temperature. It turns out that the nature of the random perturbation is very important. In particular it is important to know whether the random perturbation breaks the internal symmetries of the model or not. The two key models that differ in that respect are the *dilute Ising model* and the *random field Ising model*. The dilute Ising model has the Hamiltonian

$$H_\Lambda(\sigma) = -\sum_{\|i-j\|=1, i\vee j\in\Lambda} \sigma_i\sigma_j J_{ij} \tag{1.1}$$

where J_{ij} are independent random variables with positive mean and small variance. The $\pm$ symmetry of the model is unbroken, and it is a rather simple matter to show by an extension of the Peierls argument that the model has two phases at small enough temperatures, if the variance of the J_{ij} is small enough [ARS,Ge2]. In particular cases, this can be made very precise even using results from percolation theory [ACCN]. The point here is that the disorder acts on the energy of barriers between regions where the system in the different ground states, but the equality of the ground state energy is not affected. Thus small enough perturbations still leave the energetic suppression of "domain walls" intact and long-range order is not affected. Note however, that these statements hold only for the translation invariant Gibbs states. In the ferromagnetic Ising model there exist also non-translation invariant "Dobrushin states" in dimension $d \geq 2$. It is strongly believed that these will not exist in the dilute model in $d = 2$, no matter how small the variance of J_{ij} is, while at least some of them will be stable against small perturbations in $d \geq 3$. This believe is based on corresponding rigorous results in simplified (SOS) models of interfaces [BKu1,BKu2].

The random field Ising model has the Hamiltonian

$$H_\Lambda(\sigma) = -\sum_{\|i-j\|=1, i\vee j\in\Lambda} \sigma_i\sigma_j - \sum_{i\in\Lambda} \sigma_i h_i \tag{1.2}$$

where now h_i are i.i.d. random variable with symmetric distribution and small variance. The point here is that the spin flip symmetry is broken in a typical realization of the disorder variables (even though statistically the symmetry is preserved). This leads to a breakdown of the Peierls argument and which left the issue of the existence of a phase transition at low temperatures open to debate for a long time. In fact, in the 80's, there was a heated debate in the theoretical physics community on whether the lowest dimension for which several phases could exist in this model was 3 or 4. The indecision was due to the fact that two convincing heuristic arguments (the "Imry-Ma argument" [IM] and the theory of "dimensional reduction" [We]) gave contradicting answers. The issue was resolved in 1988 by a remarkable *rigorous* proof of the existence of two phases in $d = 3$ by J. Bricmont and

A. Kupiainen [BrKup], which followed an earlier result of J. Imbrie [I] who proved in the same dimension the existence of two *ground-states*. Shortly afterwards Aizenman and Wehr [AW1,AW2] proved the uniqueness of the Gibbs state in dimension 2. Unfortunately, there has been only limited progress on these general stability questions since then (except for the extension of the method to disordered SOS models [BKu1,BKu2] and continuous spin models [Ku1,Ku2]. All these papers concern only the case where symmetries are statistically unbroken. In ordered models, there is the well developed Pirogov-Sinai theory [PiSi] that allows the analysis of phase diagrams also in the absence of symmetries. Zahradník has started a program of developing a version of this theory for randomly perturbed models, but the announcement in some conference proceedings [Za] have not been followed by full proofs. This is in my view an absolute desideratum, and on the basis of the present technology perfectly doable.

1.1.2.2. Strong disorder.

By strong disorder I will understand situations where the random interactions have strong competing components, and, in particular, there are no "obvious" deterministic or random candidates for "ground states" that can provide a basis for a perturbative treatment. The main task in such a situation becomes than to understand and describe the emergence of a genuinely random structure in the low temperature phases. The prototypical model is the Edwards-Anderson spin glass, those Hamiltonian looks like (1.1), but where the random couplings J_{ij} are now i.i.d. random variables with mean equal (or very close to) zero and variance of the order of 1. (In the original model of Edwards and Anderson, they are taken to be uniformly distributed on the interval $[-J, J]$, but one might just as well take other distributions like symmetric Bernoulli or centered Gaussian). It is probably not doing too much injustice to anyone to say that we know nothing rigorously about the low temperature phases of this model in any dimension above one. This is to except a number of general properties of the Gibbs states that can be deduced from soft arguments of ergodic theory, which have been obtained for the greater part in recent work of Ch. Newman and D. Stein [N,N1,N2,N3,N4,N5] in an attempt to provide some hard checks into the raging dispute in the physics community on whether this model is or is not well described by mean-field theory or not. Apart from that, there is at present little reason to believe that progress can be made in the near future.

1.2. Mean field models.

1.2.1. Weak disorder.

The mean field analogues of the models discussed under 1.1.2.1., the *dilute Curie-Weiss model* and the *random field Curie-Weiss model* are reasonably simple mean field models that can be analysed with the standard methods of large deviation theory. They offer no particular challenges. However, they may still serve as illustrations for some of the effects that can occur in disordered systesm [Ku3,Ku4]. Another random generalization of the Curie-Weiss model it the family of models known as "Hopfield models"[Ho, AGS]. Their Hamiltonian is of the form

$$H_{N.M}(\sigma) \equiv -\frac{1}{N} \sum_{i,j=1}^{N} \left(\sum_{\mu=1}^{M} \xi_i^\mu \xi_j^\mu \right) \sigma_i \sigma_j \tag{1.3}$$

where ξ_i^μ are i.i.d. symmetric Bernoulli random variables. If $M = 1$, the model is equivalent to the Curie-Weiss ferromagnet by a gauge transformation. When M is finite, it can again be analysed with standard large deviation methods and offers no particular challenge, although it shows already $2M$ extremal Gibbs states. As M is allowed to grow with N the model becomes more and more interesting and challenging, while, as long as $M \leq \alpha N$ with $\alpha \ll 1$ it remains somewhat in a "weak disorder regime", even though it then has infinitely many extremal Gibbs states at low temperatures, and a "complete" analysis of the full phase structure is only possible if the temperature is not too low (as a function of α). This class of models has become a most interesting playground for developing methods to tackle disordered systems. There are a number of review papers dealing with these developments [BG1,B] and I will not deal with these models in these notes.

1.2.2. Strong disorder.

The most interesting situation in disorder arises when structure emerges out of complete randomness in a "self-organizing" way. This phenomenon appear in the most striking way in mean field models of spin-glasses. The prototypical model here is the classical Sherrington-Kirkpatrick (SK)[SK] model, with Hamiltonian

$$H_N(\sigma) \equiv -\frac{1}{\sqrt{N}} \sum_{1 \leq i < j \leq N} J_{ij} \sigma_i \sigma_j \tag{1.4}$$

where the J_{ij} are i.i.d. centered Gaussian random variables with variance 1. A variant of this model which is of some importance are the p-spin interaction SK models, where

$$H_{p,N}(\sigma) \equiv -\sqrt{\frac{p!}{N^{(p-1)}}} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \dots i_p} \sigma_{i_1} \dots \sigma_{i_p} \tag{1.5}$$

It is also of interest to consider these models with an external magnetic field term, $h\sum_i \sigma_i$, added.

All these models have been solved completely with the help of the replica method and Parisi's replica symmetry breaking scheme, which is being reviewed in Monasson's tutorial. Some of these results have recently been recovered via rigorous implementations of the so-called "cavity method", mainly through the work of M. Talagrand.

1.2.2.1. High temperatures.

In the standard SK model, there is a region in the β, h plane (bounded by the "Almeida-Thouless line" where what in the language of the replica method is called the "replica symmetric solution" is supposed to hold. it is now understood that this corresponds to the situation that the Gibbs state is "unique" in the sense that the Gibbs measure in finite volume converges to a unique *random* product measure on $\{-1,1\}^\infty$, as $N \uparrow \infty$. The probability distribution of this random measure depends on two parameters (the "mean", m, and the "overlap parameter", q) that satisfy a set of nonlinear equation that are exactly what comes out of the "replica symmetric solution" in the replica method. At present, it is possible to prove that this is indeed correct in a *subset* of the region where it this is expected to hold. Similar result are now known in a number of mean field models such as the Hopfield model and the perceptron (in this case, in a recent paper [ST], Shcherbina et al. prove such a result indeed for the entire region where it is expected to hold).

The method behind these proofs are always induction over the volume. This program goes in fact back to work of L. Pastur and M. Shcherbina [PS] that first established in this way rigorously the relation between the factorisation of the Gibbs measure and the validity of the replica symmetric ansatz. Talagrand [T6] has improved these results by proving inductively at the same time that the factorization hypothesis hold in certain parameter ranges. The proofs are, unfortunetely very demanding.

1.2.2.2. Low temperatures.

Moving to low temperatures, results become much sparser. Basically, there are two examples now where at least part of the low temperature region is explored. In the Hopfield model for small α this has been possible by some a priori control on the support properties of the Gibbs measure in some essentially "deterministic" sets [BGP,BG2]. The conditional measures on a connected component can then be shown via the cavity method to be asymptotically product, and the replica symmetric solutions can be established [BG1,BG3,T2,T5].

Even more recently, Talagrand has provided a similar result in the p-spin interaction SK models for large p [T3,T7,T8]. In this setting the Gibbs measure concentrates on a *random* set of disjoint subsets. A crucial new ingredient in this analysis are the so-called Ghirlanda-Guerra identities [GG,AC] which I will explain in great detail in the main part of the paper. However, also this situation is, from the replica theory point of view, quite trivial, as only a single step of the replica symmetry breaking proceedure is necessary, while in the standard SK model, or even in the p-spin or the Hopfield model at lower temperatures, an infinity of steps is required.

There is, however, annother class of mean field spin glasses, introduced by Derrida in the 80's, called the Generalized Random Energy models (GREM), that are for some technical reasons much simpler to analyse, although from a replica point of view they exhibit the full complexity of infinite replica symmetry breaking as the SK models. I find it rather interesting tht with the help of the Ghirlanda-Guerra identities and some relatively elementary work, we can now understand and anlyse these models in full detail and rigor and get in this way a glimpse of the beautiful structure and physical meaning of the structures predicted by the replica theory. I will devote a large part of these notes to the review of these recent results.

The remainder of these notes consists of three Section. In Section 2 I will will introduce and formalize the Gibbsian setting for random spin systems and briefly introduce the metastate formalism. Section 3 is devoted to a pedagogical toy model, the random energy model. Here we will get an occasion to introduce some important concepts in an elementary setting. In Section 4 I will apply these concepts to Derrida's models.

2. Random Gibbs measures and metastates.

We will now give a general definition of disordered lattice spin systems and discuss the basis of the Gibbsian formalisms as well as the notion of metastates in this context. This is an abbreviated version of part 2 of my MaPhySto lectures [B]. We consider a lattice $\mathbb{Z}^d$, a single site spin space $(\mathcal{S}_0, \mathcal{F}_0, \nu_0)$ and the corresponding a priori product space $(\mathcal{S}, \mathcal{F}, \nu)$. We add a (rich enough) probability space $(\Omega, \mathcal{B}, \mathbb{P})$ where Ω will always be assumed to be a Polish space. On this probability space we construct a *random interaction* as follows:

Definition 2.1: *A random interaction Φ is a family $\{\Phi_A\}_{A\subset\mathbb{Z}^d}$ of random variables on $(\Omega, \mathcal{B}, \mathbb{P})$ taking values in $B(\mathcal{S}, \mathcal{F}_A)$, i.e. measurable maps $\Phi_A : \Omega \ni \omega \to \Phi_A[\omega] \in B(\mathcal{S}, \mathcal{F}_A)$. A random interaction is called regular, if, for $\mathbb{P}$-almost all ω, for any $x \in \mathbb{Z}^d$, there exists a* *finite constant* $c[\omega]$, *such that*

$$\sum_{A \ni x} \|\Phi_A[\omega]\|_\infty \leq c[\omega] < \infty \tag{2.1}$$

A regular random interaction is called continuous if for each $A \subset \Lambda$, Φ_A *is jointly continuous in the variables* η *and* ω.

In the present section we discuss only regular random interactions. Some of the most interesting physical systems do correspond to irregular random interactions. In particular, many real spin glasses have a non-absolutely summable interaction, called the RKKY-interaction.

Remark: In most examples of interest one assumes that the random interaction has the property that Φ_A and Φ_B are *independent*, if $A \cap B = \emptyset$.

Given a random interaction, it is straightforward to define random finite-volume Hamiltonians

$$H_\Lambda[\omega](\sigma) = \sum_{A \cap \Lambda \neq \emptyset} \Phi_A[\omega](\sigma) \tag{2.2}$$

. Note that for regular random interactions, H_Λ is a random variable taking values in the space $\mathcal{B}_{ql}(\mathcal{S})$, i.e. the mapping $\omega \to H_\Lambda[\omega]$ is measurable. If moreover the Φ_A are continuous functions of ω, then the local Hamiltonians are also continuous functions of ω.

Random local specifications $\mu^{(\cdot)}_{\Lambda,\beta}[\omega]$ are again defined in complete analogy to the deterministic case, i.e.

$$\mu^{(\eta)}_{\Lambda,\beta}[\omega](d\sigma) \equiv \frac{1}{Z^\eta_{\beta,\Lambda}[\omega]} e^{-\beta H_\Lambda[\omega](\sigma_\Lambda,\eta_{\Lambda^c})} \rho_\Lambda(d\sigma_\Lambda) \delta_{\eta_{\Lambda^c}}(d\sigma_{\Lambda^c}) \tag{2.3}$$

The important point is that the maps $\omega \to \mu^{(\cdot)}_{\Lambda,\beta}[\omega]$ are again measurable in all appropriate senses. In particular:

Lemma 2.2: *Let* Φ *be a regular random interaction. Then*

(i) for all $\Lambda \subset \mathbb{Z}^d$ *and* $\mathcal{A} \in \mathcal{F}$, $\mu^{(\cdot)}_{\beta,\Lambda}(\mathcal{A})$ *is measurable function w.r.t. the product sigma-algebra* $\mathcal{F}_{\Lambda^c} \times \mathcal{B}$.

(ii) For $\mathbb{P}$*-almost all* ω, *for all* $\eta \in \mathcal{S}$, $\mu^{(\eta)}_{\Lambda,\beta}[\omega](d\sigma)$ *is a probability measure on* $\mathcal{S}$.

(iii) For almost all ω, *the family* $\left\{\mu^{(\cdot)}_{\beta,\Lambda}[\omega]\right\}_{\Lambda \subset \mathbb{Z}^d}$ *is a local specification for the interaction* $\Phi[\omega]$ *and inverse temperature* β.

(iv) If Φ is a continuous regular random interactions, then for any finite Λ, $\mu^{\eta}_{\beta,\Lambda}[\omega]$ is jointly continuous in η and ω.

We can now define random infinite-volume Gibbs measures.

