Some Mathematical Challenges in Hot Dense Plasma

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Workshop I: Computational Challenges in Hot Dense Plasma
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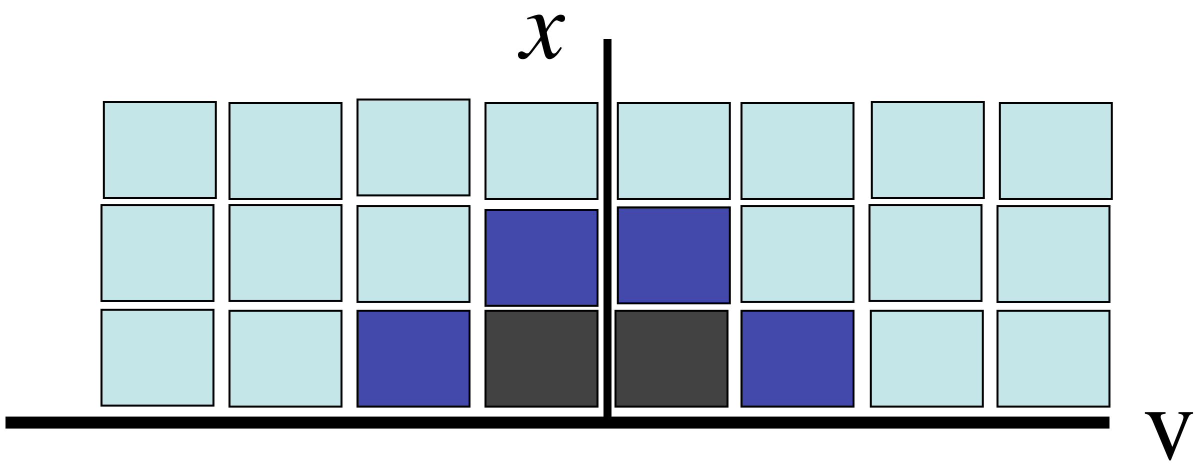
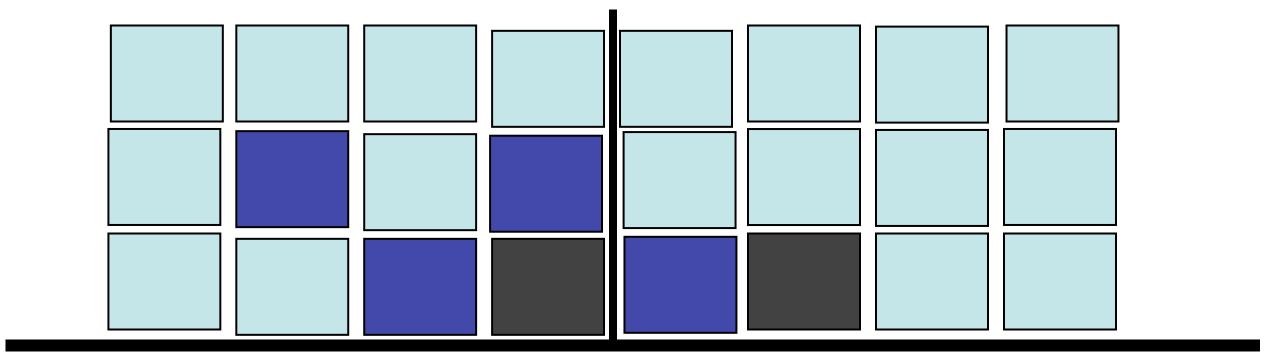
There are a few curious mathematical questions underlying some basic issues in hot dense plasma to which I wish I knew the answer.

