

Exact Conditions in Finite-Temperature DFT

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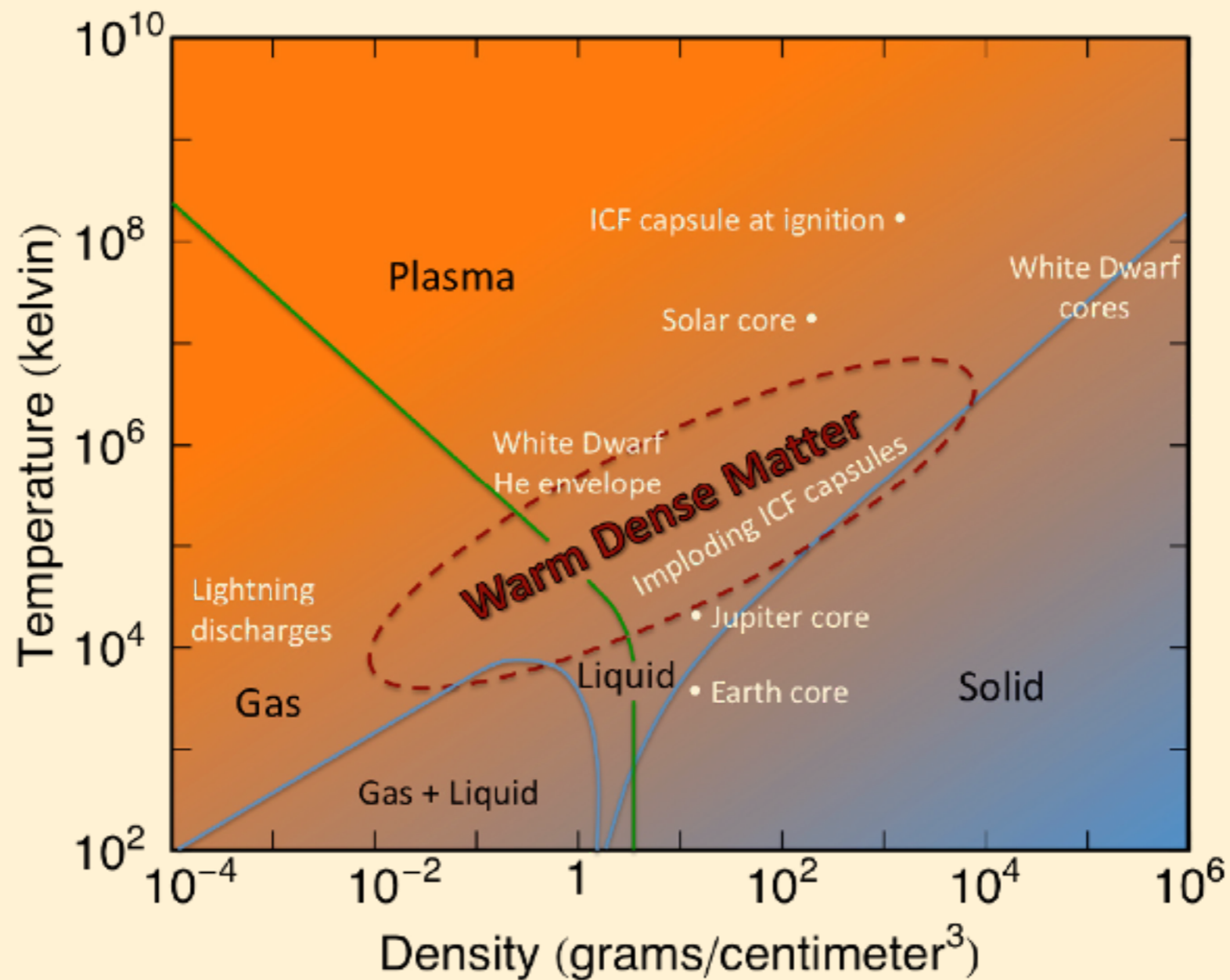
Outline

- Motivation
- Formal construction of finite-temperature (FT) density functionals (DFs)
- Some elementary facts
- Scaling in FT DF-Theory (FT-DFT)
- Conclusions and outlooks

Motivation

First-principle approach
to determine
thermodynamical properties
of many-electron
systems: universal, accurate,
and practical!