Definition 2.3: *A measurable map $\mu_\beta : \Omega \to \mathcal{M}_1(\mathcal{S},\mathcal{F})$ is called a random Gibbs measure for the regular random interaction Φ at inverse temperature β, if, for almost all ω, $\mu_\beta[\omega]$ is compatible with the local specification $\left\{\mu^{(\cdot)}_{\beta,\Lambda}[\omega]\right\}_{\Lambda\subset\mathbb{Z}^d}$ for this interaction.*

The first question concerns the existence of such random Gibbs measures. One would expect that, at least for compact state space, a simple compactness argument should prove provide existence of such random Gibbs measures. Now it is indeed obvious in the compact case that for almost all ω, any sequence $\mu^{\eta}_{\beta,\Lambda_n}[\omega]$ taken along an increasing and absorbing sequence of volumes possesses limit points, and therefore, there exist convergent subsequences $\Lambda_{n[\omega]}$ such that $\mu^{\eta}_{\beta,\Lambda_{n[\omega]}}[\omega]$ converges and the limit is a Gibbs measure for the interaction $\Phi[\omega]$. The non-trivial issue provoked by the fact that the subsequence $\Lambda_n[\omega]$ must in general depend on the realization of the disorder is, whether the measures obtained by this construction depend on ω in a *measurable* way?

This question may first sound like some irrelevant mathematical sophistication, and indeed this problem was mostly disregarded in the literature. To my knowledge this problem was first discussed in a paper by van Enter and Griffiths [vEG] and studied in more detail by Aizenman and Wehr [AW1], but it was Ch. Newman and D. Stein [NS1,NS2,N] to have brought the intrinsic *physical relevance* of this issue to light. Needless to say the issue arises only when limits along *deterministic* subsequences cannot be constructed, and this could be feared mainly in very strongly disordered systems such as spin-glasses as we will discuss later.

On more physical terms, the construction of infinite-volume Gibbs measures via limits along random subsequences can be criticised by its lack of actual approximative power. An infinite-volume Gibbs measure is supposed to approximate reasonably a very large system under controlled conditions. If, however, this approximation is only valid for certain very special finite volumes that depend on the specific realization of the disorder, while for other volumes the system is described by other measures, knowledge of just what are the infinite-volume measures is surely not enough, if nothing is known about the relevant subsequences.

As far as proving the existence of random Gibbs measures is concerned, there is a rather simple way out of the random subsequence problem. This goes by extending the local specifications to probability measures $K^{\eta}_{\beta,\Lambda}$ on $\Omega\times\mathcal{S}$ in such a way that the marginal distribution of $K^{\eta}_{\beta,\Lambda}$ on Ω is simply $\mathbb{P}$, while the conditional distribution, given $\mathcal{B}$, is $\mu^{(\eta)}_{\beta,\Lambda}[\omega]$.

Theorem 2.4: *Let Φ be a continuous regular random interaction. Let $K^{(\cdot)}_{\beta,\Lambda}$ be the corresponding measure defined as above. Then*

(i) If for some increasing and absorbing sequence, Λ_n, and some $\eta \in \mathcal{S}$ the weak limit $\lim_{n\uparrow\infty} K^{\eta}_{\beta,\Lambda_n} \equiv K^{\eta}_{\beta}$ exists, then its conditional distribution $K^{\eta}_{\beta}(\cdot|\mathcal{B}\times\mathcal{S})$ given $\mathcal{B}$ is a random Gibbs measure for the interaction Φ.

(ii) If $\mathcal{S}$ is compact, then there exist increasing and absorbing sequences Λ_n such that the hypothesis of (i) is satisfied.

Proof: The proof of this theorem is rather instructive. Let $f \in C(\mathcal{S},\mathcal{F})$ be a continuous function. We must show that

$$K^{\eta}_{\beta}(f|\mathcal{B}\times\mathcal{S})[\omega] = K^{\eta}_{\beta}(\mu^{(\cdot)}_{\beta,\Lambda}[\omega](f)|\mathcal{B}\times\mathcal{S})[\omega] \tag{2.4}$$

Let $\mathcal{B}_k$, $k \in \mathbb{N}$ be a filtration of the sigma-algebra $\mathcal{B}$ where $\mathcal{B}_k$ is generated by the interaction potentials Φ_A with $A \subset \Lambda_k$ with Λ_k some increasing and absorbing sequence of volumes. Note that

$$K^{\eta}_{\beta}(f|\mathcal{B}\times\mathcal{S})[\omega] \equiv \lim_{k\uparrow\infty}\lim_{n\uparrow\infty} K^{\eta}_{\beta,\Lambda_n}(f|\mathcal{B}_k\times\mathcal{S})[\omega] \tag{2.5}$$

Let us denote by $\mathcal{B}_k[\omega]$ the set of all $\omega' \in \Omega$ that have the same projection to $\mathcal{B}_k$ as ω, more formally

$$\mathcal{B}_k[\omega] \equiv \left\{\omega' \in \Omega \,\middle|\, \forall_{\mathcal{A}\in\mathcal{B}_k:\omega\in\mathcal{A}} : \omega' \in \mathcal{A}\right\} \tag{2.6}$$

But for any fixed Λ and n large enough,

$$\begin{aligned}\int_{\mathcal{B}_k[\omega]} \mathbb{P}[d\omega']\mu^{(\eta)}_{\beta,\Lambda_n}[\omega'](f) &= \int_{\mathcal{B}_k[\omega]} \mathbb{P}[d\omega']\mu^{(\eta)}_{\beta,\Lambda_n}[\omega']\left(\mu^{(\cdot)}_{\beta,\Lambda}[\omega'](f)\right)\\ &= \int_{\mathcal{B}_k[\omega]} \mathbb{P}[d\omega']\mu^{(\eta)}_{\beta,\Lambda_n}[\omega']\left(\mu^{(\cdot)}_{\beta,\Lambda}[\omega](f)\right)\\ &+ \int_{\mathcal{B}_k[\omega]} \mathbb{P}[d\omega']\mu^{(\eta)}_{\beta,\Lambda_n}[\omega']\left(\mu^{(\cdot)}_{\beta,\Lambda}[\omega'](f) - \mu^{(\cdot)}_{\beta,\Lambda}[\omega](f)\right)\end{aligned} \tag{2.7}$$

The first term in the last expression converges to $K^{\eta}_{\beta}(\mu^{(\eta)}_{\beta,\Lambda}[\omega](f)|\mathcal{B}\times\mathcal{S})[\omega]$, while for the last we observe that due to the continuity of the local specifications in ω, uniformly in n,

$$\begin{aligned}&\left|\int_{\mathcal{B}_k[\omega]} \mathbb{P}[d\omega']\mu^{(\eta)}_{\beta,\Lambda_n}[\omega']\left(\mu^{(\eta)}_{\beta,\Lambda}[\omega'](f) - \mu^{(\eta)}_{\beta,\Lambda}[\omega](f)\right)\right|\\ &\leq \sup_{\omega'\in\mathcal{B}_k[\omega]}\sup_{\eta\in\mathcal{S}}\left|\mu^{(\eta)}_{\beta,\Lambda}[\omega'](f) - \mu^{(\eta)}_{\beta,\Lambda}[\omega](f)\right| \downarrow 0,\end{aligned} \tag{2.8}$$

as $k \uparrow \infty$. This proves the Theorem.$\diamondsuit$

Theorem 2.4 appears to solve our problems concerning the proper Gibbsian set-up for random systems. We understand what a random infinite-volume Gibbs measure is and we can prove their existence in reasonable generality. Moreover, there is a constructive procedure that allows us to obtain such measures through a procedure of taking infinite-volume limits. However, upon closer inspection, the construction is not quite as satisfactory as it seems. The unsatisfactory point lies actually hidden in equation (2.5) that tells us what conditioning on $\mathcal{B}$ actually amounts to. In all the examples of interest, the space Ω will itself be some infinite product space $\Omega = \Omega_0^{\mathbb{Z}^d}$, and will be equipped with the product topology. The filtration $\mathcal{B}_k$ will then consist of the Borel-field of $\Omega_0^{\Lambda_k}$ for some increasing and absorbing sequence of finite volumes Λ_k. That is, the measures $K_\beta^\eta(\cdot|\mathcal{B}_k \times \mathcal{S})$ are actually averages of Gibbs measures over the values of the random interactions outside a finite region Λ_k, and so their limit still contains an averaging over the realization of the disorder "at infinity". This manifests itself in the fact that the measures $K_\beta^\eta(\cdot|\mathcal{B} \times \mathcal{S})$ will often be mixed states. In particular, this state will not actually describe the result of the observations of one sample of the material at given conditions, but rather the average over many samples that have been prepared to look alike locally. This is clearly not a very physical situation.

While we have come to understand that it may not be realistic to construct a state that predicts the outcome of observations on a single (infinite) sample, it would already be more satisfactory to obtain a probability distribution for these predictions (i.e. a random probability measure) rather than just a mean prediction (and average over probability measures). This led Aizenman and Wehr [AW1] and more emphatically Newman and Stein [NS1] to an extension of the preceding construction to a measure-valued setting. That is, rather than to consider measures on the space $\Omega \times \mathcal{S}$, they introduced measures $\mathcal{K}_{\beta,\Lambda}^\eta$ on the space $\Omega \times \mathcal{M}_1(\mathcal{S})$, defined in such a way that the marginal distribution of $\mathcal{K}_{\beta,\Lambda}^\eta$ on Ω is again $\mathbb{P}$, while the conditional distribution, given $\mathcal{B}$, is $\delta_{\mu_{\beta,\Lambda}^{(\eta)}[\omega]}$, the Dirac-measure concentrated on the corresponding local specification. We will introduce the symbolic notation

$$\mathcal{K}_{\beta,\Lambda}^\eta \equiv \mathbb{P} \times \delta_{\mu_{\beta,\Lambda}^{(\eta)}[\omega]} \tag{2.9}$$

One has the following analogue of Theorem 2.4:

Theorem 2.5: *Let Φ be a continuous regular random interaction. Let $K_{\beta,\Lambda}^{(\cdot)}$ be the corresponding measure defined as above. Then*

(i) If for some increasing and absorbing sequence, Λ_n, and some $\eta \in \mathcal{S}$ the weak limit

$\lim_{n\uparrow\infty}\mathcal{K}^\eta_{\beta,\Lambda_n} \equiv \mathcal{K}^\eta_\beta$ exists, then its conditional distribution $\mathcal{K}^\eta_\beta(\cdot|\mathcal{B}\times\mathcal{S})$ given $\mathcal{B}$ is a probability distribution on $\mathcal{M}_1(\mathcal{S})$ that, for almost all ω, gives full measure to the set of infinite-volume Gibbs measures corresponding to the interaction $\Phi[\omega]$ at inverse temperature β. Moreover,

$$K^\eta_\beta(\cdot|\mathcal{B}\times\mathcal{S}) = \mathcal{K}^\eta_\beta(\mu|\mathcal{B}\times\mathcal{S}) \tag{2.10}$$

(ii) If $\mathcal{S}$ is compact, then there exist increasing and absorbing sequences Λ_n such that the hypothesis of (i) is satisfied for any η.

Remark: The conditional measure

$$\kappa^\eta_\beta \equiv \mathcal{K}^\eta_\beta(\cdot|\mathcal{B}\times\mathcal{S}) \tag{2.11}$$

is called the Aizenman-Wehr *metastate* (following the suggestion of Newman and Stein [NS1]).

Proof: A proof of this theorem can be found in [N]. Here I will give a simple proof following [AW1]. Note that the assertion (i) will follow if for any bounded continuous function $f:\mathcal{S}\to\mathbb{R}$, and any finite $\Lambda\subset\mathbb{Z}^d$, we can show that

$$\mathbb{E}\int \mathcal{K}^\eta_\beta(d\mu|\mathcal{B}\times\mathcal{S})(\omega)\left|\mu(f)-\mu\left(\mu^{(\cdot)}_{\beta,\Lambda}[\omega](f)\right)\right| = 0 \tag{2.12}$$

But the left hand side clearly equals

$$\int \mathcal{K}^\eta_\beta(d\mu,d\omega)\left|\mu(f)-\mu\left(\mu^{\eta}_{\beta,\Lambda}[\omega](f)\right)\right| \tag{2.13}$$

Now $\mu(g)$ is trivially a continuous function of μ if g is continuous. By Lemma 1.9, $\mu^\eta_{\beta,\Lambda}[\omega](f)$ is continuous in f whenever $\Phi[\omega]$ is regular and continuous, i.e. for almost all ω. Thus, both $\mu(f)$ and $\mu\left(\mu^\eta_{\beta,\Lambda}[\omega](f)\right)$ are continuous in μ, and hence the integrand in (2.12) is a bounded continuous function of μ and ω. But then, by definition, the left-hand side of (2.13) is given by the limit

$$\begin{aligned}&\lim_{n\uparrow\infty}\int \mathcal{K}^\eta_{\beta,\Lambda_n}(d\mu,d\omega)\left|\mu(f)-\mu\left(\mu^{\eta}_{\beta,\Lambda}[\omega](f)\right)\right|\\ &=\lim_{n\uparrow\infty}\mathbb{E}\left|\mu^\eta_{\beta,\Lambda_n}[\omega](f)-\mu^\eta_{\beta,\Lambda_n}[\omega]\left(\mu^{\eta}_{\beta,\Lambda}[\omega](f)\right)\right|\end{aligned} \tag{2.14}$$

But the first term in the last line is equal to zero as soon as n is so large that $\Lambda\subset\Lambda_n$ which implies that (2.12) holds. Assertion (ii) follows by compactness. $\diamondsuit$

At this stage the reader may rightly hold his breath and ask the question whether all this abstract formalism is really necessary, or whether in reasonable situations, we will not get away without all of this? To answer this question, we need to look at specific results, and above all, at examples.

Unfortunately, there are almost no lattice spin models that we could work out to sufficient precision to be able to really construct any metastate explicitely. In fact, the only example where this has been done very recently is a ferromagnetic Ising model with random boundary conditions [vEMN]. Of course this goes a little against our original intentions. Despite the fact that we cannot work out an explicit metastate, Newman and Stein have brough up this concept precisely in the context of lattice spin glasses. Their main intention was to verify which of the general properties that come out of the mean field solution of Sherrington-Kirkpatrick model can possibly remain true in a lattice model in finite dimensions, and how the various quantities ought to be interpretated in a mathematically sound framework. This has led to considerable clarifications, at least in my view. However, since lattice models are not necessarily the main concern of this meeting, I will not follow this line, and rather turn to the analysis of mean field models there these concepts can be nicely interpreted.

3. The simplest example: The random energy model.

The random energy model, introduced by Derrida [D1,D2] can be considered as the ultimate toy model of a disordered system. In this model, rather little is left of the structure of interacting spins, but we will still be able to gain a lot of insight into the peculiarities of disordered systems by studying this simple system. For rigorous work on the REM see e.g. [Ei,OP,GMP,DW].

The REM is a model with state space $\mathcal{S}_N = \{-1, +1\}^N$. For fixed N, the Hamiltonian is given by

$$H_N(\sigma) = -\sqrt{N} X_\sigma \tag{3.1}$$

where X_σ, is a family of 2^N i.i.d. centered normal random variables.

3.2.1. The free energy.

Before turning to the question of Gibbs measures, we turn to the simpler question of analysing in some detail the partition function. In this model, the partition function is of course just the sum of i.i.d. random variables, i.e.

$$Z_{\beta,N} \equiv 2^{-N} \sum_{\sigma \in \mathcal{S}_N} e^{\beta\sqrt{N} X_\sigma} \tag{3.2}$$

One usually asks first for the exponential asymptotics of this quantity, i.e. one introduces the *free energy*,

$$F_{\beta,N} \equiv -\frac{1}{N} \ln Z_{\beta,N} \tag{3.3}$$

and tries to find its limit as $N \uparrow \infty$. Let me mention that in general mean field spin glasses, the existence of the limit even of the averaged free energyt has been a long standing open problem. While writing these note, a preprint by Guerra and Toninelli [GT] has appeared in which a simple and clever proof of the existence of the limit in a rather large class of mean field spin glass models is given.