Gardener Restacking



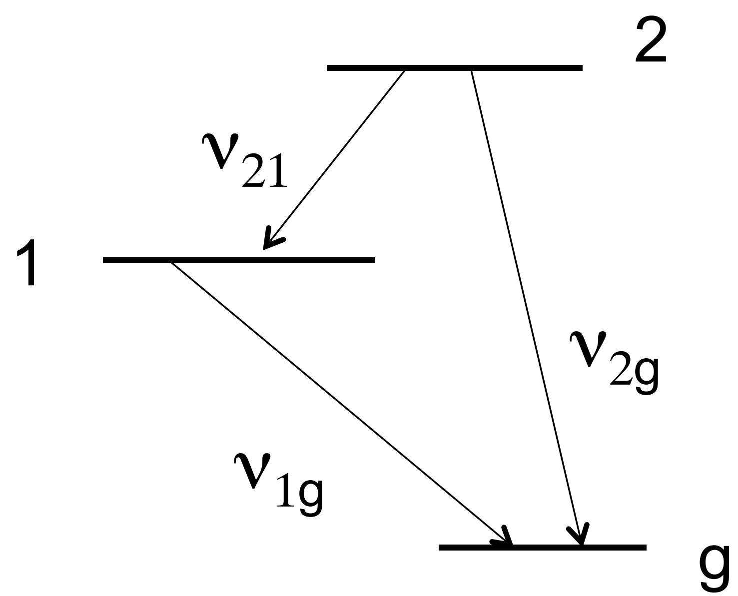
Minimum energy distribution



Initial energy distribution

Entropy conserving

Free Energy under Phase Space Rearrangement



$$\varepsilon = n_g \varepsilon_g + n_1 \varepsilon_1 + n_2 \varepsilon_2$$

minimized for: $n_g > n_1 > n_2$

more generally, minimize: $\varepsilon = \vec{n} \cdot \vec{\varepsilon}$

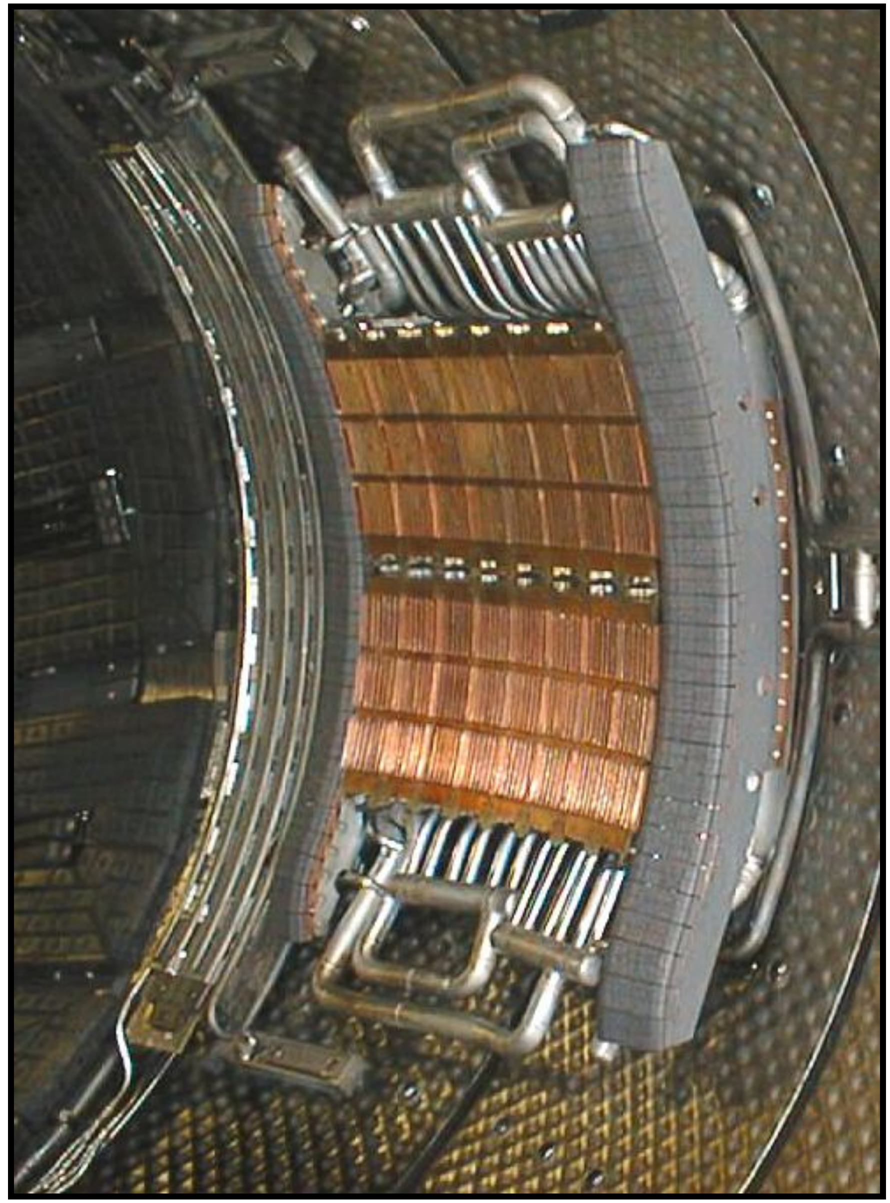
using π-pulse excitations ν_{ij}

Example: for $n_1 > n_2 > n_3$ t=0: $n_3\ n_2\ n_1$

To release free energy, apply 3 (ordered) π-pulses (to exchange densities)

ν_{21}: $n_3\ n_1\ n_2$ ν_{1g}: $n_1\ n_3\ n_2$ ν_{21}: $n_1\ n_2\ n_3$

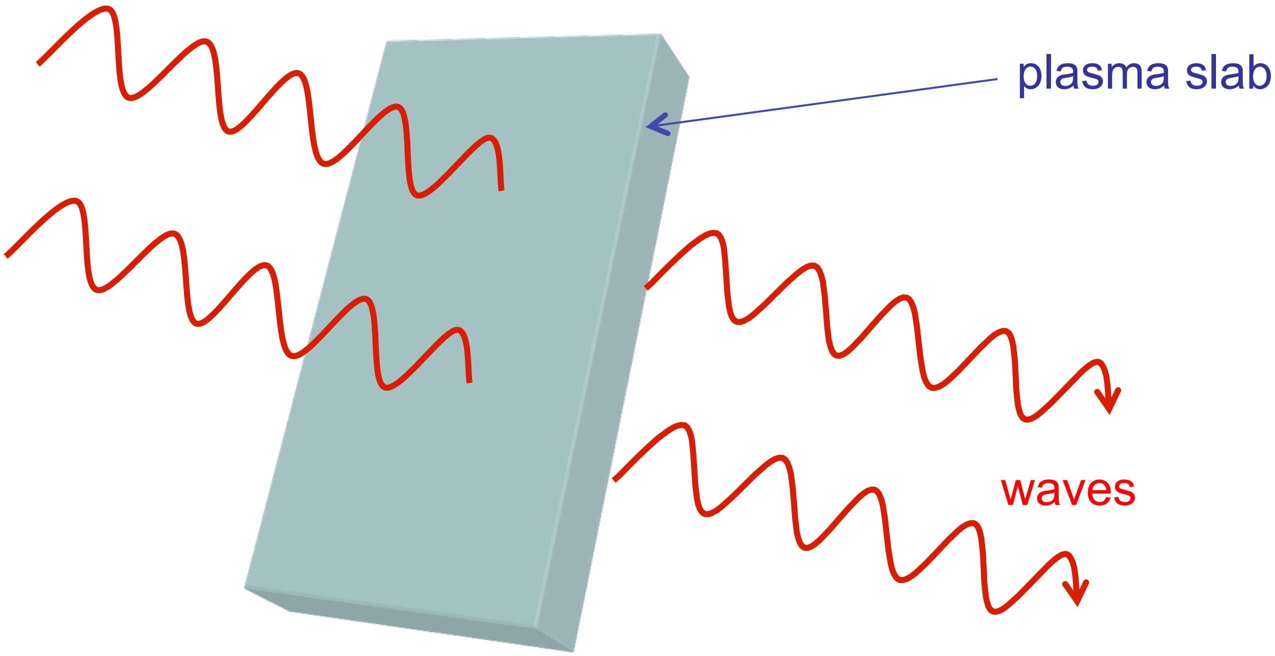
RF Methods of Heating and Current Drive in a Tokamak