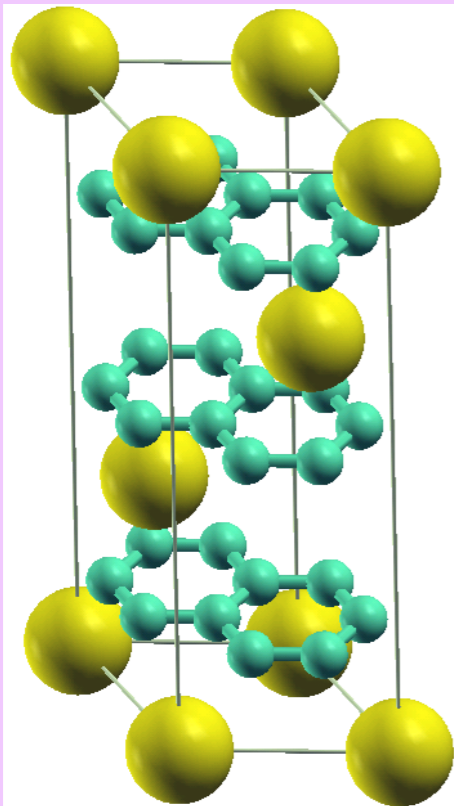
Warm Dense Matter



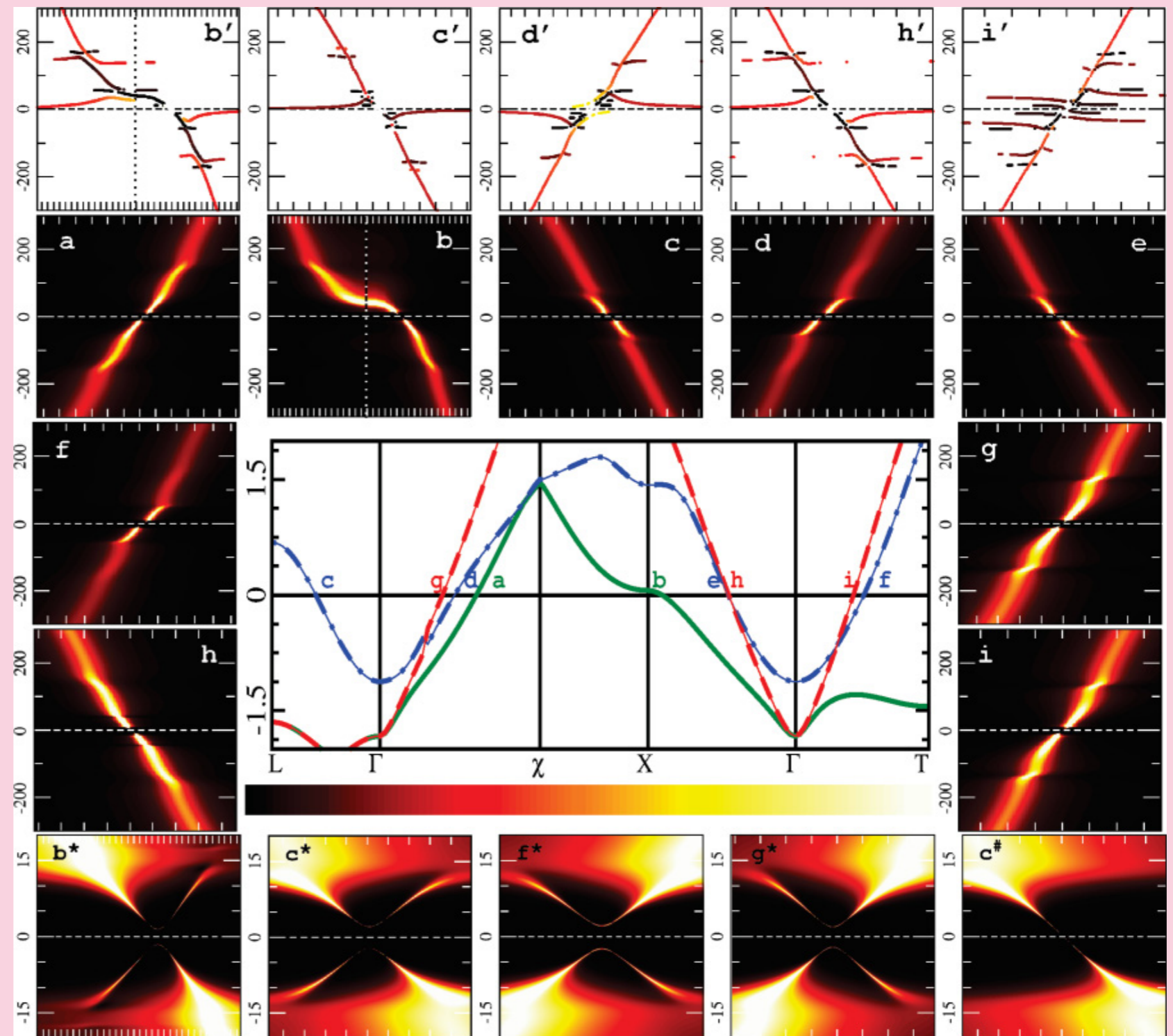
* Diagram from: <http://www.qtp.ufl.edu/ofdft/>

Superconductivity

System: CaC_6

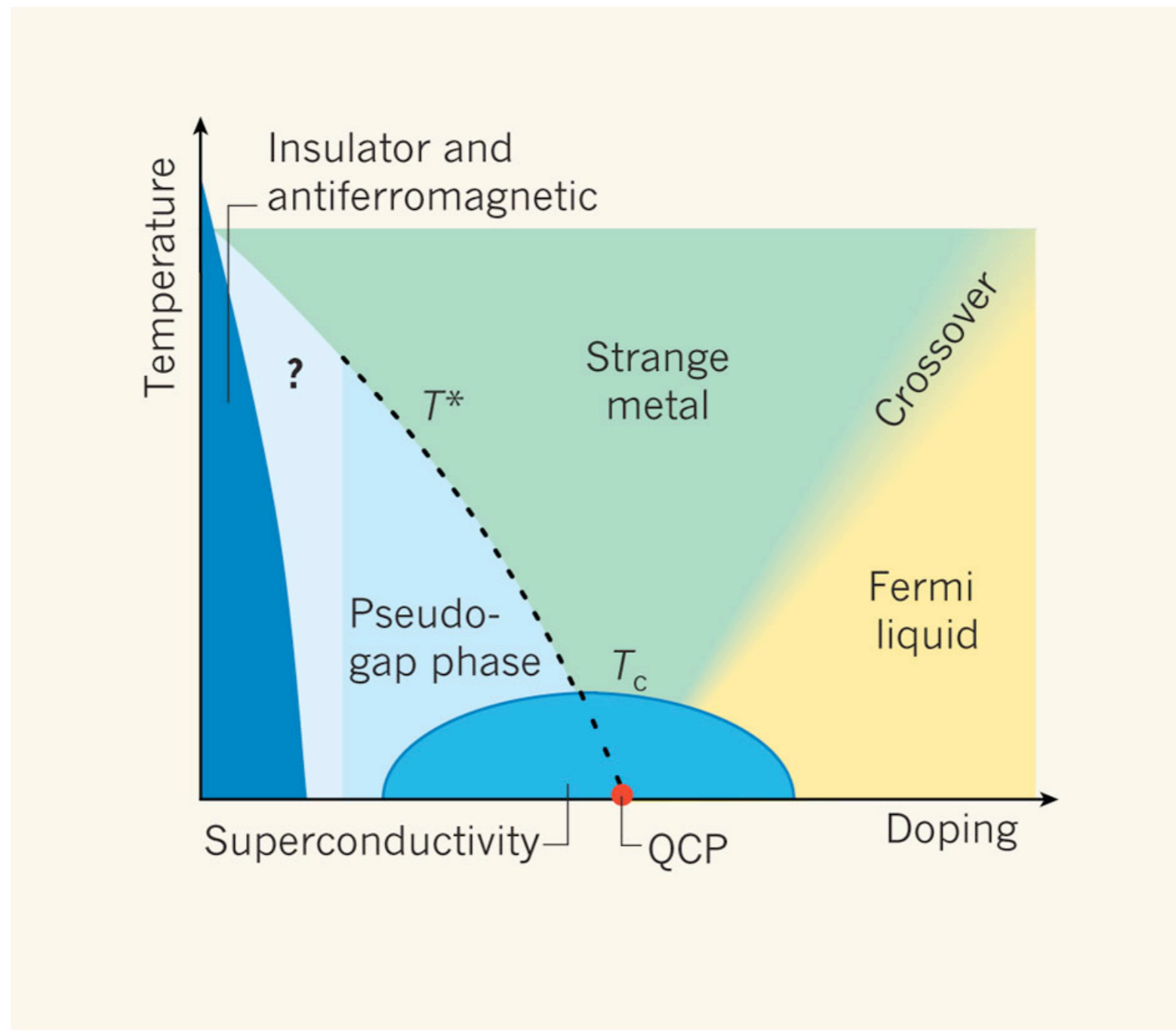


Sanna, Pittalis,
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Sharma, Ummarino,
Massidda, Gross,
PRB (2012)