In our simple model we expect of course to be able to compute this limit exactly. In fact, the first guess would be that a *law of large numbers* might hold, implying that $Z_{\beta,N} \sim \mathbb{E}Z_{\beta,N}$, and hence

$$\lim_{N\uparrow\infty} F_{\beta,N} = \lim_{N\uparrow\infty} -\frac{1}{N} \ln \mathbb{E}Z_{\beta,N} = -\frac{\beta^2}{2}, \quad \text{a.s.} \tag{3.4}$$

It turns out that this is indeed true, but only for small enough values of β, and this can be linked precisely to a critical value for the breakdown of the law of large numbers. The analysis of this problem will allow us to compute the free energy exactly.

Theorem 3.1: *In the REM,*

$$\lim_{N\uparrow\infty} \mathbb{E}F_{\beta,N} = \begin{cases} -\frac{\beta^2}{2}, & \text{for } \beta \leq \beta_c \\ -\frac{\beta_c^2}{2} - (\beta-\beta_c)\beta_c, & \text{for } \beta \geq \beta_c \end{cases} \tag{3.5}$$

where $\beta_c = \sqrt{2\ln 2}$.

Proof: We give the proof of this theorem since it allows us to show the working of an important idea, the method of truncated second moments which was introduced by M. Talagrand [T2,T3]. We will first derive a lower bound for $\mathbb{E}F_{\beta,N}$. Note first that by Jensen's inequality, $\mathbb{E}\ln Z \leq \ln \mathbb{E}Z$, and thus

$$\mathbb{E}F_{\beta,N} \geq -\frac{\beta^2}{2} \tag{3.6}$$

On the other hand we have that

$$-\mathbb{E}\frac{d}{d\beta} F_{\beta,N} = N^{-1/2}\mathbb{E}\frac{\mathbb{E}_\sigma X_\sigma e^{\beta\sqrt{N}X_\sigma}}{Z_{\beta,N}} \leq N^{-1/2}\mathbb{E}\max_{\sigma\in\mathcal{S}_N} X_\sigma \leq \beta\sqrt{2\ln 2}(1+C/N) \tag{3.7}$$

for some constant C. Moreover, since $\frac{d^2}{d\beta^2}F_{\beta,N} \leq 0$, we may combine (3.6) and (3.7) to deduce that

$$\mathbb{E}F_{\beta,N} \geq \sup_{\beta_0\geq 0} \begin{cases} -\frac{\beta^2}{2}, & \text{for } \beta \leq \beta_0 \\ -\frac{\beta_0^2}{2} - (\beta-\beta_0)\sqrt{2\ln 2}(1+C/N), & \text{for } \beta \geq \beta_0 \end{cases} \tag{3.8}$$

It is easy to see that the supremum is realized (ignore the C/N correction) for $\beta_0 = \sqrt{2\ln 2}$. This shows that the right hand side of (3.5) is a lower bound.

It remains to show the corresponding upper bound. The basic idea behind this approach is to obtain a variance estimate on the partition function[4] . Naively, one would compute

$$\begin{aligned} \mathbb{E}Z^2_{\beta,N} &= \mathbb{E}_\sigma \mathbb{E}_{\sigma'} \mathbb{E}e^{\beta\sqrt{N}(X_\sigma + X_{\sigma'})} \\ &= 2^{-2N}\left(\sum_{\sigma\neq\sigma'} e^{N\beta^2} + \sum_\sigma e^{2N\beta^2}\right) \\ &= e^{N\beta^2}\left[(1-2^{-N}) + 2^{-N}e^{N\beta^2}\right] \end{aligned} \tag{3.9}$$

where all we used is that for $\sigma \neq \sigma'$ X_σ and $X_{\sigma'}$ are independent. Now we see that the second term in the square brackets is exponentially small if and only if $\beta^2 < \ln 2$. For such values of β we have that

$$\begin{aligned} \mathbb{P}\left[\left|\ln\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}}\right| > \epsilon N\right] &= \mathbb{P}\left[\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} < e^{-\epsilon N} \text{ or } \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} > e^{\epsilon N}\right] \\ &\leq \mathbb{P}\left[\left(\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} - 1\right)^2 > \left(1-e^{-\epsilon N}\right)^2\right] \\ &\leq \frac{\mathbb{E}Z^2_{\beta,N}/(\mathbb{E}Z_{\beta,N})^2 - 1}{(1-e^{-\epsilon N})^2} \\ &\leq \frac{2^{-N} + 2^{-N}e^{N\beta^2}}{(1-e^{-\epsilon N})^2} \end{aligned} \tag{3.10}$$

which is more than enough to get (3.4). But of course this does not correspond to the critical value of β claimed in the proposition! Some reflection shows that the point here is that when computing $\mathbb{E}e^{\beta\sqrt{N}2X_\sigma}$, the dominant contribution comes from the part of the distribution of X_σ where $X_\sigma \sim 2\beta\sqrt{N}$, whereas in the evaluation of $\mathbb{E}Z_{\beta,N}$ the values of X_σ where $X_\sigma \sim \beta\sqrt{N}$ give the dominant contribution. Thus one is led to the realization that it is not the second moment of Z one should control, but rather that of a truncated version of Z, namely, for $c \geq 0$,

$$\widetilde{Z}_{\beta,N}(c) \equiv \mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma} \mathbb{1}_{X_\sigma < c\sqrt{N}} \tag{3.11}$$

An elementary computation using the standard bound, for $u > 0$,

$$\frac{1}{\sqrt{2\pi}u}e^{-u^2/2}\left(1-2u^{-2}\right) \leq \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-x^2/2}dx \leq \frac{1}{\sqrt{2\pi}u}e^{-u^2/2} \tag{3.12}$$

[4]This idea can be traced to Aizenman, Lebowitz, and Ruelle [ALR], and later Comets and Neveu [CN] who used it in the proofs of a central limit theorem for the free energy.

shows that

$$\mathbb{E}\widetilde{Z}_{\beta,N}(c) = \begin{cases} e^{\beta^2 N}\left(1 - \frac{e^{-N(-\beta)^2/2}}{\sqrt{2\pi N}(c-\beta)}(1+O(1/N)\right), & \text{if } \beta < c \\ \frac{1+O(1/N)}{\sqrt{2\pi N}(\beta-c)} e^{N\beta c - \frac{Nc^2}{2}}, & \text{if } \beta > c \end{cases} \tag{3.13}$$

Note that (3.13) shows that this truncation essentially does not influence the mean partition function, if $\beta < c$.

But now compute the mean of the square of the truncated partition function:

$$\mathbb{E}\widetilde{Z}^2_{\beta,N}(c) = (1-2^{-N})[\mathbb{E}\widetilde{Z}_{\beta,N}(c)]^2 + 2^{-N}\mathbb{E}e^{\beta\sqrt{N}2X_\sigma} 𝟙_{X_\sigma < c\sqrt{N}} \tag{3.14)}$$

where the second term satisfies (we do not mention the irrelevant $O(1/N)$ error term anymore)

$$2^{-N}\mathbb{E}e^{2\beta\sqrt{N}X_\sigma} 𝟙_{X_\sigma < c\sqrt{N}} \leq \begin{cases} 2^{-N}e^{2\beta^2 N}, & \text{if } 2\beta < c \\ 2^{-N}\frac{e^{2c\beta N - \frac{c^2 N}{2}}}{(2\beta-c)\sqrt{2\pi N}}, & \text{otherwise,} \end{cases} \tag{3.15}$$

Combined with (3.13) this implies

$$\frac{2^{-N}\mathbb{E}e^{2\beta\sqrt{N}X_\sigma} 𝟙_{X_\sigma < c\sqrt{N}}}{\left(\mathbb{E}\widetilde{Z}_{N,\beta}\right)^2} \leq \begin{cases} e^{-N(\ln 2 - \beta^2)}, & \text{if } \beta < \frac{c}{2}, \\ \frac{e^{-N(c-\beta)^2 - N(\ln 2 - \frac{c^2}{2})}}{(2\beta-c)\sqrt{N}}, & \text{if } \frac{c}{2} < \beta < c, \\ e^{(c^2/2 - \ln 2)N}\sqrt{2\pi N}\frac{(\beta-c)^2}{2\beta-c}, & \text{if } \beta > c \end{cases} \tag{3.16}$$

Therefore, for all $c < \sqrt{2\ln 2}$, and all $\beta \neq c$,

$$\mathbb{E}\left[\frac{\widetilde{Z}_{\beta,N}(c) - \mathbb{E}\widetilde{Z}_{\beta,N}(c)}{\mathbb{E}\widetilde{Z}_{\beta,N}(c)}\right]^2 \leq e^{-Ng(c,\beta)} \tag{3.17}$$

with $g(c,\beta) > 0$. Thus Chebyshev's inequality implies that

$$\mathbb{P}\left[|\widetilde{Z}_{\beta,N}(c) - \mathbb{E}\widetilde{Z}_{\beta,N}(c)| > \delta\mathbb{E}\widetilde{Z}_{\beta,N}(c)\right] \leq \delta^{-2}e^{-Ng(c,\beta)} \tag{3.18}$$

which implies in particular that

$$\lim_{N\uparrow\infty}\frac{1}{N}\mathbb{E}\ln\widetilde{Z}_{\beta,N}(c) = \lim_{N\uparrow\infty}\frac{1}{N}\ln\mathbb{E}\widetilde{Z}_{\beta,N}(c) \tag{3.19}$$

for all $c < \sqrt{2\ln 2} = \beta_c$. But this implies that for all $c < \beta_c$

$$\lim_{N\uparrow\infty}\frac{1}{N}\ln\mathbb{E}Z_{\beta,N} \geq \lim_{N\uparrow\infty}\frac{1}{N}\ln\mathbb{E}\widetilde{Z}_{\beta,N}(c) = \begin{cases} \frac{\beta^2}{2}, & \text{for } \beta < c \\ \frac{c^2}{2} + c(\beta - c), & \text{for } \beta > c \end{cases} \tag{3.20}$$

which converges to minus the right hand side of (3.5) as $c \uparrow \beta_c$. This proves the theorem.$\diamondsuit$

3.2.2. Fluctuations and limit theorems.

Knowing the free energy is important, but, as one may expect, it is not enough to understand the properties of the Gibbs measures completely. It is the analysis of the fluctuations of the free energy that will reveal, as we will see, the necessary information. In the REM this can be done using classical results from the theory of extreme value statistics. The proofs are, nonetheless, quite cumbersome, and may be found in in [BKL] or [B].

Theorem 3.2: *The partition function of the REM has the following fluctuations:*

(i) If $\beta < \sqrt{\ln 2/2}$, *then*

$$e^{\frac{N}{2}(\ln 2-\beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.21}$$

(ii) If $\beta = \sqrt{\ln 2/2}$, *then*

$$\sqrt{2}e^{\frac{N}{2}(\ln 2-\beta^2)} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{3.22}$$

(iii) Let $\alpha \equiv \beta/\sqrt{2\ln 2}$. *If* $\sqrt{\ln 2/2} < \beta < \sqrt{2\ln 2}$, *then*

$$e^{\frac{N}{2}(\sqrt{2\ln 2}-\beta)^2+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \ln \frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} \xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} e^{\alpha z}(\mathcal{P}(dz) - e^{-z}dz), \tag{3.23}$$

where $\mathcal{P}$ *denotes the Poisson point process*[5] *on* $\mathbb{R}$ *with intensity measure* $e^{-x}dx$.

(iv) If $\beta = \sqrt{2\ln 2}$, *then*

$$e^{\frac{1}{2}[\ln(N\ln 2)+\ln 4\pi]}\left(\frac{Z_{\beta,N}}{\mathbb{E}Z_{\beta,N}} - \frac{1}{2} + \frac{\ln(N\ln 2)+\ln 4\pi}{4\sqrt{\pi N\ln 2}}\right) \xrightarrow{\mathcal{D}} \int_{-\infty}^{0} e^{z}(\mathcal{P}(dz) - e^{-z}dz) + \int_{0}^{\infty} e^{z}\mathcal{P}(dz). \tag{3.24}$$

(v) If $\beta > \sqrt{2\ln 2}$, *then*

$$e^{-N[\beta\sqrt{2\ln 2}-\ln 2]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} Z_{\beta,N} \xrightarrow{\mathcal{D}} \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) \tag{3.25}$$

and

$$\ln Z_{\beta,N} - \mathbb{E}\ln Z_{\beta,N} \xrightarrow{\mathcal{D}} \ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz) - \mathbb{E}\ln \int_{-\infty}^{\infty} e^{\alpha z}\mathcal{P}(dz). \tag{3.26}$$

[5] For a thorough exposition on point processes and their connection to extreme value theory, see in particular [Re].

Remark: Note that expressions like $\int_{-\infty}^{0} e^{z}(\mathcal{P}(dz) - e^{-z}dz)$ are always understood as $\lim_{y\downarrow -\infty}\int_{y}^{0} e^{z}(\mathcal{P}(dz) - e^{-z}dz)$. All the functionals of the Poisson point process appearing are almost surely finite random variables. Note that the limit in (3.23) has infinite variance and the one in (3.25) has infinite mean.

Let us just briefly comment on how these results are obtained. In fact, (i) follows from the standard CLT for arrays of independent random variables une Lindeberg's condition.

As the Lindeberg condition fails for $2\beta^2 \geq \ln 2$, it is clear that we cannot expect a simple CLT beyond this regime. Such a failure of a CLT is always a problem related to "heavy tails", and results from the fact that extremal events begin to influence the fluctuations of the sum. It appears therefore reasonable to separate from the sum the terms where X_σ is anomalously large. For Gaussian r.v.'s it is well known that the right scale of separation is given by $u_N(x)$ defined by

$$2^N \int_{u_N(x)}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} = e^{-x} \tag{3.27}$$

which (for $x > -\ln N/\ln 2$) is equal to (see e.g. [LLR])

$$u_N(x) = \sqrt{2N\ln 2} + \frac{x}{\sqrt{2N\ln 2}} - \frac{\ln(N\ln 2) + \ln 4\pi}{2\sqrt{2N\ln 2}} + o(1/\sqrt{N}), \tag{3.28}$$

$x \in \mathbb{R}$ is a parameter. The key to most of what follows relies on the famous result on the convergence of the extreme value process to a Poisson point process. Let us now introduce the point process on $\mathbb{R}$ given by

$$\mathcal{P}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{u_N^{-1}(X_\sigma)}. \tag{3.29}$$

A classical result from the theory of extreme order statistics (see e.g. [LLR]) asserts that ***Theorem 3.3:*** *The point process $\mathcal{P}_N$ converges weakly to a Poisson point process on $\mathbb{R}$ with intensity measure $e^{-x}dx$.* The key idea is then to split the sum by a cutoff corresponding to whether X_σ is biger or smaller than $u_N(x)$; the former can then be represented as a functional of the extremal process that converges to the Poisson process, and the latter has to be controlled carefully. The computations are in fact quite tedious.