Tore Supra LH Coupler

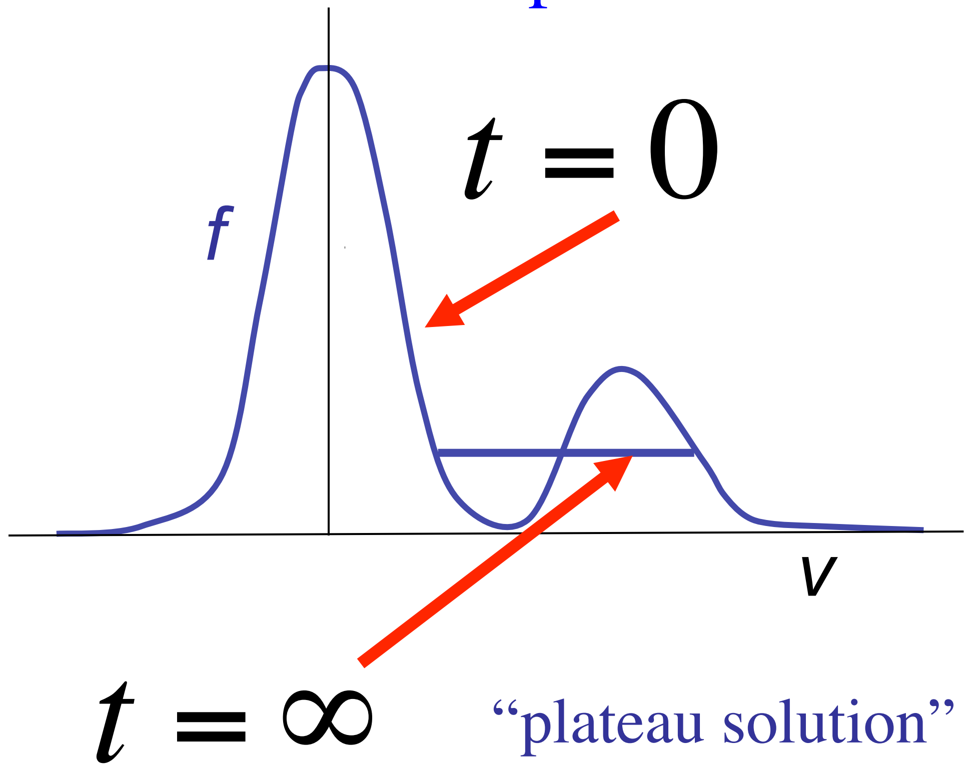
4 MW, 1000 s, 3.7 GHz

Rearrangement of Phase Space in Plasma



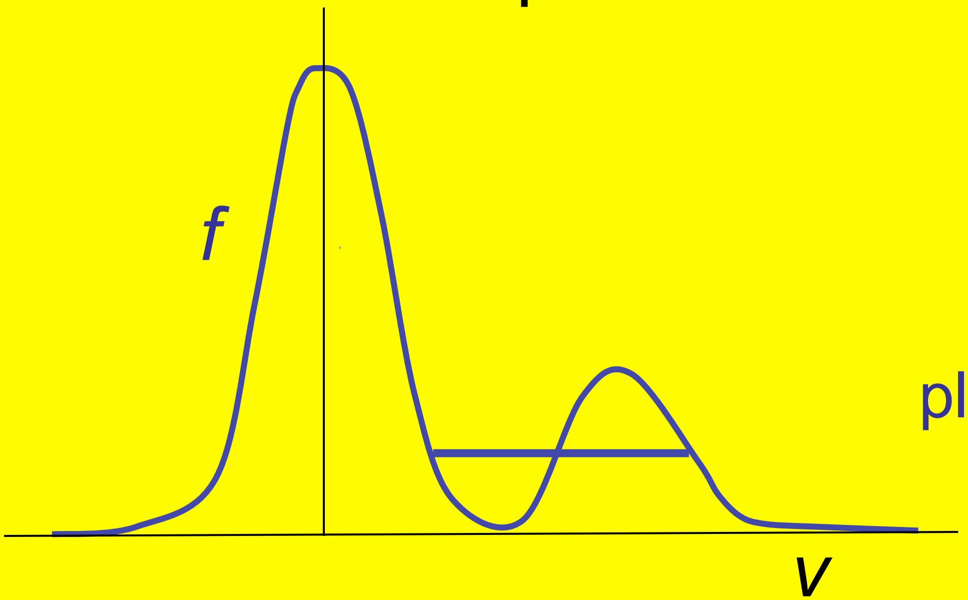
1. Current drive
2. One-way wall
3. Coupled diffusion in position-velocity: “alpha-channeling”