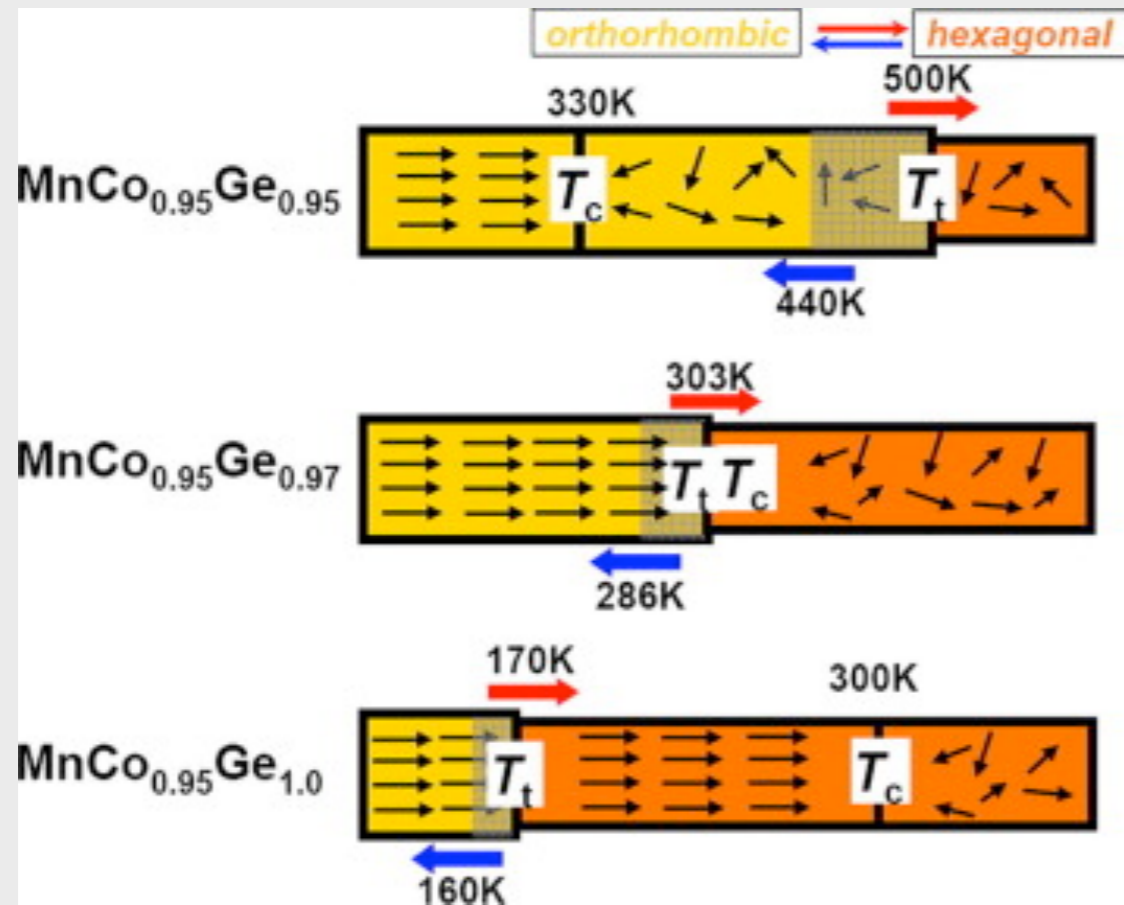
Results: "polaronic" spectral function

High- T_c Superconductors

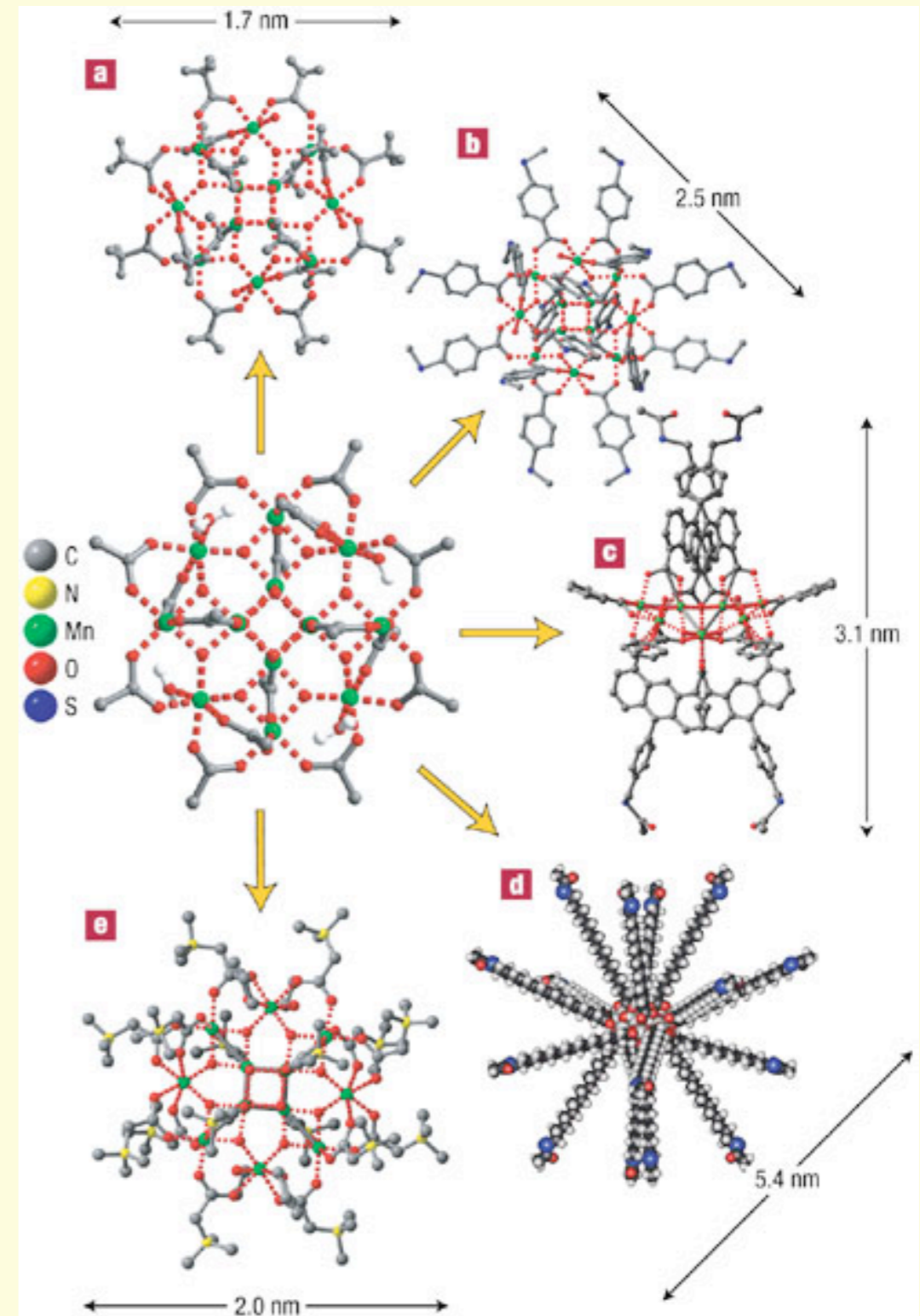


* Diagram from: Varma, *Nature* **468** (2010)

Magnetism



Magnetostructural transitions
[from Liu *et al.*, Scripta Materialia **66** (2012)]



Single-Molecule Magnets

[from Bogani *et al.*, Nature Materials **7** (2008)]

Formal Construction of FT Density Functionals

Statistical Averages

$$A[\hat{\Gamma}] = \text{Tr} \{ \hat{\Gamma} \hat{A} \} = \sum_N \sum_i w_{N,i} \langle \Psi_{N,i} | \hat{A} | \Psi_{N,i} \rangle$$

Statistical operator: $\hat{\Gamma} = \sum_{N,i} w_{N,i} | \Psi_{N,i} \rangle \langle \Psi_{N,i} |$

Pure state (orthonormal): $\left\{ | \Psi_{N,i} \rangle \right\}$

Statistical weights (normalized): $\left\{ w_{N,i} \right\}$

Grand Canonical Operator

$$\hat{\Omega} = \hat{H} - \tau \hat{S} - \mu \hat{N}$$

Hamiltonian: $\hat{H} = \hat{T} + \hat{V}_{ee} + \hat{V}$

Entropy: $\hat{S} = -k \ln \hat{\Gamma}$

and \hat{N} is the particle-number operator.

At Equilibrium

Variational principle (Gibbs):

$$\Omega_{v-\mu}^T = \min_{\hat{\Gamma}} \left\{ \Omega[\hat{\Gamma}] \right\}$$

Minimizing Statistical Operator

$$\hat{\Gamma}^0 = \sum_{N,i} w_{N,i}^0 |\Psi_{N,i}^0\rangle \langle \Psi_{N,i}^0|$$

At Equilibrium

$$\hat{\Gamma}^0 = \sum_{N,i} w_{N,i}^0 |\Psi_{N,i}^0\rangle \langle \Psi_{N,i}^0|$$

where

$$\hat{H} |\Psi_{N,i}^0\rangle = E_{N,i}^0 |\Psi_{N,i}^0\rangle$$

$$w_{N,i}^0 = \frac{\exp[-\beta(E_{N,i}^0 - \mu N)]}{\sum_{N,i} \exp[-\beta(E_{N,i}^0 - \mu N)]}$$