Iw we write

$$Z_{\beta,N} = Z_{\beta,N}^x + (Z_{\beta,N} - Z_{\beta,N}^x) \tag{3.30}$$

for $\beta \geq \sqrt{2\ln 2}$

$$Z_{\beta,N} - Z_{\beta,N}^x = e^{N\left[\beta\sqrt{2\ln 2} - \ln 2\right] - \frac{\alpha}{2}[\ln(N\ln 2) + \ln 4\pi]} \sum_{\sigma\in\mathcal{S}_N} \mathbb{1}_{\{u_N^{-1}(\sigma) > x\}} e^{\alpha u_N^{-1}(X_\sigma)} \tag{3.31}$$

so that for any $x \in \mathbb{R}$,

$$(Z_{\beta,N} - Z^x_{\beta,N})e^{-N\left[\beta\sqrt{2\ln 2}-\ln 2\right]+\frac{\alpha}{2}[\ln(N\ln 2)+\ln 4\pi]} \xrightarrow{\mathcal{D}} \int_x^\infty e^{\alpha z}\mathcal{P}(dz). \tag{3.32}$$

The remaining term is shown to converge to zero under this scaling. $\diamondsuit$

3.2.3. The Gibbs measure.

With our preparation on the fluctuations of the free energy, we have accumulated enough understanding about the partition function that we can deal with the Gibbs measures. Clearly, there are a number of ways of trying to describe the asymptotics of the Gibbs measures. Recalling the general discussion on random Gibbs measures, it should be clear that we are seeking a result on the convergence in distribution of random measures. To be able to state such a results, we have to introduce a topology on the spin configuration state that makes it uniformly compact. The usual topology we used to do this was the product topology, and this clearly would be an option here. However, given what we already know about the partition function, this topology does not appear suited to give describe the measure appropriately. Recall that at low temperatures, the partition function was dominated by a 'few' spin configurations with exceptionally large energy. This is a feature that should remain visible in a limit theorem. The question we therefore must address in mean field models is how to describe a limiting measure on an infinite dimensional cube that properly reflects the symmetry (under permutation) of the finite dimensional object, in other words that views this object in an unbiased way.

A first attempt consists in mapping the hypercube to the interval $[-1,1]$ via

$$\mathcal{S}_N \ni \sigma \to r_N(\sigma) \equiv \sum_{i=1}^N \sigma_i 2^{-i} \in [-1,1] \tag{3.33}$$

Define the pure point measure $\tilde{\mu}_{\beta,N}$ on $[-1,1]$ by

$$\tilde{\mu}_{\beta,N} \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{r_N(\sigma)}\mu_{\beta,N}(\sigma) \tag{3.34}$$

Our results will be expressed in terms of the convergence of these measures. It will be understood in the sequal that the space of measures on $[-1,1]$ is equipped with the topology of weak convergence, and all convergence results hold with respect to this topology.

As the diligent reader will have expected, in the high temperature phase the limit is the same as for $\beta = 0$, namely

Theorem 3.4: *If $\beta \leq \sqrt{2\ln 2}$, then*

$$\tilde{\mu}_{\beta,N} \to \frac{1}{2}\lambda, \quad a.s. \tag{3.35}$$

where λ denotes the Lebesgue measure on $[-1,1]$.

Proof: Note that we have to prove that for any finite collection of intervals $I_1, \dots, I_k \subset [-1,1]$, the family of random variables $\{\tilde{\mu}_{\beta,N}(I_1), \dots, \tilde{\mu}_{\beta,N}(I_k)$ converges jointly almost surely to $\frac{1}{2}|I_1|, \dots, \frac{1}{2}|I_k|$. But by construction these random vectors are independent, so that this will follow automatically, if we can prove the result in the case $k = 1$. Our strategy is to get first very sharp estimates for a family of special intervals.

In the sequel we will always assume that $N \geq n$. We will denote by Π_n the canonical projection from $\mathcal{S}_N$ to $\mathcal{S}_n$. To simplify notation, we will often write $\sigma_n \equiv \Pi_n \sigma$ when no confusion can arise. For $\sigma \in \mathcal{S}_N$, set

$$a_n(\sigma) \equiv r_n(\Pi_n \sigma) \tag{3.36}$$

and

$$I_n(\sigma) \equiv [a_n(\sigma) - 2^{-n}, a_n(\sigma) + 2^{-n}) \tag{3.37}$$

Note that the union of all these intervals forms a disjoint covering of $[-1,1)$. Obviously, these intervals are constructed in such a way that

$$\tilde{\mu}_{\beta,N}(I_n(\sigma)) = \mu_{\beta,N}\left(\{\sigma' \in \mathcal{S}_N : \Pi_n(\sigma') = \Pi_n(\sigma)\}\right) \tag{3.38}$$

The first step in the proof consists in showing that the masses of all the intervals $I_n(\sigma)$ are remarkably well approximated by their uniform mass.

Lemma 3.5: *Set $\beta' \equiv \sqrt{\frac{N}{N-n}}\beta$. For any $\sigma \in \mathcal{S}_n$,*

(i) If $\beta' \leq \sqrt{\frac{\ln 2}{2}}$,

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma)) - 2^{-n}| \leq 2^{-n} e^{-(N-n)(\ln 2 - \beta'^2)} Y_{N-n} \tag{3.39}$$

where Y_N has bounded variance, as $N \uparrow \infty$.

(ii) If $\sqrt{\frac{\ln 2}{2}} < \beta' < \sqrt{2\ln 2}$,

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma)) - 2^{-n}| \leq 2^{-n} e^{-(N-n)(\sqrt{2\ln 2} - \beta')^2/2 - \alpha \ln(N-n)/2} Y_{N-n} \tag{3.40}$$

where Y_N is a random variable with bounded mean modulus.

(iii) If $\beta=\sqrt{2\ln 2}$, then, for any n fixed,

$$|\tilde{\mu}_{\beta,N}(I_n(\sigma))-2^{-n}|\to 0 \quad \textit{in probability} \tag{3.41}$$

Remark: Note that in the sub-critical case, the results imply convergence to the uniform product mesure on $\mathcal{S}$ in a *very strong sense.* In particular, the base-size of the cylinders considered (i.e. n) can grow proportionally to N, *even if almost sure convergence uniformly for all cylinders is required!* This is unusually good. However, one should not be deceived by this fact: even though seen from the cylinder masses the Gibbs measures look like the uniform measure, seen from the point of view of individual spin configurations the picture is quite different. In fact, the measure concentrates on an *exponentially* small fraction of the full hypercube, namely those $O(\exp(N(\ln 2-\beta^2/2)))$ vertices that have energy $\sim \beta N$ (Exercise!). It is just the fact that this set is still exponentially large, as long as $\beta<\sqrt{2\ln 2}$, and is very uniformly dispersed over $\mathcal{S}_N$, that produces this somewhat paradoxical effect. The rather weak result in the critical case is not artificial. In fact it is not true that almost sure convergence will hold. This follows e.g. from Theorem 1 in [GMP]. One should of course anticipate some signature of the phase transition at the critical point.

Proof: The proof of this lemma is a simple application of the first three points in Theorem 3.2. Just note that the partial partition functions

$$Z_{\beta,N}(\sigma_n)\equiv \mathbb{E}_{\sigma'} e^{\beta\sqrt{N}X_{\sigma'}} \mathbb{1}_{\Pi_n(\sigma')=\sigma_n} \tag{3.42}$$

are independent and have the same distribution as $2^{-n}Z_{\beta',N-n}$. But

$$\tilde{\mu}_{\beta,N}(I_n(\sigma_n))=\frac{Z_{\beta,N}(\sigma_n)}{[Z_{\beta,N}-Z_{\beta,N}(\sigma_n)]+Z_{\beta,N}(\sigma_n)} \tag{3.43}$$

Note that $Z_{\beta,N}(\sigma_n)$ and $[Z_{\beta,N}-Z_{\beta,N}(\sigma_n)]$ are independent. It should now be obvious how to conclude the proof with the help of Theorem 3.2. $\diamondsuit$

Once we have the excellent approximation of the measure on all of the intervals $I_n(\sigma)$, almost sure convergence of the measure in the weak topology is a simple consequence. Of course, this is just a coarse version of the finer results we have, and much more precise information on the quality of approximation can be inferred from Lemma 3.5. But since the high-temperature phase is not our prime concern, we will not go further in this direction.

Somehow much more interesting is the behaviour of the measure at low temperatures that we will discuss now. Let us introduce the Poisson point process $\mathcal{R}$ on the strip $[-1,1]\times\mathbb{R}$ with intensity measure $\frac{1}{2}dy\times e^{-x}dx$. If (Y_k,X_k) denote the atoms of this process, define a new point process $\mathcal{W}_\alpha$ on $[-1,1]\times(0,1]$ whose atoms are (Y_k,w_k), where

$$w_k \equiv \frac{e^{\alpha X_k}}{\int \mathcal{R}(dy,dx)e^{\alpha x}} \tag{3.44}$$

for $\alpha>1$. Let us note that the process $\widehat{\mathcal{W}}=\sum_k w_k$ is known in the literature as the *Poisson-Dirichlet process* with parameter α [K].

With this notation we have that

Theorem 3.6: *If $\beta>\sqrt{2\ln 2}$, with $\alpha=\beta/\sqrt{2\ln 2}$,*

$$\tilde\mu_{\beta,N}\stackrel{\mathcal{D}}{\to}\tilde\mu_\beta \equiv \int_{[-1,1]\times(0,1]} \mathcal{W}_\alpha(dy,dw)\delta_y w \tag{3.45}$$

Proof: With $u_N(x)$ defined in (3.28), we define the point process $\mathcal{R}_N$ on $[-1,1]\times\mathbb{R}$ by

$$\mathcal{R}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{(r_N(\sigma),u_N^{-1}(X_\sigma))} \tag{3.46}$$

A standard result of extreme value theory (see [LLR], Theorem 5.7.2) is easily adapted to yield that

$$\mathcal{R}_N \stackrel{\mathcal{D}}{\to}\mathcal{R}, \quad \text{as } N\uparrow\infty \tag{3.47}$$

where the convergence is in the sense of weak convergence on the space of sigma-finite measures endowed with the (metrizable) topology of vague convergence. Note that

$$\mu_{\beta,N}(\sigma)=\frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\sum_\sigma e^{\alpha u_N^{-1}(X_\sigma)}}=\frac{e^{\alpha u_N^{-1}(X_\sigma)}}{\int\mathcal{R}_N(dy,dx)e^{\alpha x}} \tag{3.48}$$

Since $\int\mathcal{R}_N(dy,dx)e^{\alpha x}<\infty$ a.s., we can define the point process

$$\mathcal{W}_N \equiv \sum_{\sigma\in\mathcal{S}_N} \delta_{\left(r_N(\sigma),\frac{\exp(\alpha u_N^{-1}(X_\sigma))}{\int\mathcal{R}_N(dy,dx)\exp(\alpha x)}\right)} \tag{3.49}$$

on $[-1,1]\times(0,1]$. Then

$$\tilde\mu_{\beta,N}=\int\mathcal{W}_N(dy,dw)\delta_y w \tag{3.50}$$

The only non-trivial point in the convergence proof is to show that the the contribution to the partition functions in the denominator from atoms with $u_N(X_\sigma) < x$ vanishes as $x \downarrow -\infty$. But this is precisely what we have shown to be the case in the proof of part (v) of Theorem 3.2. Standard arguments then imply that first $\mathcal{W}_N \stackrel{\mathcal{D}}{\to} \mathcal{W}$, and consequently, (3.45). $\diamondsuit$

Remark: Note that Theorem 3.6 contains in particular the convergence of the Gibbs measure in the product topology on $\mathcal{S}_N$, since cylinders correspond to certain subintervals of $[-1,1]$. On the other hand, it implies that the point process of weights $\sum_{\sigma\in\mathcal{S}_N} \delta_{\mu_{\beta,N}(\sigma)}$ converges in law to the marginal of $\mathcal{W}_N$ on $(0,1]$ which is the process introduced by Ruelle [Ru4]. The formulation of Theorem 3.6 is moreover very much in the spirit of the metastate approach to random Gibbs measures. The limiting measure is a measure on a continuous space, and each point measure on this set may appear as "pure state". The "metastate", i.e. the law of the random measure $\tilde\mu_\beta$ is a probability distribution concentrated on the countable convex combinations of pure states randomly chosen by a Poisson point process from an uncountable collection, while the coefficients of the convex combination are again random and selected via another point process. The only aspect of metastates that is missing here is that we have not "conditioned on the disorder". The point is, however, that there is no natural filtration of the disorder space compatible with, say, the product topology, and thus in this model we have no natural urge to "fix the disorder locally"; note that it is possible to represent the i.i.d. family X_σ as a sum of "local" couplings, i.e. let J_I, for any $I\subset\mathbb{N}$ be i.i.d. standard normal variables. Then we can represent $X_\sigma = 2^{-N/2}\sum_{I\subset\{1,\dots,N\}}\sigma_I J_I$; obviously these variables become independent of any of the J_I, with I fixed, so that conditioning on them would not change the metastate.

Let us discuss the properties of the limiting process $\tilde\mu_\beta$. It is not hard to see that with probability one, the support of $\tilde\mu_\beta$ is the entire interval $[-1,1]$. On the other hand, its mass is concentrated on a countable set, i.e. the measure is pure point. To see this, consider the rectangle $A_\epsilon \equiv (\ln\epsilon,\infty)\times[-1,1]$. Clearly, the process $\mathcal{R}$ restricted to this set has finite total intensity given by ϵ^{-1}. i.e. the number total number of atoms in that set is a Poissonian random variable with parameter ϵ^{-1}. Now if we remove the projection of these finitely many random points from $[-1,1]$, we will show that the total mass that remains goes to zero with ϵ. Clearly, the remaining mass is given by

$$\int_{[-1,1]\times(-\infty,\ln\epsilon)} \mathcal{R}(dy,dx)\frac{e^{\alpha x}}{\int\mathcal{P}(dx')e^{\alpha x'}} = \int_{-\infty}^{\ln\epsilon}\mathcal{P}(dx)\frac{e^{\alpha x}}{\int\mathcal{P}(dx')e^{\alpha x'}} \tag{3.51}$$

We want to get a lower bound in probability on the denominator. The simplest possible bound is obtained by estimating the probability of the integral by the contribution of the largest atom which of course follows the double-exponential distribution. Thus

$$\mathbb{P}\left[\int \mathcal{P}(dx)e^{\alpha x} \le Z\right] \le e^{-e^{-\ln Z/\alpha}} = e^{-Z^{-\frac{1}{\alpha}}} \tag{3.52}$$

Setting $\Omega_Z \equiv \{\mathcal{P} : \int \mathcal{P}(dx)e^{\alpha x} \le Z\}$, we conclude that, for $\alpha > 1$,

$$\begin{aligned}
\mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} > \gamma\right] &\le \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)\frac{e^{\alpha x}}{\int \mathcal{P}(dx')e^{\alpha x'}} > \gamma,\, \Omega_Z^c\right] + \mathbb{P}[\Omega_Z] \\
&\le \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x} > \gamma Z,\, \Omega_Z^c\right] + \mathbb{P}[\Omega_Z] \\
&\le \mathbb{P}\left[\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x} > \gamma Z\right] + \mathbb{P}[\Omega_Z] \\
&\le \frac{\mathbb{E}\int_{-\infty}^{\ln \epsilon} \mathcal{P}(dx)e^{\alpha x}}{\gamma} + \mathbb{P}[\Omega_Z] \\
&\le \frac{\epsilon^{\alpha-1}}{(\alpha-1)\gamma Z} + e^{-Z^{-\frac{1}{\alpha}}}
\end{aligned} \tag{3.53}$$

Obviously, for any positive γ it is possible to choose Z as a function of ϵ in such a way that the right hand side tends to zero. But this implies that with probability one, all of the mass of the measure $\tilde{\mu}_\beta$ is carried by a countable set, implying that $\tilde{\mu}_\beta$ is pure point.