“Bump-on-tail” Instability

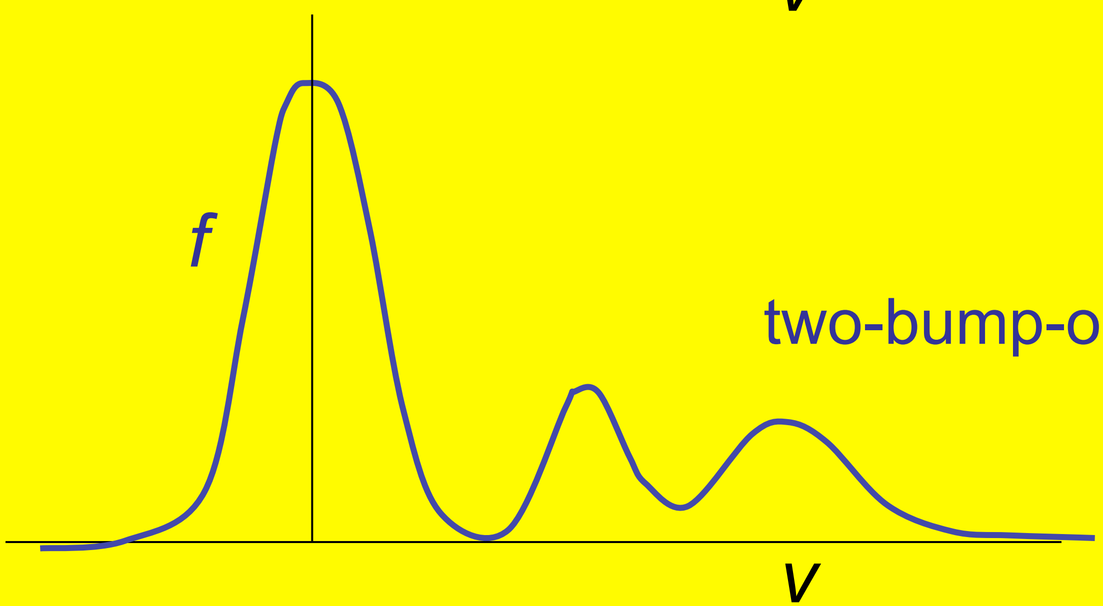


Free energy is due to equalizing population inversion
Not entropy conserving

“Bump-on-tail” Instability



plateau solution



two-bump-on-tail

multiple steady state solutions

Free Energy under *constrained* Phase Space Rearrangement

minimize: $\varepsilon = \vec{n} \cdot \vec{\varepsilon}$

under ***phase space conservation***

for atoms, use π-pulse excitations ν_{ij}: solution sequence $(\nu_{ij1}, \nu_{ij2}, \ldots)$

for plasma, use Hamiltonian forces: "Gardner restacking"

under ***diffusion constraint*** the free energy is <u>not so easily found</u>

for example: apply sequence (ν_{10}, ν_{21}) under diffusion constraint

	ε_0	ε_1	ε_2
Initial	n_0	n_1	n_2
Step1	$(n_1 + n_0)/2$	$(n_1 + n_0)/2$	n_2
Step2	$(n_1 + n_0)/2$	$(n_1 + n_0)/4 + n_2/2$	$(n_1 + n_0)/4 + n_2/2$

Fisch and Rax, 1993

Example

		$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 4$
Initial	$W_0 = 22$	$n_0 = 0$	$n_1 = 2$	$n_2 = 5$
Step 1	$W_1 = 12$	5/2	2	5/2
Step 2	$W_2 = 45/4$	5/2	9/4	9/4

Apply (ν_{20}, ν_{21})

		$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 4$
Initial	$W_0 = 22$	0	2	5
Step 1	$W_1 = 21$	1	1	5
Step 2	$W_2 = 13$	3	1	3
Step 3	$W_3 = 10$	3	2	2

Apply $(\nu_{10}, \nu_{20}, \nu_{21})$

Better strategy

Strategy 1: Diffuse particles first between similar population levels

Example (continued)

		$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 4$
Initial	$W_0 = 22$	0	2	5
Step 1	$W_1 = 35/2$	0	7/2	7/2
Step 2	$W_2 = 21/2$	7/4	7/2	7/4
Step 2	$W_3 = 77/8$	21/8	21/8	7/4

Apply $(\nu_{21},\nu_{20},\nu_{10})$

Best strategy

		$\varepsilon_0 = 0$	$\varepsilon_1 = 1$	$\varepsilon_2 = 4$
Step 1	$W_1 = 35/2$	0	7/2	7/2
Step 2	$W_2 = 63/4$	7/4	7/4	7/2
Step 3	$W_3 = 49/4$	21/8	7/4	21/8
Step 4	$W_4 = 175/16$	21/8	35/16	35/16

$(\nu_{21}, \nu_{10},\nu_{20},\nu_{21})$

Poor strategy

Strategy 2: Deplete particles first from high energy levels

Statement of the Problem

Discrete: Find the sequence $\{v_{ij}\}$ that minimizes: $W = \vec{n} \cdot \vec{\varepsilon}$