$$\beta = \frac{1}{k\tau}$$

Following DF-Fashion

Alternatively:

$$\Omega_{v-\mu}^T = \min_n \left\{ F^T[n] + \int d^3r n(\mathbf{r}) (v(\mathbf{r}) - \mu) \right\}$$

Mermin, Phys. Rev., 137, A1441 (1965)

**“Thermal Properties of
the Inhomogeneous Electron Gas”**

* See also: Kohn and Sham, Phys. Rev. **140** (1965)

At Equilibrium

Variational principle:

$$\Omega_{v-\mu}^T = \min_{\hat{\Gamma}} \left\{ \Omega[\hat{\Gamma}] \right\}$$

Minimizing Statistical Operator

$$\hat{\Gamma}^0 = \sum_{N,i} w_{N,i}^0 |\Psi_{N,i}^0\rangle \langle \Psi_{N,i}^0|$$

Constraint-Search Construction

$$\Omega_{v-\mu}^\tau = \min_n \left\{ F^\tau[n] + \int d^3r n(\mathbf{r}) (v(\mathbf{r}) - \mu) \right\}$$

Alternatively:

$$\begin{aligned} F^\tau[n] &:= \min_{\hat{\Gamma} \rightarrow n} \left\{ T[\hat{\Gamma}] + V_{\text{ee}}[\hat{\Gamma}] - \tau S[\hat{\Gamma}] \right\} \\ &= \min_{\hat{\Gamma} \rightarrow n} F^\tau[\hat{\Gamma}] = F^\tau[\hat{\Gamma}^\tau[n]] \end{aligned}$$

Constraint-Search Construction

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Constraint-Search Construction

The *non-interacting* analog is

$$\begin{aligned} F_S^\tau[n] &:= \min_{\hat{\Gamma} \rightarrow n} \left\{ T[\hat{\Gamma}] - \tau S[\hat{\Gamma}] \right\} \\ &= \min_{\hat{\Gamma} \rightarrow n} K^\tau[\hat{\Gamma}] = K^\tau[\hat{\Gamma}_S^\tau[n]] \end{aligned}$$

We refer to $K^\tau[\hat{\Gamma}]$ as the “kentropy” to emphasize both the kinetic and entropy contributions.

