Geometry and topology of turbulence in active nematics


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Defects dynamics

Courtesy of Z. Dogic
Cytoskeletal fluids

Dense mixtures of microtubules and kinesin behaves as fluids of mutually propelled rods: *active liquid crystals.*
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Defects dynamics

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Reynolds number = $10^{-5} - 10^{-4}$
Richardson cascade

Big whorls have little whorls
That feed on their velocity;
And little whorls have lesser whorls
And so on to viscosity.

energy injection

energy dissipation
Modeling active LCs

In nematics, a material element is characterized by four physical quantities:

\[ Q = S \left( nn^T - \frac{1}{2} I \right) \]
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- Nematic director \( n \)

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- Density $\rho$
- Nematic director $\mathbf{n}$
- Nematic order parameter $S$

$Q = S (\mathbf{n} \mathbf{n}^T - \frac{1}{2} I)$
Modeling active LCs

In nematics, a material element is characterized by four physical quantities:

- Density  \( \rho \)
- Nematic director  \( n \)
- Nematic order parameter  \( S \)
- Flow velocity  \( v \)

\[
Q = S\, (nn^T - \frac{1}{2}I)
\]
Active and passive stresses

The flow velocity obeys to the Navier-Stokes equation:

\[
\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot (\mathbf{\sigma}^p + \mathbf{\sigma}^a)
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\begin{align*}
\text{passive stress (viscous + elastic)}
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Active stress:

$$\mathbf{F}^a \sim \mathbf{n}$$

Passive stress (viscous + elastic):

$$\sigma^a = \alpha Q$$

Pedley & Kessler (1992), Simha & Ramaswamy (2002)
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Contractile vs extensile active stress

Contractile
\[ \alpha > 0 \]

Extensile
\[ \alpha < 0 \]
An hydrodynamic equation for the nematic tensor $Q$ is obtained phenomenologically (Olmsted & Goldbart ’92):

$$\frac{DQ}{Dt} = \lambda S\mathbf{u} + [Q, \omega] + \gamma^{-1} \frac{\delta F_{LdG}}{\delta Q}$$
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\[ u = \frac{1}{2} (\nabla v + \nabla v^T) \quad \text{strain-rate} \]

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**Coupling with the flow**

$$u = \frac{1}{2} (\nabla v + \nabla v^T)$$ strain-rate

$$\omega = \frac{1}{2} (\nabla v - \nabla v^T)$$ vorticity

**Relaxational dynamics**

$$\theta \propto 1/\lambda$$
Active nematic hydrodynamics


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\frac{DQ}{Dt} = \lambda Su + [Q, \omega] + \gamma^{-1} \frac{\delta F_{\text{LdG}}}{\delta Q}
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\[
\rho \frac{Dv}{Dt} = \eta \nabla^2 v - \nabla p + \nabla \cdot (\sigma^e + \alpha Q), \quad \nabla \cdot v = 0
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**Defects core radius**

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\]

**Active length scale**

\[
\ell_a = \sqrt{\frac{K}{|\alpha|}}
\]

where active and elastic stresses balance
Dynamics of active nematics

\[ \ell_a \gg L \quad \ell_a \sim L \quad \ell_a \ll L \]

- Stationary Flow
- Laminar Flow
- Periodic Flow
- Turbulence

Activity
Extensile: $\gamma = 10, \alpha = -3$
Curtesy of Zvonimir Dogic
Topological defects play a pivotal role! (Thampi, Golestanian & Yeomans 2013)
Vortex areal density: $n(a)$

$$da \, n(a) = \# \text{ of vortices whose area is in } [a, a+da]$$
Vortex statistics

![Graph showing vortex statistics](image)

- Number of vortices $n(a) \Delta a$
- Vortex area $a/L^2$
- $|\alpha|/\Sigma \times 10^3$

Lines for different values:
- Black: 4
- Red: 6
- Blue: 8
Vortex statistics

\[ a_{\text{min}} \quad \text{active range} \quad \frac{\pi}{16} L^2 \]

![Graph showing the distribution of vortex statistics](image-url)

- Number of vortices vs. Vortex area \( a/L^2 \)
- Plot with different values of \( |\alpha|/\Sigma \times 10^3 \) for \( 4, 6, \) and \( 8 \)
Vortex statistics

\[ a_{\text{min}} \quad \text{active range} \quad \frac{\pi}{16} L^2 \]

\[ n(a) = \frac{N}{Z} e^{-a/a^*} \]

Number of vortices \( n(a) \Delta a \)

|\[|\alpha|/\Sigma \times 10^3\]|
|---|
| 4 |
| 6 |
| 8 |

Vortex area \( a/L^2 \)
Vortex statistics

\[ a_{\text{min}} = a^* \sim \ell_a^2 \]

\[ \times 10^{-3} \]

\[ \frac{a^*}{L^2}, \frac{a_{\text{min}}}{L^2} \]

\[ (\ell_a/L)^2 \times 10^{-3} \]
Average vorticity

Average vorticity of a individual vortices:  \( \omega_v \approx \alpha / \eta \)
Velocity and vorticity PDF

![Graphs showing PDF distributions for velocity and vorticity](image-url)
Energy and enstrophy

Notice that for active laminar flows: \( \nu \sim \alpha \)
Using the vortex areal density:
Energy and enstrophy

Using the vortex areal density:

\[ \frac{1}{2} \langle \omega^2 \rangle \approx \frac{1}{2L^2} \int da \, n(a) \, a \, \omega_v^2 \]
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Using the vortex areal density:

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(total area occupied by the vortices)
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Upon increasing the activity, the vortices becomes faster but smaller.
Energy and enstrophy spectra

For 2D inertial turbulence \( \Omega(k) \sim k^{-1}, \quad E(k) \sim k^{-3} \)
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Then, neglecting the spatial correlation between vortices (i.e. mean-field approximation) and using \( n(a) \):
Mean-field theory

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$$E(\kappa) = \kappa^{-2} \Omega(\kappa) \sim \kappa^{-4}$$
Correlation function

\[ C_{\omega \omega}(r) \quad \text{Numerics} \]
\[ C_{\nu \nu}(r) \quad \text{MFT} \]
The local vortex geometry is intrinsically coupled with the topological structure of the director. Thus, topological defects have the same statistics of active vortices.

\[ \eta \nabla^2 \mathbf{v} - \nabla p + \alpha \nabla \cdot \mathbf{Q} = 0 \]
Defects statistics

![Graphs showing MSD, number of defects, mean free path, and rate vs. activity. The graphs illustrate the relationship between various defect statistics and time and activity.](image)
Defects density

\[ N_{\text{defects}} \sim N_{\text{vortices}} \sim \ell_a^{-2} \sim \alpha \]
Where $n(a)$ comes form?

Let’s think about the active flow as an “ensemble” of vortices:
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Maximizing \( W \) yields the original exponential distribution:

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where: \( \bar{a} = \mathcal{A}/N \sim \ell_a^2 \)
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http://wwwhome.lorentz.leidenuniv.nl/~giomi

Thanks!