Continuous: Let

$$\frac{\partial f(v,t)}{\partial t} = \int K(v,v',t)\left[f(v',t) - f(v,t)\right]$$

$$K(v,v',t) = K(v',v,t)$$

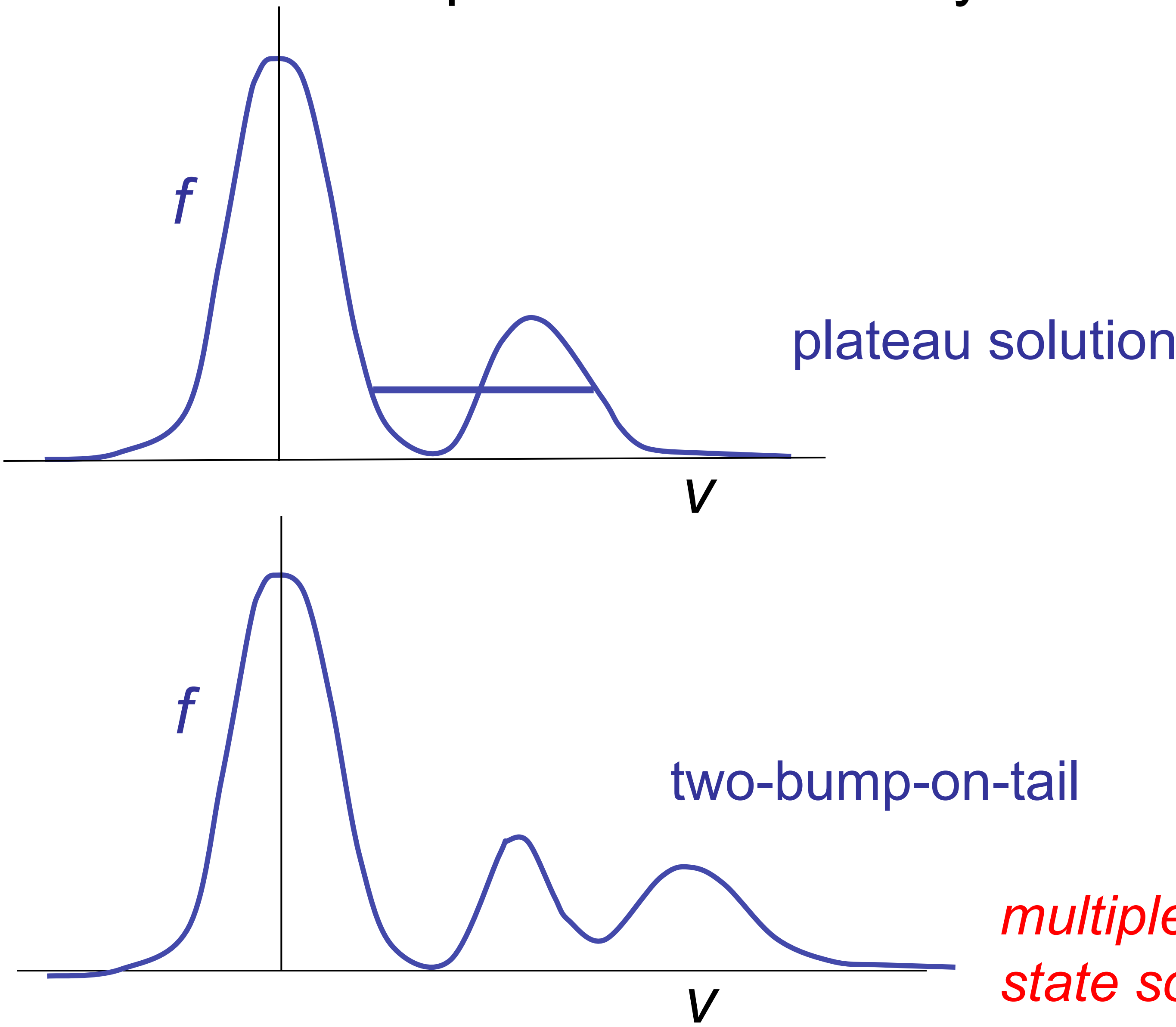
$$K(v,v',t) \geq 0$$

$$W(t) = \int \varepsilon(v) f(v,t) dv$$

Then find K that minimizes $W(t \rightarrow \infty)$.