Kohn-Sham System at FT

Through the KS system, one gets

$$F^\tau[n] = F_S^\tau[n] + U^\tau[n] + \Omega_X^\tau[n] + \Omega_C^\tau[n]$$

where

$$F_S^\tau[n] := K^\tau[\hat{\Gamma}_S^\tau[n]]$$

$$\Omega_X^\tau[n] := V_{ee}[\hat{\Gamma}_S^\tau[n]] - U^\tau[n]$$

$$\Omega_C^\tau[n] := F^\tau[\hat{\Gamma}^\tau[n]] - F^\tau[\hat{\Gamma}_S^\tau[n]] - V_{ee}[\hat{\Gamma}_S^\tau[n]]$$

Kohn-Sham System at FT

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Kohn-Sham System at FT

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$$F^\tau[n] = F_s^\tau[n] + U^\tau[n] + \Omega_x^\tau[n] + \Omega_c^\tau[n]$$

where

$$F_s^\tau[n] := K^\tau[\hat{\Gamma}_s^\tau[n]]$$

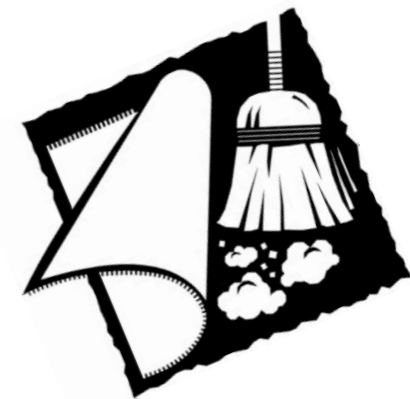
$$\Omega_x^\tau[n] := V_{ee}[\hat{\Gamma}_s^\tau[n]] - U^\tau[n]$$

$$\Omega_c^\tau[n] := F^\tau[\hat{\Gamma}^\tau[n]] - F^\tau[\hat{\Gamma}_s^\tau[n]] - V_{ee}[\hat{\Gamma}_s^\tau[n]]$$

Kohn-Sham System at FT

Through the KS system, one gets

$$F^\tau[n] = F_s^\tau[n] + U^\tau[n] + \Omega_x^\tau[n] +$$



where

$$F_s^\tau[n] := K^\tau[\hat{\Gamma}_s^\tau[n]]$$

$$\Omega_x^\tau[n] := V_{ee}[\hat{\Gamma}_s^\tau[n]] - U^\tau[n]$$



$$:= F^\tau[\hat{\Gamma}^\tau[n]] - F^\tau[\hat{\Gamma}_s^\tau[n]] - V_{ee}[\hat{\Gamma}_s^\tau[n]]$$

Under the “Carpet”

$$\Omega_C^\tau[n] := E_C^\tau[n] - \tau S_C^\tau[n] = K_C^\tau[n] + U_C^\tau[n]$$

where

$$E_C^\tau[n] := T_C^\tau[n] + U_C^\tau[n]$$

$$T_C^\tau[n] := T[\hat{\Gamma}^\tau[n]] - T[\hat{\Gamma}_s^\tau[n]]$$

$$U_C^\tau[n] := V_{ee}[\hat{\Gamma}^\tau[n]] - V_{ee}[\hat{\Gamma}_s^\tau[n]]$$

Under the "Carpet"

$$\Omega_C^\tau[n] := E_C^\tau[n] - \tau S_C^\tau[n] = K_C^\tau[n] + U_C^\tau[n]$$

where

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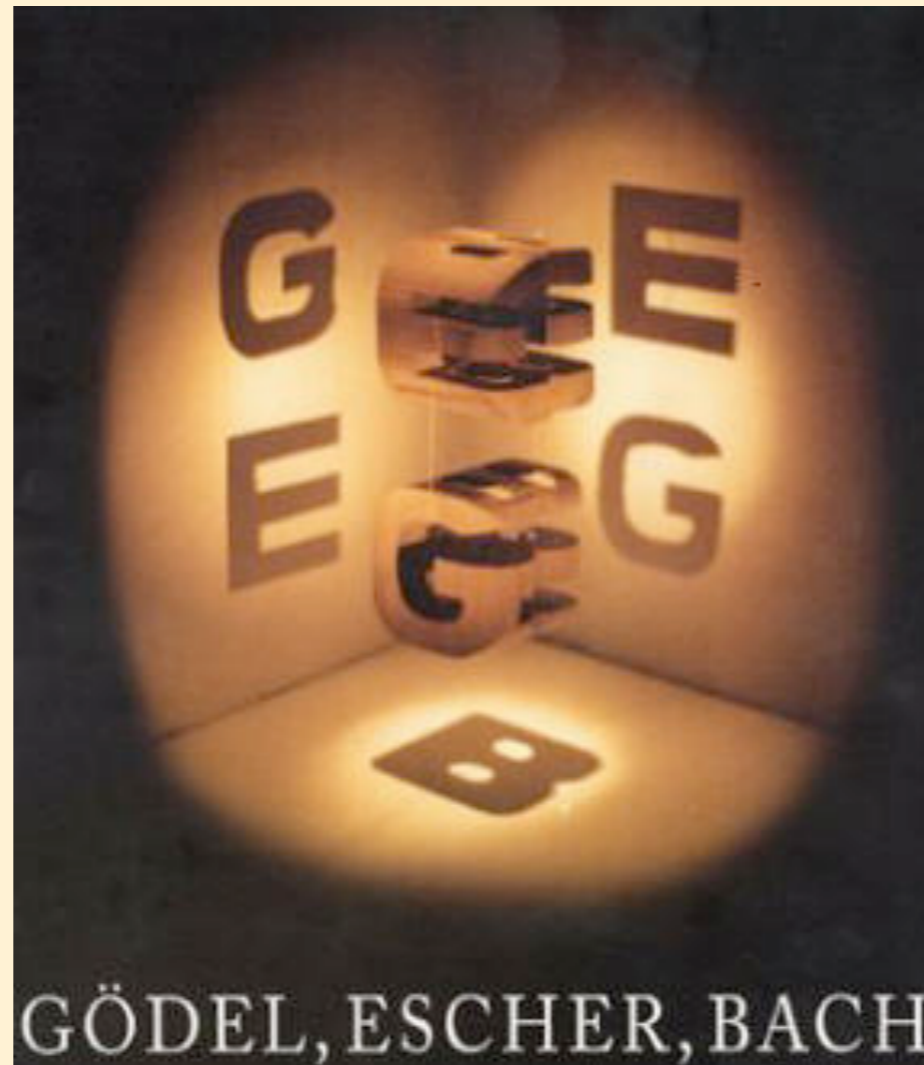
$$T_C^\tau[n] := T[\hat{\Gamma}^\tau[n]] - T[\hat{\Gamma}_s^\tau[n]]$$