So we see that the phase transition in the REM expresses itself via a change of the properties of the infinite volume Gibbs measure mapped to the interval from Lebesgue measure at high temperatures to a random dense pure pure point measure at low temperatures.

3.2.4. The replica overlap.

While the random measure description of the phase transition in the REM appears rather nice, one would argue that it ignores fully the geometry of the statespace as a hypercube. A neat object to measure look at in this respect would be the mass dustribution around a given configuration,

$$m_\sigma(t) \equiv \mu_{\beta,N}\left(R_N(\sigma,\sigma') \ge t\right) \tag{3.54}$$

where the σ is fixed and the measure μ refers to the configuration σ'. $m_\sigma(\cdot)$ is a probability distribution function on $[-1,1]$. As a function of σ, this is a measure values random variable. Taking the overage of this quantity again with respect to the Gibbs distribution of σ, we obtain the popular "overlap distribution",

$$f_{\beta,N}[\omega](dz) \equiv \mu_{\beta,N}\left(m_\sigma(dz)\right) = \mu_{\beta,N}[\omega] \otimes \mu_{\beta,N}[\omega]\left(R_N(\sigma,\sigma') \in dz\right) \tag{3.55}$$

AS we have seen in the discussion of the metastate, it pays to pass again to a measure valued description, that it to say, the following object

$$\mathcal{K}_{\beta,N} \equiv \sum_{\sigma\in\mathcal{S}_N} \mu_{\beta,N}(\sigma)\delta_{m_\sigma(\cdot)} \tag{3.56}$$

contains more information than just the overlap distribution. In fact, it tells us the probability to see a given miss distribution around oneself, if one is distributed with the Gibbs measure. Of course we have that

$$f_{\beta,N}[\omega](\cdot) = \int \mathcal{K}_{\beta,N}(dm)m(\cdot) \tag{3.57}$$

Of course, in the REM, one is not likely to see anything very exciting, the overlap distribution is asympototically concentrated on the values 0 and 1 only:

Theorem 3.7:

(i) For all $\beta<\sqrt{2\ln 2}$

$$\lim_{N\uparrow\infty} f_{\beta,N} = \delta_0, \quad a.s. \tag{3.58}$$

(ii) For all $\beta>\sqrt{2\ln 2}$

$$f_{\beta,N} \stackrel{\mathcal{D}}{\to} \delta_0\left(1-\int \mathcal{W}(dy,dw)w^2\right) + \delta_1\int \mathcal{W}(dy,dw)w^2 \tag{3.59}$$

(iii) The random measures $\mathcal{K}_{\beta,N}$ converge to a random probability distribution $\mathcal{K}_\beta$ that is supported on the atomic measures with support on $\{0,1\}$, more precisely if $\beta>\sqrt{2\ln 2}$,

$$\mathcal{K}_\beta = \int \mathcal{W}(dy,dw)w\delta_{w\delta_1+(1-w)\delta_0} \tag{3.60}$$

while for $\beta<\sqrt{2\ln 2}$, $\mathcal{K}_\beta$ is the Dirac mass on the Dirac mass concentrated at 0.

Proof: We will write for any $I\subset[-1,1]$

$$f_{\beta,N}(I) = Z_{\beta,N}^{-2}\mathbb{E}_\sigma\mathbb{E}_{\sigma'}\sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} e^{\beta\sqrt{N}(X_\sigma+X_{\sigma'})} \tag{3.61}$$

First of all, the denominator is bounded from below by $[\widetilde{Z}_{\beta,N}(c)]^2$, and by (3.18), with probability of order $\delta^{-2}\exp(-Ng(c,\beta))$, this in turn is larger than $(1-\delta)^2[\mathbb{E}\widetilde{Z}_{\beta,N}(c)]^2$. Now let first $\beta<\sqrt{2\ln 2}$. Assume first that $I\subset(0,1)\cup[-1,0)$. We conclude that

$$\begin{aligned}\mathbb{E}f_{\beta,N}(I) &\leq \frac{1}{(1-\delta)^2}\mathbb{E}_\sigma\mathbb{E}_{\sigma'}\sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} 1+\delta^{-2}e^{-g(c,\beta)N}\\ &= \frac{1}{\sqrt{2\pi N}}\frac{1}{(1-\delta)^2}\sum_{t\in I}\frac{2e^{-N\phi(t)}}{1-t^2} + \delta^{-2}e^{-g(c,\beta)N}\end{aligned} \tag{3.62}$$

for any $\beta < c < \sqrt{2\ln 2}$, where $\phi : [-1,1] \to \mathbb{R}$ denotes the Cramèr entropy function

$$\phi(t) = \frac{(1+t)}{2}\ln(1+t) + \frac{(1-t)}{2}\ln(1-t) \tag{3.63}$$

Here we used of course that, firstly, if $(1-t)N = 2\ell$, $\ell = 0, \dots, N$, then

$$\mathbb{E}_\sigma \mathbb{E}_{\sigma'} \mathbb{1}_{R_N(\sigma,\sigma')=t} = 2^{-N}\binom{N}{\ell} \tag{3.64}$$

and, secondly, Stirling's approximation which implies that

$$\binom{N}{\ell} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{N}{\ell(N-\ell)}}\frac{N^N}{\ell^\ell (N-\ell)^{N-\ell}}(1+o(1)) \tag{3.65}$$

valid if $\ell \sim xN$ with $x \in (0,1)$. Under our assumptions on I, we see immediately from this representation that the right hand side of (3.62) is clearly exponentially small in N. If $1 \in I$, the additional term coming from $t = 1$ is precisely the term that we have estimated in (3.16), so that again this gives an exponentially small contribution. This shows that the measure $f_{\beta,N}$ concentrates asymptotically on the point 0. This proves (3.58).

Now let $\beta > \sqrt{2\ln 2}$. Here we use the sharper truncations introduced in 3.2.2. Note first that for any interval I

$$\left| f_{\beta,N}(I) - Z_{\beta,N}^{-2}\mathbb{E}_\sigma \mathbb{E}_{\sigma'} \sum_{\substack{t\in I\\ R_N(\sigma,\sigma')=t}} \mathbb{1}_{X_\sigma, X_{\sigma'} \geq u_N(x)} e^{\beta\sqrt{N}(X_\sigma + X_{\sigma'})} \right| \leq \frac{2Z_{\beta,N}^x}{Z_{\beta,N}} \tag{3.66}$$

The proof of Theorem 3.2 shows that the right hand side of (3.66) tends zero in probability as first $N \uparrow \infty$ and then $x \downarrow -\infty$. On the other hand, for $t \neq 1$

$$\begin{aligned} &\mathbb{P}\left[\exists_{\sigma,\sigma,:R_N(\sigma,\sigma')=t}\, X_\sigma > u_N(x) \wedge X'_\sigma > u_N(x)\right] \\ &\leq \mathbb{E}_\sigma \mathbb{1}_{R_N(\sigma,\sigma')=t}\, 2^{-2N}\mathbb{P}\left[X_\sigma > u_N(x)\right]^2 = \frac{2}{\sqrt{2\pi N}\sqrt{1-t^2}} e^{-\phi(t)N} e^{2x} \end{aligned} \tag{3.67}$$

by the definition of $u_N(x)$ (see (3.27)). This implies again that any interval $I \subset (0,1)\cup[-1,0)$ will have zero mass. To conclude the proof it will be enough to compute $f_{\beta,N}(1)$. Clearly

$$f_{\beta,N}(1) = \frac{2^{-N}\mathbb{Z}_{2\beta,N}}{Z_{\beta,N}^2} \tag{3.68}$$

By Theorem 3.2, (v), one sees easily that

$$f_{\beta,N}(1) \xrightarrow{\mathcal{D}} \frac{\int e^{2\alpha z}\mathcal{P}(dz)}{\left(\int e^{\alpha z}\mathcal{P}(dz)\right)^2} \tag{3.69}$$

Expressing the left hand side of (3.69) in terms of the point process $\mathcal{W}_\alpha$ defined in (3.44) yields the expression for the mass of the atom at 1; since the only other atom is at zero the full results follows from the fact that $f_{\beta,N}$ is a probability measure.

The assertions on the measure $\mathcal{K}_{\beta,N}$ are essentially a corollary of the preceeding results. The fact that f_β is a sum of δ_0 and δ_1 implies immediately that the probability that m_σ is not such a sum tends to zero. The explicit formula (3.60) is then quite straightforward. $\diamondsuit$

3.2.5. Multi-overlaps and Ghirlanda–Guerra identities.

It will be interesting to see that the random measures $\mathcal{K}_\beta$ can be controlled with the help of some remarkable algebraic identities that in fact allow us to avoid the detailed analysis of fluctuations performed in Section 3.2.2.

Let us first note that the convergence of the measures $\mathcal{K}_{\beta,N}$ can be controlled through their moments, which can be written als follows:

$$\begin{aligned}
&\mathbb{E}\left(\int \mathcal{K}_{\beta,N}(dm)m^{k_1}\cdots\int \mathcal{K}_{\beta,N}(dm)m^{k_l}\right)\\
&=\mathbb{E}\mu_{\beta,N}^{\otimes l}\left(m_{\sigma^1}^{k_1}(\cdot)\dots m_{\sigma^l}^{k_l}(\cdots)\right)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\cdots+k_l}\Big(R_N(\sigma^1,\sigma^{l+1})\in\cdot,\dots,R_N(\sigma^1,\sigma^{l+k_1})\in\cdot,\dots,\\
&\qquad\dots,R_N(\sigma^l,\sigma^{l+k_1+\cdots+k_{l-1}+1})\in\cdot,\dots,R_N(\sigma^l,\sigma^{l+k_1+\cdots+k_l})\in\cdot\Big)
\end{aligned}\tag{3.70}$$

The right hand side is a (marginal of) the distribution of the $m(m-1)$ replica overlaps under the averaged product Gibbs measure on $m=l+k_1+\cdots+k_{l-1}+1$ independent replicas of the spin variables. Thus, if we can show that these multi-replica distributions converge, as $N\uparrow\infty$, then the convergence of the measures $\mathcal{K}_{\beta,N}$ will be proven. This is a general fact, which has notheing to do with the particular model we look at. In the REM, of course, considerable simplification will take place since we know that the overlap takes only the values 0 and one in the limit, and thus instead of looking at the entore distributions, it will be enough to look at the atoms when overlaps equal to 1. That is to say it will be enough in our case to consider the numbers

$$
\begin{aligned}
&\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(R_N(\sigma^1,\sigma^{l+1})=1,\dots,R_N(\sigma^1,\sigma^{l+k_1})=1,\dots,\\
&\qquad \dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_{l-1}+1})=1,\dots,R_N(\sigma^l,\sigma^{l+k_1+\dots+k_l})=1\Big)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(\sigma^1=\sigma^{l+1},\dots,\sigma^1=\sigma^{l+k_1},\dots,\\
&\qquad \dots,\sigma^l=\sigma^{l+k_1+\dots+k_{l-1}+1},\dots,\sigma^l=\sigma^{l+k_1+\dots+k_l}\Big)\\
&=\mathbb{E}\mu_{\beta,N}^{l+k_1+\dots+k_l}\Big(\sigma^1=\sigma^{l+1}=\dots=\sigma^{l+k_1},\dots,\\
&\qquad \dots,\sigma^l=\sigma^{l+k_1+\dots+k_{l-1}+1}=\dots=\sigma^{l+k_1+\dots+k_l}\Big)
\end{aligned}
\tag{3.71}
$$

As we will show now, the multi-overlaps are not independent, but satisfy recursion relations that are due to rather general principles. It will be instructive to look at them in this simple context. These identities have been known in the physics literature and a more rigorous analysis is given in a paper by Girlanda and Guerra [GG]. Equivalent relations were in fact derived somewhat earlier by Aizenman and Contucci [AC].The importance of these relations has been underlined by Talagrand [T4,T7]. Let us begin with the simplest instance of these relations.

Proposition 3.8: *For any value of β,*

$$
\mathbb{E}\frac{d}{d\beta}F_{\beta,N}=-\beta(1-\mathbb{E}f_{\beta,N}(1))
\tag{3.72}
$$

Proof: Obviously,

$$
\mathbb{E}\frac{d}{d\beta}F_{\beta,N}=-N^{-1}\mathbb{E}\frac{\mathbb{E}_\sigma\sqrt{N}X_\sigma e^{\beta\sqrt{N}X_\sigma}}{\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma}}
\tag{3.73}
$$

Now if X is standard normal variable, and g any function of at most polynomial growth, then

$$
\mathbb{E}[Xg(X)]=\mathbb{E}g'(X)
\tag{3.74}
$$

Using this identity in the right hand side of (3.73) with respect to the average over X_σ, we get immediately that

$$
\begin{aligned}
\mathbb{E}\frac{\mathbb{E}_\sigma\sqrt{N}X_\sigma e^{\beta\sqrt{N}X_\sigma}}{\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma}}&=N\beta\mathbb{E}\left(1-\frac{2^{-N}\mathbb{E}_\sigma e^{2\beta\sqrt{N}X_\sigma}}{(\mathbb{E}_\sigma e^{\beta\sqrt{N}X_\sigma})^2}\right)\\
&=N\beta\mathbb{E}\left(1-\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{\sigma^1=\sigma^2})\right)
\end{aligned}
\tag{3.75}
$$

which is obviously the claim of the lemma.◇

In exactly the same way one can prove the following generalisation:

Lemma 3.9: *Let $h : \mathcal{S}_N^n \to \mathbb{R}$ be any bounded function of n spins. Then*

$$\begin{aligned} &\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k}h(\sigma^1,\dots,\sigma^n)\right) \\ &= \beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^{n}\mathbb{1}_{\sigma^k=\sigma^l} - n\mathbb{1}_{\sigma^k=\sigma^{n+1}}\right)\right) \end{aligned} \tag{3.76}$$

Proof: Left as an exercise.◇

The strength of Lemma 3.9 comes out when combined with a factorization result that in turn is a consequence of self-averaging.