Note the H-theorem: $$\frac{d}{dt}\int f(v,t)^2 dv \leq 0$$

“Bump-on-tail” Instability

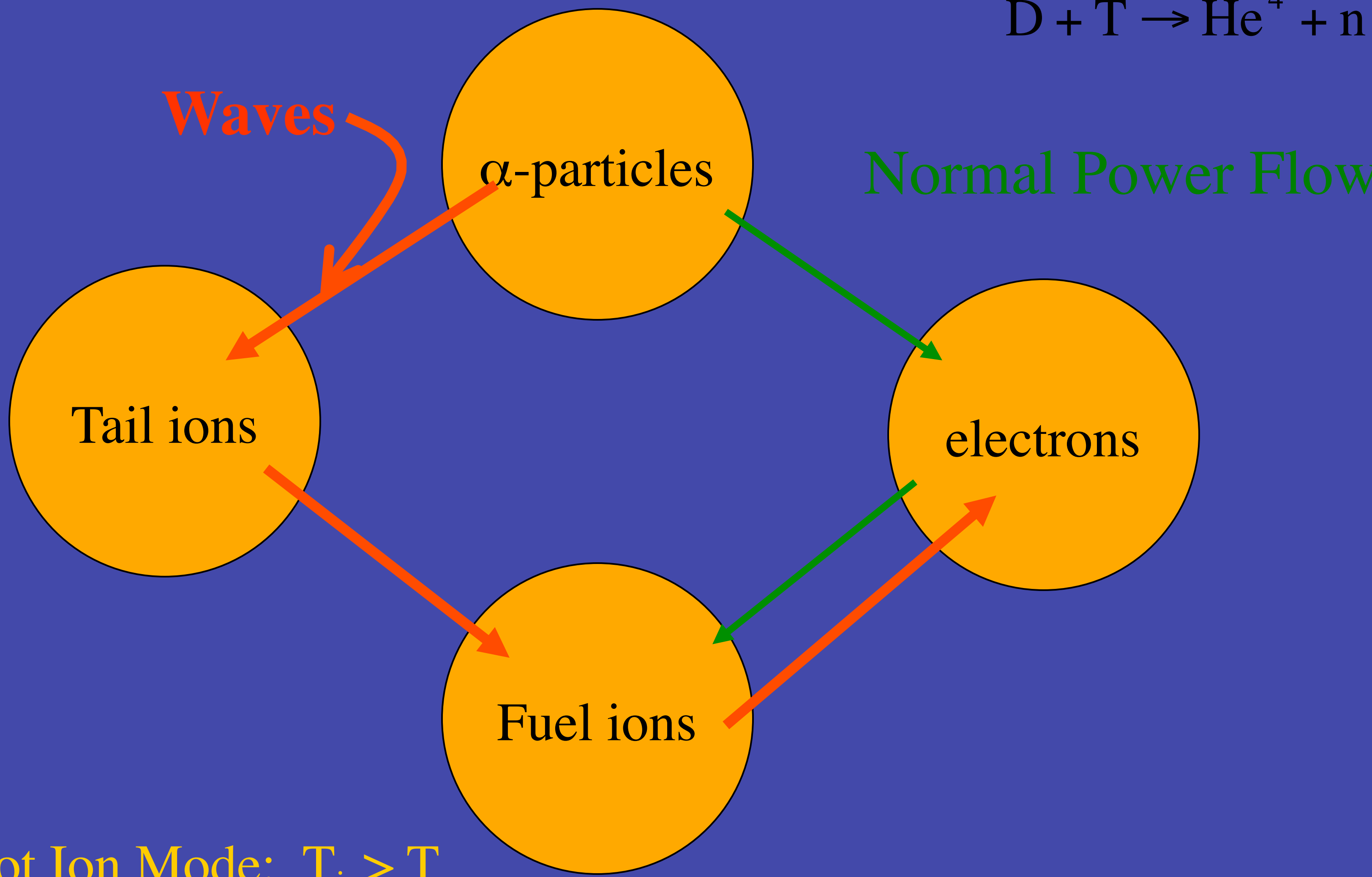


multiple steady state solutions

Power Flow in a Fusion Reactor

Advantages of "α-Channeling"