$$U_C^\tau[n] := V_{ee}[\hat{\Gamma}^\tau[n]] - V_{ee}[\hat{\Gamma}_s^\tau[n]]$$

$$S_C^\tau[n] := S[\hat{\Gamma}^\tau[n]] - S[\hat{\Gamma}_s^\tau[n]]$$

$$K_C^\tau[n] := K[\hat{\Gamma}^\tau[n]] - K[\hat{\Gamma}_s^\tau[n]]$$

How do they really look?



Cover of the book:
“Gödel, Escher, Bach: An Eternal Golden Braid”
by Douglas Hofstadter (1979)

Some Elementary Facts

Signs of Basic Quantities

Consider

$$K_C^\tau[n] := K[\hat{\Gamma}^\tau[n]] - K[\hat{\Gamma}_S^\tau[n]]$$

Since $\hat{\Gamma}_S^\tau[n]$ minimizes $K^\tau[\hat{\Gamma}]$:

$$K_C^\tau[n] \geq 0$$

Signs of Basic Quantities

Consider

$$\Omega_C^\tau[n] := F^\tau[\hat{\Gamma}^\tau[n]] - F^\tau[\hat{\Gamma}_S^\tau[n]] - V_{ee}[\hat{\Gamma}_S^\tau[n]]$$

Since $\hat{\Gamma}^\tau[n]$ minimizes $F^\tau[\hat{\Gamma}]$:

$$\Omega_C^\tau[n] \leq 0$$

Signs of Basic Quantities

Since $K_C^\tau[n] = T_C^\tau[n] - \tau S_C^\tau[n] \geq 0$

and $\Omega_C^\tau[n] = E_C^\tau[n] - \tau S_C^\tau[n] \leq 0$

combining with

$$\Omega_C^\tau[n] = K_C^\tau[n] + U_C^\tau[n]$$

gives:

$$U_C^\tau[n] \leq 0$$

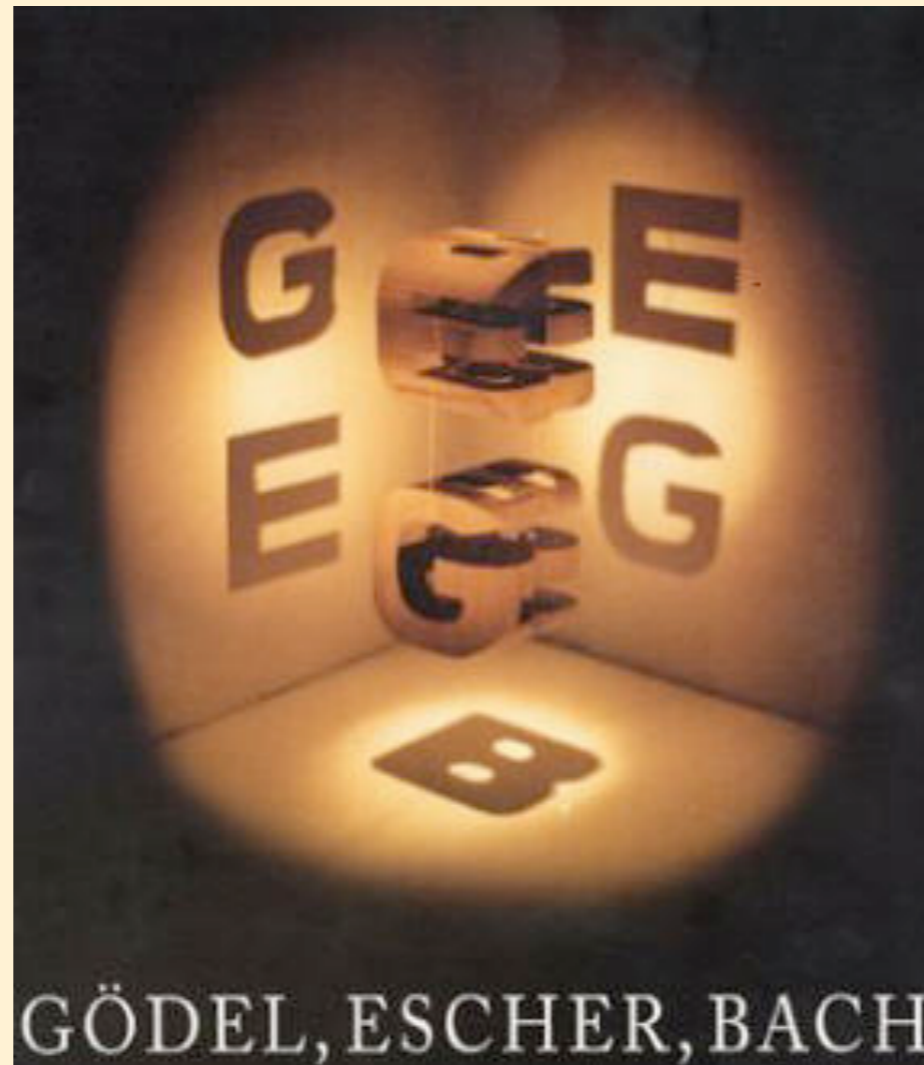
Signs of Basic Quantities

Also, it is true

$$\Omega_{\mathbf{x}}^{\tau}[n] \leq 0$$

** See also: Goerling's lectures in this workshop.*

How do they really look?



Cover of the book:
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Scaling in FT-DFT

Scaling Wavefunctions

Reminder:

$$\Psi^\gamma(\mathbf{r}_1, \dots, \mathbf{r}_N) := \gamma^{\frac{3}{2}N} \Psi(\gamma\mathbf{r}_1, \dots, \gamma\mathbf{r}_N)$$

Scale coordinates and keep normalization constant.