Lemma 3.10: *Let h be as in the previous lemma. For all but possibly a countable number of values of β,*

$$\lim_{N\uparrow\infty}\frac{1}{\sqrt{N}}\left|\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k}h(\sigma^1,\dots,\sigma^n)\right) - \mathbb{E}\mu_{\beta,N}\left(X_{\sigma^k}\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right| = 0 \tag{3.77}$$

Proof: Let us write

$$\begin{aligned} &\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k}h(\sigma^1,\dots,\sigma^n)\right) - \mathbb{E}\mu_{\beta,N}\left(X_{\sigma^k}\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right)^2 \\ &= \left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(\left(X_{\sigma^k} - \mathbb{E}\mu_{\beta,N}^{\otimes n}X_{\sigma^k}\right)h(\sigma^1,\dots,\sigma^n)\right)\right)^2 \\ &\leq \mathbb{E}\mu_{\beta,N}^{\otimes n}\left(X_{\sigma^k} - \mathbb{E}\mu_{\beta,N}^{\otimes n}X_{\sigma^k}\right)^2\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)^2 \end{aligned} \tag{3.78}$$

where the last inequality is the Cauchy–Schwarz inequality applied to the joint expectation with respect to the Gibbs measure and the disorder. Obviously the first factor in the last line is equal to

$$\begin{aligned} &\mathbb{E}\left(\mu_{\beta,N}(X_\sigma^2) - [\mu_{\beta,N}(X_\sigma)]^2\right) + \mathbb{E}\left(\mu_{\beta,N}(X_\sigma) - \mathbb{E}\mu_{\beta,N}(X_\sigma)\right)^2 \\ &= -\beta^{-2}\mathbb{E}\frac{d^2}{d\beta^2}F_{\beta,N} + N\beta^{-2}\mathbb{E}\left(\frac{d}{d\beta}F_{\beta,N} - \mathbb{E}\frac{d}{d\beta}F_{\beta,N}\right)^2 \end{aligned} \tag{3.79}$$

We know that $F_{\beta,N}$ converges as $N\uparrow\infty$ and that the limit is infinitely differentiable for all $\beta \geq 0$, except at $\beta = \sqrt{2\ln 2}$; moreover, $-F_{\beta,N}$ is convex in β. Then standard results of convex analysis imply that

$$\limsup_{N\uparrow\infty}\left(-\mathbb{E}\frac{d^2}{d\beta^2}F_{\beta,N}\right) = -\frac{d^2}{d\beta^2}\lim_{N\uparrow\infty}\mathbb{E}F_{\beta,N} \tag{3.80}$$

which is finite for all $\beta \neq \sqrt{2\ln 2}$. Thus, the first term in (3.79) will vanish when divided by N. To see that the coefficient of N of the second term gives a vanishing contribution, we use the general fact that if the variance of family of a convex (or concave) functions tends to zero, then the same is true for its derivative, except possibly on a countable set of values of their argument. In Theorem 3.2 we have more than established that the variance of $F_{\beta,N}$ tends to zero, and hence the result of the Lemma is proven. ◇

If we combine Proposition 3.8, Lemma 3.9, and Lemma 3.10 we arrive immediately at

Proposition 3.11: *For all but a countable set of values β, for any bounded function h : $\mathcal{S}_N^n \to \mathbb{R}$,*

$$\begin{aligned}\lim_{N\uparrow\infty}\Bigg|&\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\mathbb{1}_{\sigma^k=\sigma^{n+1}}\right)\\ &-\frac{1}{n}\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^{n}\mathbb{1}_{\sigma^l=\sigma^k}+\mathbb{E}\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{\sigma^1=\sigma^2})\right)\right)\Bigg|=0\end{aligned}\tag{3.81}$$

Together with the fact that the product Gibbs measures are concentrated only on the sets where the overlaps take values 0 and 1, (3.81) permits to compute the distribution of all higher overlaps in terms of the two-replica overlap. E.g., if we put

$$A_n \equiv \lim_{N\uparrow\infty}\mathbb{E}\mu_{\beta,N}^{\otimes n}(\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n})\tag{3.82}$$

then (3.81) with $h=\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n}$ provides the recursion

$$\begin{aligned}A_{n+1}&=\frac{n-1}{n}A_n+\frac{1}{n}A_nA_2=A_n\left(1-\frac{1-A_2}{n}\right)\\ &=\prod_{k=2}^{n}\left(1-\frac{1-A_2}{k}\right)A_2\\ &=\frac{\Gamma(n+A_2)}{\Gamma(n+1)\Gamma(A_2)}\end{aligned}\tag{3.83}$$

Note that we can use alternatively Theorem 3.4 to compute, for the non-trivial case $\beta > \sqrt{2\ln 2}$,

$$\lim_{N\uparrow\infty}\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{\sigma^1=\sigma^2=\dots=\sigma^n})=\int\mathcal{K}_\beta(dm)[m(1)]^{n-1}\tag{3.84}$$

so that (3.83) implies a formula for the mean of the n-th moments of $\mathcal{W}$,

$$\mathbb{E}\int\mathcal{W}(dy,dw)w^n=\frac{\Gamma(n+A_2)}{\Gamma(n+1)\Gamma(A_2)}\tag{3.85}$$

where $A_2 = \mathbb{E}\int \mathcal{W}(dy, dw)w^2$. This result has been obtained by a direct computation by Ruelle ([Ru], Corollary 2.2), but its derivation via the Ghirlanda–Guerra identities shows a way to approach this problem in a different manner that has the potential to give results in more complicated situations.[6]

4. The Derrida models.

The reader of the previous chapter may think that that was 'much ado about nothing'. First, it was all about independent random variables, second, we used heavy tools to describe structure that is in fact very simple. We will now move towards a class of models that have been introduced 17 years ago by Derrida as "simplified" spin glass models. It turns out that while these models exhibit structure that is as complex as (and in fact almost identical to) in the Sherrington-Kirkpatrick type spin glasses, they can now be analysed with full rigor whith the help of the tools I have explained in the previous section. The results of these Section cover recent work with Irina Kurkova [BK1,BK2]. The purpose of this section is to explain how the remarkable universal structures predicted by Parisi's replica symmetry breaking scheme arise as a limiting object in a spin glass model. For further analysis of the limting object itself we refer to a paper by Bolthausen and Sznitman [BoSz].

4.1. Definitions and basics.

As we have already pointed out in the introduction, from a mathematial point of view it is natural to embed the SK models in the general stting of models based on Gaussian processes on the hypercubes $\mathcal{S}_N$. The special feature of the SK models in that context is than that their covariance depends only on the "overlap", $R_N(\sigma, \sigma') = \frac{1}{N}(\sigma, \sigma')$.

Derrida introduced annother class of models that he called *Generalized Random Energy models* (GREM) that can be constructed in full analogy to the SK class by introducing annother function charcterizing distance that is to replace the overlap R_N, namely

$$\delta_N(\sigma, \sigma') \equiv \frac{1}{N}\left(\min(i : \sigma_i \neq \sigma'_i) - 1\right) \tag{4.1}$$

To be precise, δ_N is an *ultrametric* valueation on the set $\mathcal{S}_N$. An ultrametric distance would be given e.g. by a function $D(\sigma, \sigma') = \exp(-\delta_N(\sigma, \sigma'))$. We will now consider centered Gaussian processes X_σ on $\mathcal{S}_N$ those covariance is given as

$$\mathrm{cov}\,(X_\sigma, X - \sigma') = \mathbb{E}X_\sigma X_{\sigma'} = A(\delta_N(\sigma, \sigma')) \tag{4.2}$$

[6] More generally, one may dervive recursion formulas for more general moments of Ruelle's process that show that the identies (3.81) determine completely the process of Ruelle in terms of the two-overlap distribution function.

where A is a probability distribution function on the interval $[0,1]$.

In fact, the original models of Derrida correspond to the special case when A is the distribution function of a random variable that takes only finitely many values, i.e. when A is a monotone increasing step function with finitely many steps. However, Derrida also considered limits when the number of these steps tend to infinty.

The choice of the distance δ_N has a number of remarkable effect that help to make these models truely solvable. In particular, it allows to introduce a continuous time martingale $X_\sigma(t)$ those marginal at $t=1$ coincides with X_σ. This process is simply a Gaussian process on $\mathcal{S}_N \times [0,1]$ with covariance

$$\operatorname{cov}\left(X_\sigma(t), X-\sigma'(t')\right) = t \wedge t' \wedge A(\delta_N(\sigma,\sigma')) \tag{4.3}$$

In particular, this gives rise to the integral representation of X_σ as

$$X_\sigma = \int_0^1 dX_\sigma(t) \tag{4.4}$$

where the increments satisfy

$$\mathbb{E} dX_\sigma(t) dX_{\sigma'}(t') = dt ds \delta(t-t') 1\!\!1_{A(\delta_N(\sigma,\sigma'))>t} \tag{4.5}$$

If A is a step function, this gives rise to a representation in the form

$$X_\sigma \equiv \sqrt{a_1} X_{\sigma_1} + \sqrt{a_2} X_{\sigma_1\sigma_2} + \dots + \sqrt{a_n} X_{\sigma_1\sigma_2\dots\sigma_n}, \quad \text{if } \sigma = \sigma_1\sigma_2\dots\sigma_n, \tag{4.6}$$

where a_i is the increment of A at the step point $q_i = \sum_{j=1} i \frac{\ln \alpha_j}{\ln 2}$, and $\sigma = \sigma - 1\sigma_2 \dots \sigma_n$ with $\sigma_i \in \{-1,1\}^{\ln \alpha_i N}$.

Note that in the SK class, neither is it possible to construct such a represntation, nor are step functions allowed as covariances.

The representation (4.7) allows explicit computations of the partition function. This was done first by Derrida and Gardner [DG1], and in full generality (and with full rigor) by Cappocaccia, Cassandro, and Picco [CaCaPi]. While we will not reproduce this calculations (they are in spirit not very different from those in the REM and make use of (4.6) to set up a recursive scheme), we will state their result in a particularly useful form.

Let us denote the convex hull of the function $A(x)$ by $\bar{A}(x)$. We will also need the left-derivative of this function, $\bar{a}(x) \equiv \lim_{\epsilon\downarrow 0} \epsilon^{-1}(\bar{A}(x) - \bar{A}(x-\epsilon))$ which exists for all values of $x \in (0,1]$.

Theorem 4.1: *Whenever A is a step function with finitely many steps, the free energy $F_{\beta,N} \equiv \frac{1}{N}\ln Z_{\beta,N}$ converges almost surely to the non-random limit F_β given by*

$$F_\beta = \sqrt{2\ln 2}\beta \int_0^{x(\beta)} \sqrt{\bar{a}(x)}dx + \frac{\beta^2}{2}(1 - \bar{A}(x(\beta))) \tag{4.7}$$

where

$$x(\beta) \equiv \sup\left(x|\bar{a}(x) > \frac{2\ln 2}{\beta^2}\right) \tag{4.8}$$

It is also very easy to derive from (4.7) an explicit fromula for the distance-distribution function

$$f_{\beta,N}(x) \equiv \mu_{\beta,N}^{\otimes 2}(\delta_N(\sigma,\sigma') < x) \tag{4.9}$$

This just makes use of the fact that

Proposition 4.2: *For any value of β, and any $i = 1,\dots,n$,*

$$\mathbb{E}\frac{d}{d\sqrt{a}_i}F_{\beta,N} = -\beta^2\sqrt{a_i}\mathbb{E}f_{\beta,N}(q < q_i) \tag{4.10}$$

with the convention that $q_0 = 0$ and $q_n = 1$.

This implies in fact immediately that

Theorem 4.3: *Whenever A is a step function with finitely many steps, the $f_{\beta,N}$ converges in mean to the limiting function*

$$\mathbb{E}f_\beta(x) = \begin{cases} \beta^{-1}\sqrt{2\ln 2}/\sqrt{\bar{a}(x)}, & \text{if } x \leq x_\beta \\ 1, & \textit{if } x > x_\beta \end{cases} \tag{4.11}$$

It is obvious that if A_n is a sequence of step functions that converges to a limiting function A, then the sequences of free energies and distance distributions converge. It in not very difficult to show ([BK2]) that these limits then are in fact the free energies and distribution functions for the corresponding models with arbirtrary A.

4.2. Gibbs measures and point processes.

As in the case of the REM, Ruelle [Ru] had proposed an effective model for the thermodynamic limit of the GREM in terms of Poisson processes, or rather *"Poisson cascades"*, i.e. nested sequences of Poisson processes, without establishing a rigorous relation between the two models. Ruelle also constructed limiting objects of his processes when the number of "levels" (i.e. n) tends to infinity. The connection between Ruelle's models and the GREMs with finitely many levels have been made rigorous in [BK1]. While again in spirit the proofs are similar to those in the REM, they require considerably more computations.

However, it is quite remarkable that via the Ghirlanda-Guerra relations, one can construct (at least in principle) the thermodynamic limit on the level of the measures on the mass distribution without much explicit computation even in the case of arbitrary A.

It will be convenient to introduce here the analogues of the random measures $\mathcal{K}$ defined above where the overlap R_N is replaced by the distance δ_N. I.e. we set now

$$m_\sigma(x) \equiv \mu_{\beta,N}(\sigma : \delta_N(\sigma',\sigma) > x) \tag{4.12}$$

and

$$\mathcal{K}_{\beta,N} \equiv \sum_{\sigma\in\mathcal{S}_N} \mu_{\beta,N}(\sigma)\delta_{m_\sigma(\cdot)} \tag{4.13}$$

In the case when A is a step function with finitely many steps, one can control the convergence of $\mathcal{K}_{\beta,N}$ to a limit rather explicitely. We will present the corresponding results, without proof, below.

In the general case, this will no longer be possible. However, the Ghirlanda-Guerra identities will allow again to prove the existence of the limit and to decribe its properties. The key point to notice is that to prove convergence, it is enough to prove convergence of all expressions of the form

$$\begin{aligned}\mathbb{E}\Bigg(\bigg(\int \mathcal{K}_{\beta,N}(dm)m(\Delta_{11})^{r_{11}}\dots m(\Delta_{1j_1})^{r_{1j_1}}\bigg)^{q_1}\dots\\ \dots\bigg(\int \mathcal{K}_{\beta,N}(dm)m(\Delta_{l1})^{r_{l1}}\dots m(\Delta_{lj_l})^{r_{lj_l}}\bigg)^{q_l}\Bigg)\end{aligned} \tag{4.14}$$

where $\Delta_{ij}\subset[0,1]$ and q_i, r_{ij} are integers.

The key point will be to establish again the Ghirlanda-Guerra identities. In this the process $X_\sigma(t)$ plays a crucial rôle.

In this context, it will be convenient to use the process

$$Y_\sigma(t) \equiv X_\sigma(A(t) \tag{4.15}$$

Theorem 4.4: *For any $n \in \mathbb{N}$ and any $x \in [0,1]\backslash x_\beta$,*

$$\lim_{N\uparrow\infty}\Bigg|\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\mathbb{1}_{A(\delta_N(\sigma^k,\sigma^{n+1}))\geq x}\right) - \frac{1}{n}\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l\neq k}^{n}\mathbb{1}_{A(\delta_N(\sigma^k,\sigma^l))\geq x} + \mathbb{E}\mu_{\beta,N}^{\otimes 2}(\mathbb{1}_{A(\delta_N(\sigma^1,\sigma^2))\geq x})\right)\right)\Bigg| = 0 \tag{4.16}$$

Proof: As a first step we need the following lemma.