$$D + T \rightarrow He^4 + n$$



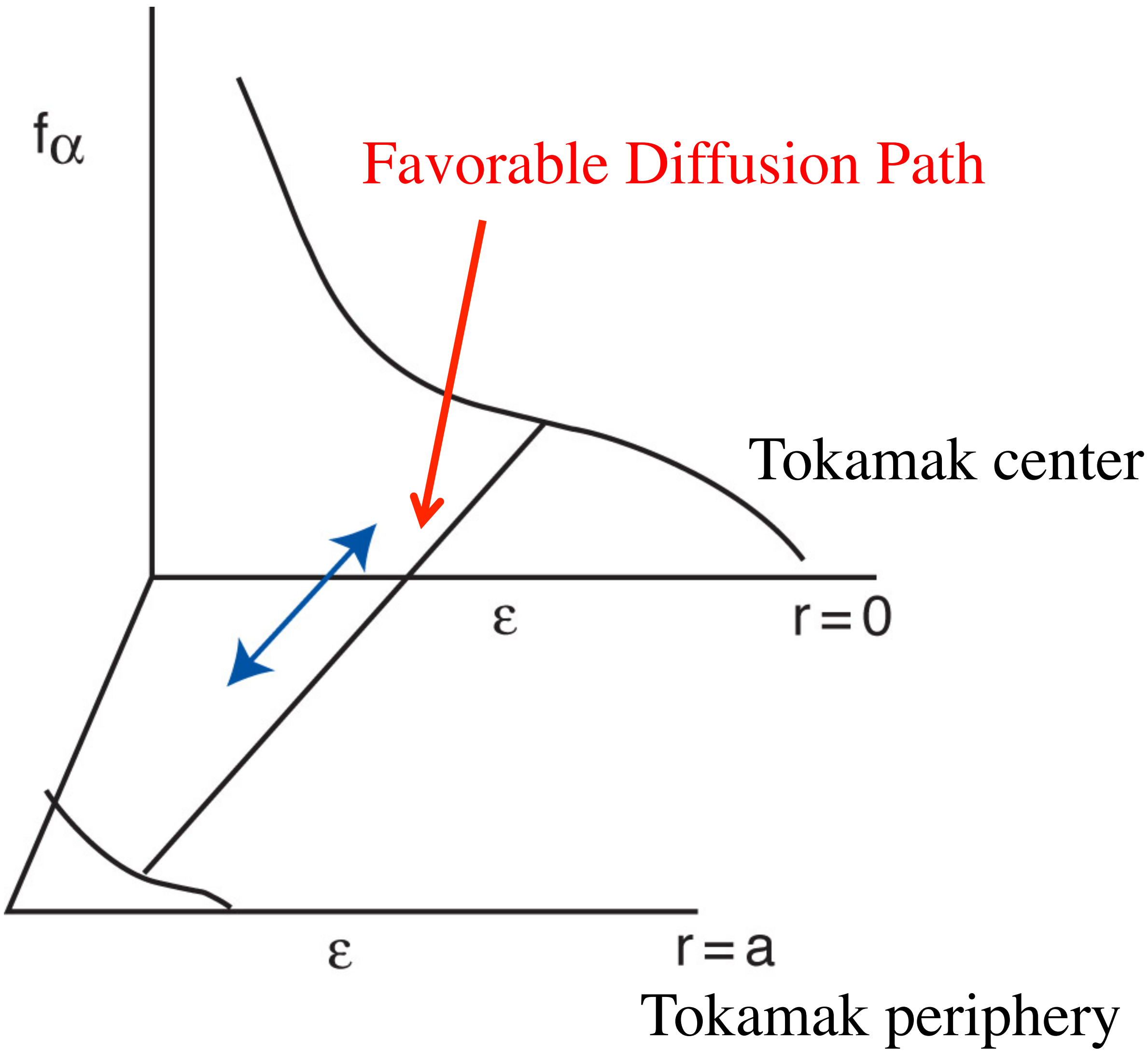
Get Hot Ion Mode: $T_i > T_e$

75% of α power to ions $\Rightarrow P_f \rightarrow 2\,P_f$

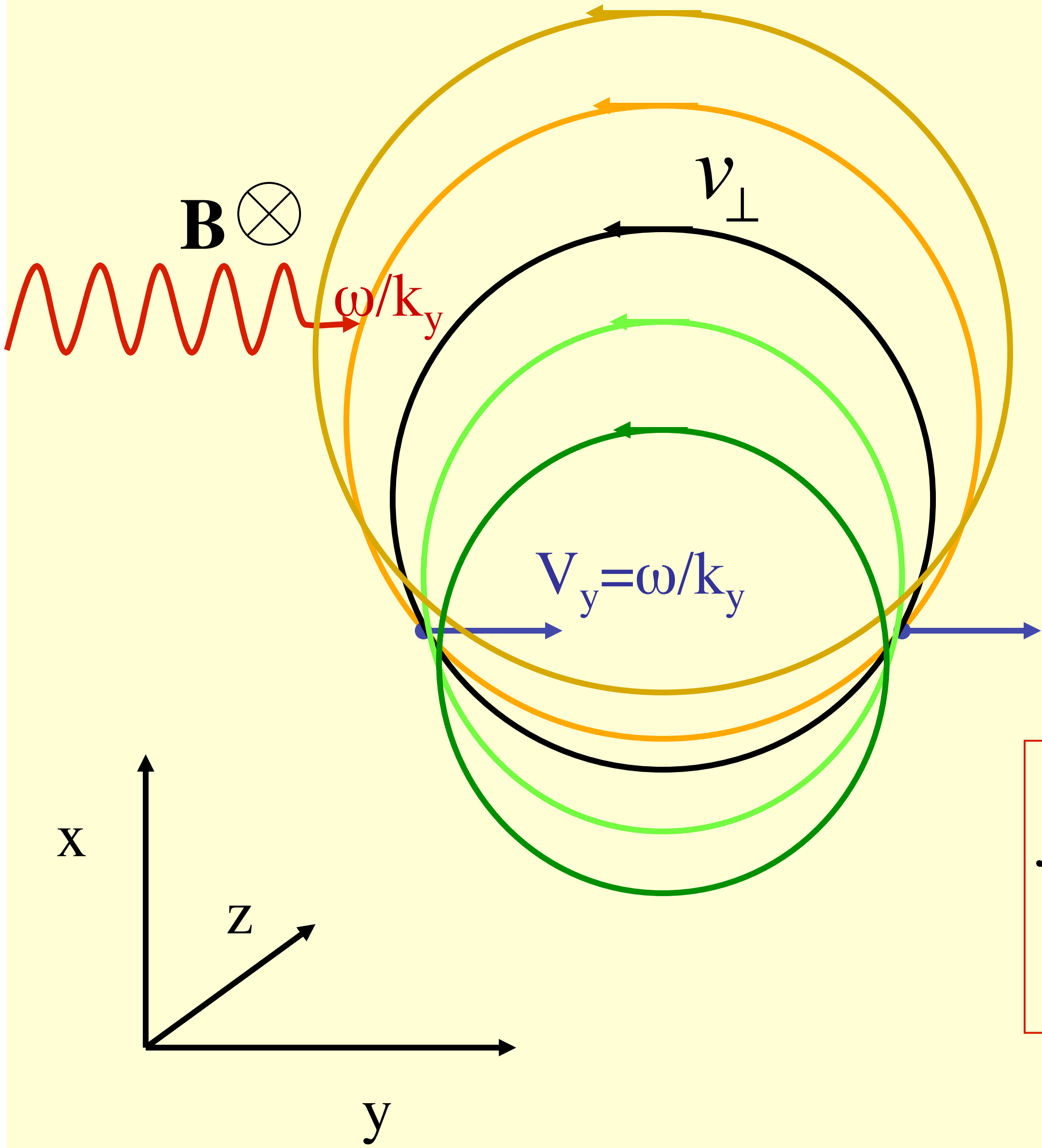
Extracting Free Energy

Fisch and Rax, 1992

motivated by:
Wong and Ono, 1984



Diffusion Paths



$$v_y \rightarrow v_y + \Delta v_y$$

$$x_{gc} \rightarrow x_{gc} + \Delta v_y / \Omega$$

$$\Omega \equiv eB/m$$

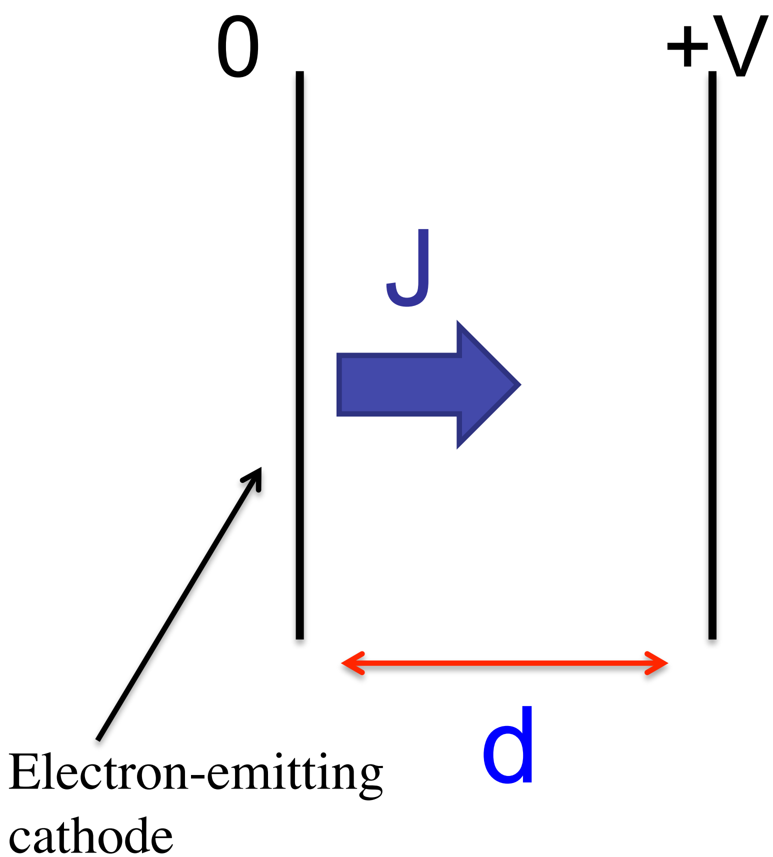
$$\Delta E = m v_y \Delta v_y$$

$$x_{gc} \rightarrow x_{gc} + \frac{\Delta E}{m\Omega \, \omega / k_y}$$