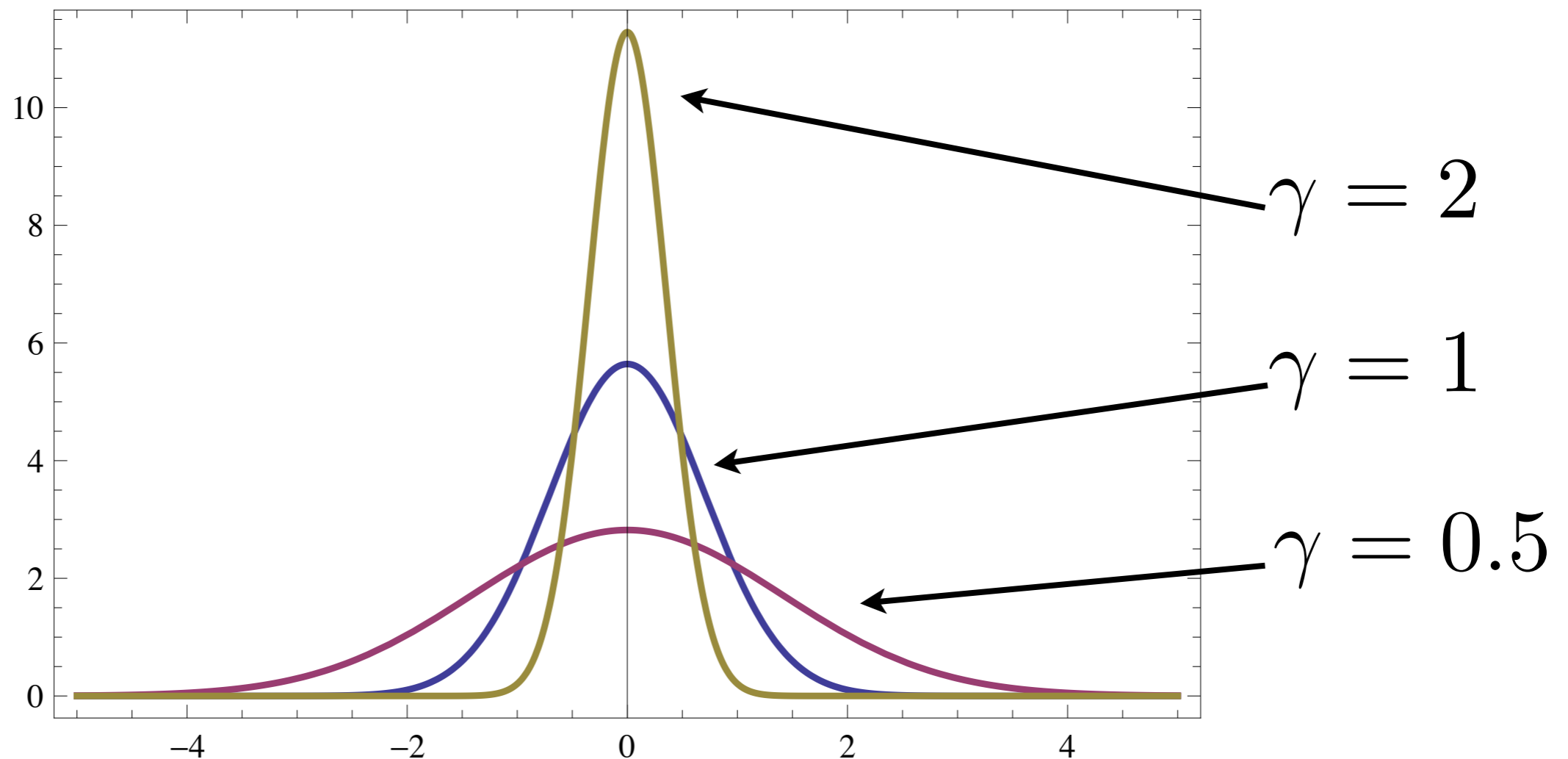
Scaling Wavefunctions

Correspondingly: $n_\gamma(\mathbf{r}) = \gamma^3 n(\gamma\mathbf{r})$

Example:

$N = 10$

$\text{Dim} = 1$



Scaling Ensembles

We propose

$$\hat{\Gamma}_\gamma := \sum_N \sum_i w_{N,i} |\Psi_{N,i}^\gamma\rangle \langle \Psi_{N,i}^\gamma|$$

Pure states are scaled as usual

$$\Psi^\gamma(\mathbf{r}_1, \dots, \mathbf{r}_N) = \langle \mathbf{r}_1, \dots, \mathbf{r}_N | \Psi^\gamma \rangle$$

Statistical weight ARE KEPT FIXED!

Scaling of Statistical Averages

Therefore:

$$T[\hat{\Gamma}_\gamma] = \gamma^2 T[\hat{\Gamma}]$$

$$V_{ee}[\hat{\Gamma}_\gamma] = \gamma V_{ee}[\hat{\Gamma}]$$

$$N[\hat{\Gamma}_\gamma] = N[\hat{\Gamma}]$$

$$S[\hat{\Gamma}_\gamma] = S[\hat{\Gamma}]$$

* Use Statistical Operators built with orthonormal pure states.

What happens to
the FT Density functionals?

Non-interacting Case

Because

$$K^\tau[\hat{\Gamma}_\gamma] = \gamma^2 \left(T[\hat{\Gamma}] - \frac{\tau}{\gamma^2} S[\hat{\Gamma}] \right) = \gamma^2 K^{\tau/\gamma^2}[\hat{\Gamma}]$$

We get:

$$\hat{\Gamma}_S^\tau[n_\gamma] = \hat{\Gamma}_{S,\gamma}^{\tau/\gamma^2}[n]$$

Non-interacting Case

Homogeneous scaling relations:

$$F_S^\tau [n_\gamma] = \gamma^2 F_S^{\tau/\gamma^2} [n]$$
$$S_S^\tau [n_\gamma] = S_S^{\tau/\gamma^2} [n]$$

* See also: Dufty and Trickey, *Phys. Rev. B* **84** (2011)

Non-interacting Case

Interesting consequences of

$$F_S^\tau[n_\gamma] = \gamma^2 F_S^{\tau/\gamma^2}[n]$$

$$S_S^\tau[n_\gamma] = S_S^{\tau/\gamma^2}[n]$$

High-dens. ~ Low-temp.

$$T_S[n] = \lim_{\gamma \rightarrow \infty} F_S^\tau[n_\gamma]/\gamma^2$$

Low-dens. ~ High-temp.

$$S_S^\infty[n] = - \lim_{\gamma \rightarrow 0} F_S^\tau[n_\gamma]/\tau$$

Non-interacting Case

Interesting consequences of

$$F_S^\tau [n_\gamma] = \gamma^2 F_S^{\tau/\gamma^2} [n]$$

Temp-dependence generated through scaling:

$$F_S^{\tau'} [n] = \frac{\tau'}{\tau} F_S^\tau [n \sqrt{\tau/\tau'}]$$

What about the interacting
functionals?