Lemma 4.5: *For any $t \in (0,1]$, and let $h : \mathcal{S}_N^n \to \mathbb{R}$ be any bounded function of n spins*

$$\begin{aligned}&\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(dY_{\sigma^k}(t)h(\sigma^1,\dots,\sigma^n)\right)\\ &= \beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^{n}\mathbb{1}_{\delta_N(\sigma^k,\sigma^l)\geq t} - n\mathbb{1}_{\delta_N(\sigma^k,\sigma^{n+1})\geq t}\right)\right)dA(t)\end{aligned} \tag{4.17}$$

Proof: The proof makes use of the Gaussian integration by parts formula

$$\begin{aligned}\mathbb{E}dX_\sigma(t)f\left(\int dX_{\sigma'}(s)\right) &= \mathbb{E}f'\left(\int dX_{\sigma'}(s)\right)\int\mathbb{E}dX_\sigma(t)dX_{\sigma'}(s)\\ &= \mathbb{E}f'\left(X_{\sigma'}\right)\mathbb{1}_{A(\delta_N(\sigma,\sigma'))\geq t}dt\end{aligned} \tag{4.18}$$

where f is any differentiable function. Applying this to the left hand side Note that the left hand side of (4.17) can be written as

$$N^{-1/2}\mathbb{E}\mathbb{E}_{\sigma^1\dots\sigma^n}h(\sigma^1,\dots,\sigma^n)dX_{\sigma^k}(t)\prod_{l=1}^{n}f\left(X_{\sigma^l}\right) \tag{4.19}$$

with

$$f\left(X_{\sigma^l}\right) = \frac{e^{\beta\sqrt{N}X_{\sigma^l}(1)}}{\mathbb{E}_{\sigma^l}e^{\beta\sqrt{N}X_{\sigma^l}(1)}} \tag{4.20}$$

Using (4.18) gives readily

$$\begin{aligned}&\frac{1}{\sqrt{N}}\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(dY_{\sigma^k}(t)h(\sigma^1,\dots,\sigma^n)\right)\\ &= \beta\mathbb{E}\mu_{\beta,N}^{\otimes n+1}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^{n}\mathbb{1}_{A(\delta_N(\sigma^k,\sigma^l))\geq t} - n\mathbb{1}_{A(\delta_N(\sigma^k,\sigma^{n+1}))\geq t}\right)\right)dt\end{aligned} \tag{4.21}$$

Realizing that $A(\delta_N(\sigma,\sigma')) < A(t)$ is equivalent to $\delta_N(\sigma,\sigma') < t$ whenever $A(t)$ is not constant then yields the claim of the lemma. $\diamondsuit$

The more important step is the proof is contained in the next lemma.

Lemma 4.6: *Let h be as in the previous lemma. Except possibly when $t = x_\beta$,*

$$\begin{aligned}\lim_{N\uparrow\infty}\frac{1}{\sqrt{N}}\Big|&\mathbb{E}\mu_{\beta,N}^{\otimes n}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))h(\sigma^1,\dots,\sigma^n)\right)\\ &-\mathbb{E}\mu_{\beta,N}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\Big|=0\end{aligned} \tag{4.22}$$

Proof: Let us write

$$\begin{aligned}&\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)-\mathbb{E}\mu_{\beta,N}\left(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon)\right)\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\ &=\left(\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}^{\otimes n}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)h(\sigma^1,\dots,\sigma^n)\right)\right)^2\\ &\leq\mathbb{E}\mu_{\beta,N}^{\otimes n}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}^{\otimes n}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\mathbb{E}\mu_{\beta,N}^{\otimes n}\left(h(\sigma^1,\dots,\sigma^n)\right)^2\end{aligned} \tag{4.23}$$

where the last inequality is the Cauchy–Schwarz inequality applied to the joint expectation with respect to the Gibbs measure and the disorder. Obviously the first factor in the last line is equal to

$$\begin{aligned}&\mathbb{E}\mu_{\beta,N}\left((Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\\ &\quad+\mathbb{E}\left(\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))-\mathbb{E}\mu_{\beta,N}(Y_{\sigma^k}(t)-Y_{\sigma^k}(t-\epsilon))\right)^2\end{aligned} \tag{4.24}$$

Now let us introduce the deformed process

$$X_\sigma^u \equiv X_\sigma + u\left(Y_\sigma(t)-Y_\sigma(t-\epsilon)\right) \tag{4.25}$$

If we denote by $F_{\beta,N}^u$ the free energy corresponding to this deformed process, the last line of (4.24) can be represented as

$$\beta^{-2}\mathbb{E}\frac{d^2}{du^2}F_{\beta,N}^u+N\beta^{-2}\mathbb{E}\left(\frac{d}{du}F_{\beta,N}^u-\mathbb{E}\frac{d}{du}F_{\beta,N}^u\right)^2 \tag{4.26}$$

At this point we need a concentration result on the free energy. which we state here without proof.

Lemma 4.7: *For any β, and any covariance distribution A, for any $\epsilon \geq 0$*

$$\mathbb{P}\left[|F_{\beta,N} - \mathbb{E}F_{\beta,N}| > r\right] \leq 2\exp\left(-\frac{r^2 N}{2\beta^2}\right) \tag{4.27}$$

We know that $F^u_{\beta,N}$ converges as $N \uparrow \infty$ and that the limit is infinitely differentiable as a function of u, except possibly when $x_\beta = t$; moreover, $-F^u_{\beta,N}$ is convex in the variable u. This can be seen by explicit computation using the expression (4.7) for the free energy. Then a standard result of convex analysis (see [Ro], Theorem 25.7) imply that

$$\limsup_{N\uparrow\infty}(-\mathbb{E}\frac{d^2}{du^2}F^u_{\beta,N}) = -\frac{d^2}{du^2}\lim_{N\uparrow\infty}\mathbb{E}F^u_{\beta,N} \tag{4.28}$$

which is finite at zero except possibly if $x_\beta = t$. Thus, the first term in (3.79) will vanish when divided by N. To see that the coefficient of N of the second term gives a vanishing contribution, we use the general fact that if the variance of family of a convex (or concave) functions tends to zero, then the same is true for its derivative, provided the second derivative of the expectation is bounded (see e.g. Lemma 8.9 in [BG], or Proposition 4.3 in [TaHopf]).

But by Lemma 4.7 the variance of $F_{\beta,N}$ tends to zero, and (4.28) implies that $\mathbb{E}\frac{d^2}{du^2}F^u_{\beta,N}$ is bounded for large enough N whenever $\frac{d^2}{du^2}\mathbb{E}F^u_\beta$ is finite. Hence the result of the lemma is proven. $\diamondsuit$

To prove the theorem we use integrate (4.17) and then use (4.22) on the left hand side. This gives

$$\begin{aligned}\lim_{N\uparrow\infty}\Bigg(&\frac{1}{\sqrt{N}}\mathbb{E}\mu^{\otimes n}_{\beta,N}\left(Y_{\sigma^k}(t) - Y_{s^k}(t-\epsilon)\right)\mathbb{E}\mu^{\otimes n}_{\beta,N}\left(h(\sigma^1,\dots,\sigma^n)\right)\\ &-\beta\int_{t-\epsilon}^{t}\left(\mathbb{E}\mu^{\otimes n+1}_{\beta,N}\left(h(\sigma^1,\dots,\sigma^n)\left(\sum_{l=1}^{n}\mathbb{1}_{\delta_N(\sigma^k,\sigma^l)\geq s} - n\mathbb{1}_{\delta_N(\sigma^k,\sigma^{n+1})\geq s}\right)\right)\right)dA(s)\Bigg) = 0\end{aligned} \tag{4.29}$$

Finally, we use once more (4.17) with $n = 1$ to express $\mathbb{E}\mu^{\otimes n}_{\beta,N}\left(Y_{\sigma^k}(t) - Y_{s^k}(t-\epsilon)\right)$ in terms of the two replica distribution. The final result follows by trivial algebraic manipulations and the fact that ϵ is arbitrary. $\diamondsuit\diamondsuit$

Following [GG], we now define the family of measures $\mathbb{Q}^{(n)}_N$ on the space $[0,1]^{n(n-1)/2}$.

$$\mathbb{Q}^{(n)}_{\beta,N}(\underline{\delta_N} \in \mathcal{A}) \equiv \mathbb{E}\mu^{\otimes n}_{N,\beta}\left[\underline{\delta_N} \in \mathcal{A}\right] \tag{4.30}$$

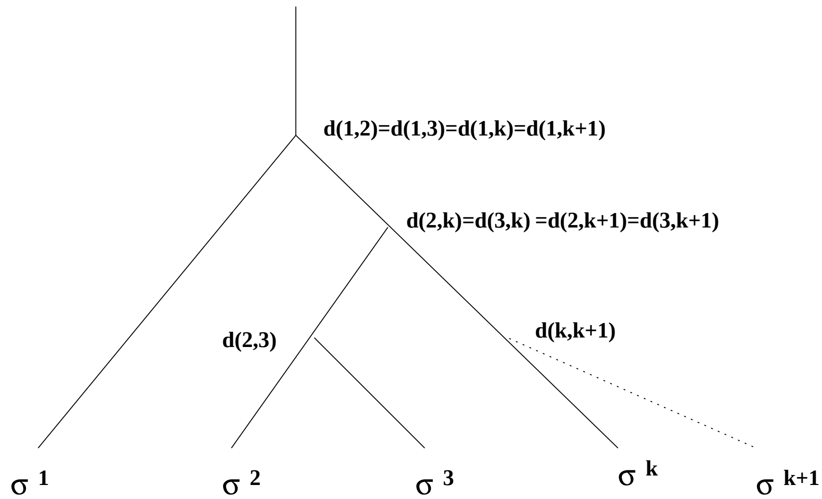
where $\underline{\delta_N}$ denotes the vector of replica distances whose components are $\delta_N(\sigma^l, \sigma^k)$, $1 \leq l < k \leq n$. Denote by $\mathcal{B}_k$ the sigma-algebra generated by the first $k(k-1)/2$ coordinates, and let A be a Borel set in $[0,1]$.

Theorem 4.8: *The family of measures $\mathbb{Q}^{(n)}_{\beta,N}$ converge to limiting measures $\mathbb{Q}^{(n)}_{\beta}$ for all finite n, as $N \uparrow \infty$. Moreover, these measures are uniquely determined by the distance distribution functions f_β. They satisfy the identities*

$$\mathbb{Q}^{(n+1)}_{\beta}\left(d_{k,n+1} \in A | \mathcal{B}_n\right) = \frac{1}{n}\mathbb{Q}^{(2)}_{\beta}(A) + \frac{1}{n}\sum_{l \neq k}^{n} \mathbb{Q}^{(n)}_{\beta}\left(d_{k,l} \in A | \mathcal{B}_n\right) \tag{4.31}$$

for any Borel set A.

Proof: Choosing h as the indicator function of any desired event in $\mathcal{B}_k$, one sees that (4.16) implies (4.31). This actually implies that in the limit $N \uparrow \infty$, the family of measures $\mathbb{Q}^{(n)}_{\beta,N}$ is entirely determined by the two-replica distribution function. While this may not appear obvious, it follows when taking into account the ultrametric property of the function δ_N. This is most easily seen by realising that the prescription of the mutual distances between k spin configurations amounts to prescribing a tree (start all k configurations at the origin and continue on top of each other as long as the coordinates coincide, then branch of). To determine the full tree of $k+1$ configurations, it is sufficient to know the overlap of configuration $\sigma^{(k+1)}$ with the configuration it has maximal overlap with, since then all overlaps with all other configurations are determined. But the corresponding probabilities can be computed recursively via (4.31).



The distance d(k,k+1) determines all other distances d(j,k+1)

Now we have already seen that $\mathbb{Q}^{(2)}_{\beta,N} = \mathbb{E}\tilde{f}_{\beta,N}$ converges. Therefore the relation (4.31) implies the convergence of all distributions $\mathbb{Q}^{(n)}_{\beta,N}$, and proves the relation (4.31) hold for the limiting measures. $\diamondsuit$

Now it is clear that all expressions of the form (3.54) (with R_N replaced by δ_N) can be expressed in terms of the measures $\mathbb{Q}^{(k)}_{\beta,N}$ for k sufficiently large (we leave this as an exercise for the reader to write down). Thus, Theorem 4.8 implies in turn the convergence of the process $\mathcal{K}_{\beta,N}$ to a limit $\mathcal{K}_\beta$.

A remarkable feature takes place again if we are only interested in the marginal process $K_\beta(t)$ for fixed t. This process is a simple point process on $[0,1]$ and is fully determined in terms of the moments

$$\begin{aligned}
&\mathbb{E}\left(\int K_{\beta,N}(t)(dx)x^{r_1}\cdots\int K_{\beta,N}(t)(dx)x^{r_j}\right)\\
&=\mathbb{E}\mu_{\beta,N}^{\otimes r_1+\cdots+r_j+j}\Big(\delta_N(\sigma^1,\sigma^{j+1})>t,\dots,\delta_N(\sigma^1,\sigma^{j+r_1})>t,\dots,\\
&\quad\dots,\delta_N(\sigma^j,\sigma^{j+r_1+\cdots+r_{j-1}+1})>t,\dots\delta_N(\sigma^j,\sigma^{j+r_1+\cdots+r_j})>t\Big)
\end{aligned}\tag{4.32}$$

This restricted family of moments satisfies via the Ghirlanda-Guerra identities exactly the same recursion as in the case of the REM. This implies:

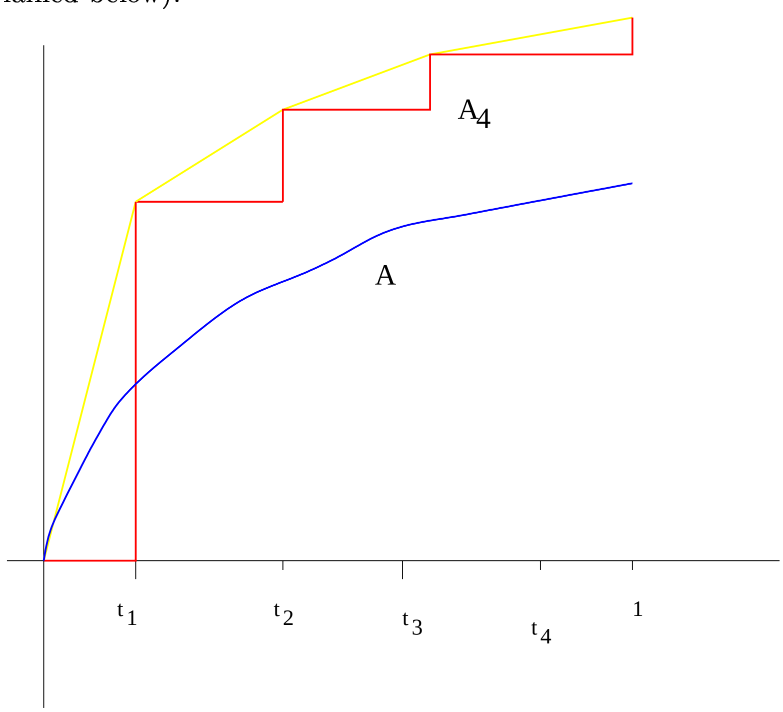
Theorem 4.9: *Assume that t is such that $\mathbb{E}\mu_\beta^{\otimes 2}(\delta(\sigma,\sigma')>t)>0$. Then the random* *measure $K_\beta(t)$ is a pure point measure obtained from a normalized Poisson process (i.e. from Poisson-Dirichlet process).*

In fact much more is true. We can consider the processes on arbitrary finite dimensional marginals, i.e.

$$K_{\beta,N}(t_1,\dots,t_m) \equiv \sum_{\sigma\in\mathcal{S}_N} \mu_{\beta,N}(\sigma)\delta_{m_\sigma(t_1),\dots,m_\sigma(t_m)} \tag{4.33}$$

for $0 < t_1 < \cdots < t_m < 1$. The point is that this process is entirely determined by the expressions (4.14) with the Δ_{ij} all of the form $(t_i, 1]$ for t_i in the fixed set of values $t_1,\dots,t_m$. This in turn implies that the process is determined by the multi-replica distribution functions $\mathcal{Q}^{(n)}_{\beta,N}$ restricted to the discrete set of events $\{\delta_N(\sigma^i,\sigma^j) > t_k\}$. Since these numbers are totally determined through the Ghirlanda-Guerra identities, they depend only on the values of the two-replica distribution function on those values. In particular, they are the same for processes with different correlation functions A, provided they give the same replica distribution function at these values! This allows us to construct, given A, and the set $t_1,\dots,t_n$, a new function $\widetilde{A}_n$ that is a step function with n steps for which $K_\beta(t_1,\dots,t_m)$ will be perfectly identical.

But the model with covariance $\widetilde{A}_n$ is a GREM with finitely many levels, for which the process $\mathcal{K}_\beta$ can be constructed explicitely in terms of Poisson cascades (this is shown in [BK1], and will be explained below).



In this sense we obtain an astonishingly explicit description of the limiting mass distribution function $\mathcal{K}_\beta$.

Probability cascades in the GREM with finitely many levels.

Let us now briefly explain the structure of the process $\mathcal{K}_\beta$ in the case when A_n is a step function with steps of hight a_i at the values $t_i \equiv \frac{\ln \alpha_i}{\ln 2}$. To avoid complications, we will assume that the linear interpolation of this function is convex, and that all points t_i belong to the extremal set of the convex hull.

Remark: I will not give the proofs here, that are somewhat involved, in particular when the general case is considered. They can be found in [BK1]. The following summary of results is in fact just a cooked down version of the complete analysis of the GREM with finitely many hierarchies given there.

Note that in this setting the Gaussian process X_σ can be represented as a sum

$$X_\sigma = \sqrt{a_1} Y_{\sigma_1} + \dots + \sqrt{a_n} Y_{\sigma_1 \dots, \sigma_n} \tag{4.34}$$

where $\sigma_i \in \{-1,1\}^{N \ln \alpha_i}$, and all $Y_{\sigma_1 \dots \sigma_i}$ are i.i.d. standard normal r.v.'s.

We introduce the function $u_{\ln \alpha, N}(x)$, $x \in \mathbb{R}$, such that $P(X > u_{\ln \alpha, N}(x)) \sim \alpha^{-N} e^{-x}$, as $N \uparrow \infty$, where X is a standard Gaussian random variable. It can be written explicitly as (see e.g. [LLR], page 12?)

$$u_{\ln \alpha, N}(x) = \sqrt{2 \ln \alpha N} + \frac{x}{\sqrt{2 \ln \alpha N}} - \frac{\ln N + \ln \ln \alpha + \ln 4\pi}{2\sqrt{2 \ln \alpha N}}. \tag{4.35}$$

Note that then for all i,

$$\sum_{\sigma_i} \delta_{u^{-1}_{\ln \alpha_i, N}(Y_{\sigma_1 \dots \sigma_{i-1} \sigma_i})} \to \mathcal{P}_i \tag{4.36}$$

where $\mathcal{P}_i$ are all independent Poisson point processes on $\mathbb{R}$ with intensity measure $e^{-x}dx$. Then under the assumptions on A, the following result holds:

Theorem 4.10: *Then the following point processes on* $\mathbb{R}^k$

$$\mathcal{P}_N^{(k)} \equiv \sum_{\sigma_1} \delta_{u^{-1}_{\ln \alpha_1, N}(Y_{\sigma_1})} \sum_{\sigma_2} \delta_{u^{-1}_{\ln \alpha_2, N}(Y_{\sigma_1 \sigma_2})} \cdots \sum_{\sigma_k} \delta_{u^{-1}_{\ln \alpha_k, N}(Y_{\sigma_1 \sigma_2 \dots \sigma_k})} \to \mathcal{P}^{(k)}$$

converge weakly to point process $\mathcal{P}^{(k)}$ *on* $\mathbb{R}^k$*, which is characterised by the following generating functions:*

$$\begin{aligned} F_{\Delta_1 \times \dots \times \Delta_k}(z) &\equiv \mathbb{E} z^{\sum_{x_1} \mathbb{1}_{\{x_1 \in \Delta_1\}} \cdots \sum_{x_k} \mathbb{1}_{\{x_k \in \Delta_k\}}} \\ &= f_{1,\Delta_1}(f_{2,\Delta_2}(f_{3,\Delta_3} \cdots (f_{k-1,\Delta_{k-1}}(f_{k,\Delta_k}(z))) \cdots)), \qquad |z| < 1 \end{aligned} \tag{4.37}$$

where $f_{i,\Delta_i}(z) = e^{K_i(z-1)(e^{-a_i}-e^{-b_i})}$, $\Delta_i = (a_i, b_i]$ *with* $a_i, b_i \in \mathbb{R}$ *or* $b_i = \infty$, $i = 1, 2, \ldots, k$.

Moreover, the following independence properties of the counting random variables of the process $\mathcal{P}^{(k)}$, $\sum_{x_1} \mathbb{1}_{\{x_1 \in \Delta_1^j\}} \cdots \sum_{x_k} \mathbb{1}_{\{x_k \in \Delta_k^j\}}$, *corresponding to the intervals* $\Delta_1^j \times \cdots \times \Delta_k^j$, $\Delta_i^j = [a_i^j, b_i^j)$, $j = 1, 2, \ldots, k$, $k > 1$, *hold true:*

(i) If the first components of these intervals are disjoint, i.e. $a_1^1 \leq b_1^1 \leq a_1^2 \leq b_1^2 \leq \cdots a_1^k \leq b_1^k$, *then these r.v. are independent.*

(ii) If the first $l-1$ *components of these intervals coincide and the* l*th components are disjoint, i.e.* $\Delta_i^1 = \cdots = \Delta_i^k$ *for* $i = 1, \ldots, l-1$ *and* $a_l^1 \leq b_l^1 \leq a_l^2 \leq b_l^2 \leq \cdots a_l^k \leq b_l^k$, *then these r.v. are conditionally independent under condition that* $\sum_{x_1} \mathbb{1}_{\{x_1 \in \Delta_1\}} \cdots \sum_{x_{l-1}} \mathbb{1}_{\{x_{l-1} \in \Delta_{l-1}\}}$ *is fixed.*

Remark: This theorem was proven for $k = 2$ in [GMP].

We would like to clarify an intuitive construction of the process $\mathcal{P}$. If $k = 1$, this is just a Poisson point process on $\mathbb{R}$ with intensity measure $K_1 e^{-x} dx$. To construct $\mathcal{P}$ on $\mathbb{R}^2$ for $k = 2$ we place the process $\mathcal{P}$ for $k = 1$ on the axis of the first coordinate and through each of its points draw a straight line parallel to the axis of the second coordinate. Then we put on each of these lines independently a Poisson point process with intensity measure $K_2 e^{-x} dx$. These points on $\mathbb{R}^2$ form the process $\mathcal{P}$ with $k = 2$. Whenever $\mathcal{P}$ is constructed for $k-1$, we place it on the plane of the first $k-1$ coordinates and through each of its points draw a straight line parallel to the axis of the nth coordinate. On each of these lines we put after independently a Poisson point process with intensity measure $K_k e^{-x} dx$. These points constitute $\mathcal{P}$ on $\mathbb{R}^k$. Indeed, the projection of $\mathcal{P}^{(k)}$ in $\mathbb{R}^k$ to the plane of the first ℓ coordinates is distributed as the process $\mathcal{P}^{(\ell)}$ in $\mathbb{R}^\ell$.

We are now also in the position to formulate a result on the extreme order statistics of the random variables X_σ.

$\gamma_l \equiv \sqrt{a_l}/\sqrt{2 \ln \alpha_l}$, $l = 1, 2, \ldots, n$. By our assumption on A, $\gamma_1 > \gamma_2 > \cdots > \gamma_n$. Define the function $U_{J,N}$ by

$$U_{J,N}(x) \equiv \sum_{l=1}^{n} \left(\sqrt{2Na_l \ln \bar{\alpha}_l} - N^{-1/2} \gamma_l (\ln(N(\ln \alpha_l)) + \ln 4\pi)/2 \right) + N^{-1/2} x \tag{4.38}$$

and the point process

$$\mathcal{E}_N \equiv \sum_{\sigma \in \{-1,1\}^N} \delta_{U_{J,N}^{-1}(X_\sigma)}. \tag{4.39}$$

Then the following holds true:

Theorem 4.11: *(i) The point process $\mathcal{E}_N$ converges weakly, as $N \uparrow \infty$, to the point process on $\mathbb{R}$*

$$\mathcal{E} \equiv \int_{\mathbb{R}^n} \mathcal{P}^{(n)}(dx_1, \dots, dx_n) \delta_{\sum_{l=1}^m \gamma_l x_l} \tag{4.40}$$

where $\mathcal{P}^{(n)}$ is the Poisson cascade introduced in Theorem (1.3).

Next we state a convergence result for the partition function that is analogous to the low-temperature result Theorem 3.2, (v), in the REM.

One would be tempted to believe that the process that is relevant for the extremal process will again be the right one to choose. However, this will be the case only for large enough β. However, only the first $l(\beta)$ levels of the process participate, where

$$l(\beta) \equiv \max\{l \geq 1 : \beta^2 \gamma_l > 1\} \tag{4.41}$$

and $l(\beta) \equiv 0$ if $\beta^2 \gamma_l \leq 1$.

The following theorem yields the fluctuations of the partition function and connects the GREM to Ruelle's processes.

Theorem 4.12: *With the definitions above, under our hypothesis on A,*

$$\begin{aligned} &e^{\sum_{j=1}^{l(\beta)} \left(-\beta N \sqrt{2a_j \ln \alpha_j} + \beta\gamma_j [\ln(N \ln \alpha_j) + \ln 4\pi]/2 + N \ln \alpha_j\right) - N \sum_{i=l(\beta)+1}^n \beta^2 a_i/2} Z_{\beta,N} \\ &\xrightarrow{\mathcal{D}} C(\beta) \int_{\mathbb{R}^{l(\beta)}} e^{\beta\gamma_1 x_1 + \beta\gamma_2 x_2 + \dots + \beta\gamma_{l(\beta)} x_{l(\beta)}} \mathcal{P}^{(l(\beta))}(dx_1 \dots dx_{l(\beta)}). \end{aligned} \tag{4.42}$$

This integral is over the process $\mathcal{P}^{(l(\beta))}$ on $\mathbb{R}^{l(\beta)}$ constructed in Theorem (1.2) . The constant $C(\beta)$ satisfies

$$C(\beta) = 1, \quad \text{if} \quad \beta\gamma_{l(\beta)+1} < 1, \tag{4.43}$$

and

$$C(\beta) = P\Big(\bigcap_{\substack{i: l(\beta)+1 \leq i \leq l(\beta)+1 \\ (a_{l(\beta)+1} + \dots + a_i)/a_{l(\beta)+1} = \ln(\alpha_{l(\beta)+1} \cdots \alpha_i)/\ln \bar{\alpha}_{l(\beta)+1}}} (\sqrt{a_{l(\beta)+1}} Z_{l(\beta)+1} + \dots + \sqrt{a_i} Z_i < 0) \Big) \tag{4.44}$$

if $\beta\gamma_{l(\beta)+1} = 1$
where $Z_{l(\beta)+1}, \dots, Z_{l(\beta)+1}$ are independent standard Gaussian r.v. Moreover

$$\ln Z_{N,\beta} - \mathbb{E} \ln Z_{N,\beta} \xrightarrow{\mathcal{D}} \ln C(\beta) \int_{\mathbb{R}^{l(\beta)}} e^{\beta\gamma_1 x_1 + \beta\gamma_2 x_2 + \dots + \beta\gamma_{l(\beta)} x_{l(\beta)}} \mathcal{P}(dx_1 \dots dx_{l(\beta)}).$$

Let us introduce the sets

$$B_l(\sigma) \equiv \{\sigma' \in \mathcal{S}_N : \delta_N(\sigma,\sigma') \geq q_l\} \tag{4.45}$$

We define point processes $\mathcal{W}^m_{\beta,N}$ on $(0,1]^m$ given by

$$\mathcal{W}^m_{\beta,N} \equiv \sum_\sigma \delta_{(\mu_{\beta,N}(B_1(\sigma)),\dots,\mu_{\beta,N}(B_m(\sigma)))} \frac{\mu_{\beta,N}(\sigma)}{\mu_{\beta,N}\left(B_m(\sigma)\right)} \tag{4.46}$$

as well as their projection on the last coordinate,

$$\mathcal{R}^m_{\beta,N} \equiv \sum_\sigma \delta_{\mu_{\beta,N}(B_m(\sigma))} \frac{\mu_{\beta,N}(\sigma)}{\mu_{\beta,N}\left(B_m(\sigma)\right)} \tag{4.47}$$

It is easy to see that the processes $\mathcal{W}^m_{\beta,N}$ satisfy

$$\mathcal{W}^m_{\beta,N}(dw_1,\dots,dw_m) = \int_0^1 W^{m+1}_{\beta,N}(dw_1,\dots,dw_m,dw_{m+1})\frac{w_{m+1}}{w_m} \tag{4.48}$$

where the integration is of course over the last coordinate w_{m+1}. Note that these processes will in general not all converge, but will do so only when for some σ, $\mu_\beta(B_m(\sigma))$ is strictly positive. From our experience with the partition function, it is clear that this will be the case precisely when $m \leq l(\beta)$. In fact, we will prove that

Theorem 4.13: *If $m \leq l(\beta)$, the point process $\mathcal{W}^m_{\beta,N}$ on $(0,1]^m$ converges weakly to the point process $\mathcal{W}^m_\beta$ whose atoms $w(i)$ are given in terms of the atoms $(x_1(i),\dots,x_m(i)$ of the point process $\mathcal{P}^{(m)}$ by*

$$\begin{aligned}&(w_1(i),\dots,w_m(i))\\ &=\left(\frac{\int \mathcal{P}^{(m)}(dy)\delta(y_1-x_1(i))e^{\beta(\gamma,y)}}{\int \mathcal{P}^{(m)}(dy)e^{\beta(\gamma,y)}},\dots,\frac{\int \mathcal{P}^{(m)}(dy)\delta(y_1-x_1(i))\dots\delta(y_m-x_m(i))e^{\beta(\gamma,y)}}{\int \mathcal{P}^{(m)}(dy)e^{\beta(\gamma,y)}}\right)\end{aligned} \tag{4.49}$$

and the point processes $\mathcal{R}^{(m)}_{\beta,N}$ converge to the point process $\mathcal{R}^{(m)}_\beta$ whose atoms are the last component of the atoms in (4.49).

Of course the most complete object we can reasonably study is the process $\widehat{\mathcal{W}}_\beta \equiv \mathcal{W}^{l(\beta)}_\beta$. Analogously, we will set $\widehat{\mathcal{R}}_\beta \equiv \mathcal{R}^{l(\beta)}_\beta$.

The point processes $\widehat{\mathcal{W}}^{(m)}_\beta$ takes values on vectors whose components form increasing sequences in $(0,1]$. Moreover, these atoms are naturally clustered in a hierarchical way. These processes were introduced by Ruelle [Ru] and called *probability cascades*.

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