Scaling of Statistical Averages

$$T[\hat{\Gamma}_\gamma] = \gamma^2 T[\hat{\Gamma}]$$

$$V_{ee}[\hat{\Gamma}_\gamma] = \gamma V_{ee}[\hat{\Gamma}]$$

$$N[\hat{\Gamma}_\gamma] = N[\hat{\Gamma}]$$

$$S[\hat{\Gamma}_\gamma] = S[\hat{\Gamma}]$$

** Use Statistical Operators built with orthonormal pure states.*

Interacting Case

Since:

$$K^\tau[\hat{\Gamma}_\lambda] + \lambda V_{ee}[\hat{\Gamma}_\lambda] = \lambda^2 \left\{ K^{\tau/\lambda^2}[\hat{\Gamma}] + V_{ee}[\hat{\Gamma}] \right\}$$

and

$$F^{\tau,\lambda}[n] = \min_{\hat{\Gamma} \rightarrow n} \left\{ T[\hat{\Gamma}] + \lambda V_{ee}[\hat{\Gamma}] - \tau S[\hat{\Gamma}] \right\}$$

We get:

$$\hat{\Gamma}^{\tau,\lambda}[n_\lambda] = \hat{\Gamma}_\lambda^{\tau/\lambda^2}[n]$$

Interacting Case

Homogeneous scaling relations:

$$F^{\tau, \lambda}[n_\lambda] = \lambda^2 F^{\tau / \lambda^2}[n]$$

$$S^{\tau, \lambda}[n_\lambda] = S^{\tau / \lambda^2}[n]$$

* See also: lecture given by Dufty in March meeting APS (2012)

Homogeneous Scaling

Interesting consequence of

$$F^{\tau, \lambda}[n_\lambda] = \lambda^2 F^{\tau/\lambda^2}[n]$$

$$S^{\tau, \lambda}[n_\lambda] = S^{\tau/\lambda^2}[n]$$

High-dens. ~ Low-temp.

$$F[n] = \lim_{\lambda \rightarrow \infty} F^{\tau, \lambda}[n_\lambda]/\lambda^2$$

Low-dens. ~ High-temp.

$$S^\infty[n] = - \lim_{\lambda \rightarrow 0} F^{\tau, \lambda}[n_\lambda]/\tau$$

Homogeneous Scaling

Interesting consequence of

$$F^{\tau, \lambda}[n_\lambda] = \lambda^2 F^{\tau / \lambda^2}[n]$$

Temp-dependence generated through scaling:

$$F^{\tau'}[n] = \frac{\tau'}{\tau} F^{\tau, \sqrt{\tau / \tau'}}[n \sqrt{\tau / \tau'}]$$

For Fixed Coupling

What if...

***we don't scale the strength
of the interaction?***

For Fixed Coupling

Inequality!

$$K^\tau[n_\gamma] + V_{ee}^\tau[n_\gamma] \leq \gamma^2 K^\tau/\gamma^2[n] + \gamma V_{ee}^\tau/\gamma^2[n]$$

this is because:

$$\Gamma^\tau[n_\gamma] \neq \Gamma_\gamma^\tau/\gamma^2[n]$$

Lots of Inequalities!

For $\lambda = 1$ and $\gamma \geq 1$

$$K^T[n_\gamma] \leq \gamma^2 K^T/\gamma^2[n]$$

$$V_{ee}^T[n_\gamma] \geq \gamma V_{ee}^T/\gamma^2[n]$$

$$K_C^T[n_\gamma] \leq \gamma^2 K_C^T/\gamma^2[n]$$

$$U_C^T[n_\gamma] \geq \gamma U_C^T/\gamma^2[n]$$

$$\Omega_C^T[n_\gamma] \geq \gamma \Omega_C^T/\gamma^2[n]$$

Exchange

Homogeneous scaling relation

$$\Omega_{\mathbf{X}}^{\tau}[n_{\gamma}] = \gamma \Omega_{\mathbf{X}}^{\tau/\gamma^2}[n]$$

EXX and GGAs

Homogeneous scaling relation:

$$\Omega_{\mathbf{x}}^{\tau}[n_{\gamma}] = \gamma \Omega_{\mathbf{x}}^{\tau/\gamma^2}[n]$$

Constraint for FT-GGA construction

$$\omega_{\mathbf{x}}^{\text{GGA}\tau}(n(\mathbf{r}), |\nabla n|(\mathbf{r})) = e_{\mathbf{x}}^{\text{unif}}(n(\mathbf{r})) F_{\mathbf{x}}(s(\mathbf{r}), \tilde{\tau}(\mathbf{r}))$$

$$\tilde{\tau}(\mathbf{r}) \approx \tau/n^{2/3}(\mathbf{r})$$

$$s \approx |\nabla n|/(n^{4/3})$$

* See also: poster presented by Karasiev in this workshop.

XC from Coupling Integration

Coupling constant integration

$$\Omega_{\text{XC}}^{\tau}[n] = \int_0^1 d\lambda V_{ee}^{\tau,\lambda}[n] - U^{\tau}[n]$$

We can avoid in this way to deal with the correlation “Kentropy”.

X from XC

$$\Omega_X^\tau[n] = \lim_{\gamma \rightarrow \infty} \Omega_{XC}^{\gamma^2 \tau}[n_\gamma] / \gamma$$

This is useful to define/extract the “exchange” part of a given XC approximation.

Conclusions and Outlooks

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Derived several exact properties of Finite-Temperature DFT functionals useful for constraining approximations.

What's next?

- (a) Derivation of new approximations;
- (b) APPLICATIONS.

Acknowledgments

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THANK YOU
FOR YOUR ATTENTION!