Fisch and Rax, 1992

Herrmann and Fisch, 1997

Child-Langmuir Law
(rigorous upper bound)



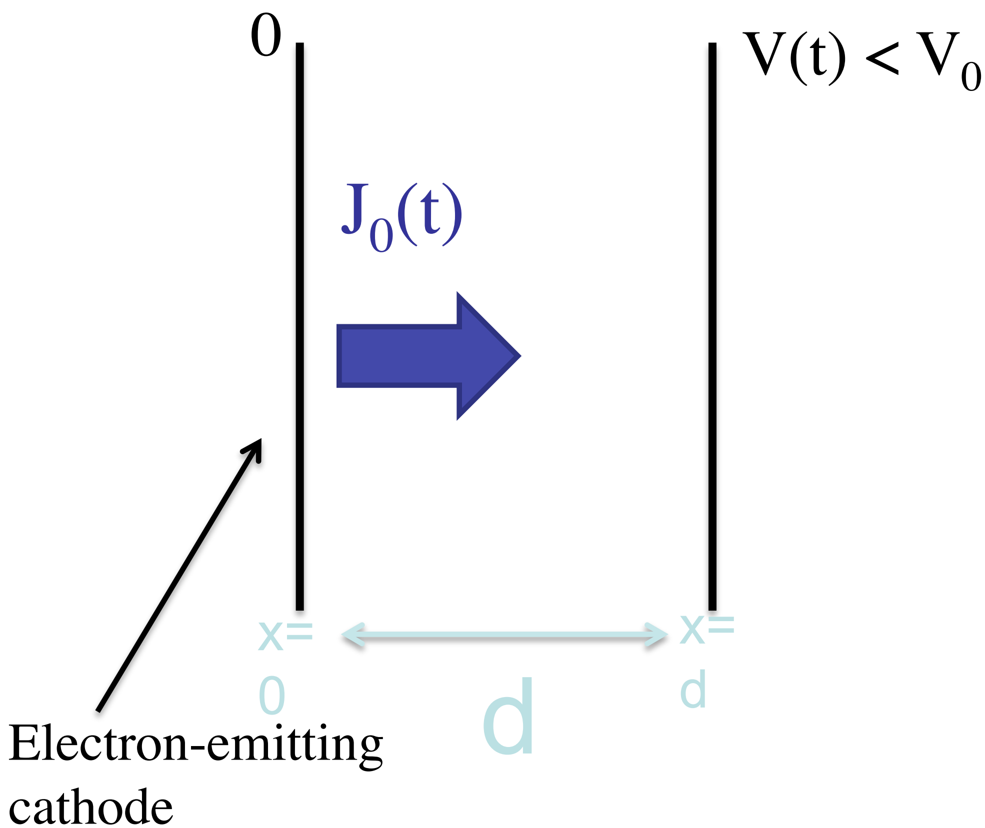
steady-state, maximum current density that can pass through a diode, Child-Langmuir Law:

$$J_{CL} = \frac{4}{9}\varepsilon_0 \sqrt{\frac{2e}{m}}\frac{V^{3/2}}{d^2}$$

Generalizations of Child-Langmuir

- nonzero injection velocity, and Maxwellian distribution (Langmuir, 1923)
- Relativistic (Chetvertkov, 1985)
- Time-Varying voltage to reduce transients (Kadish, Peter, Jones 1985)
- Quantum (Y. Y. Lau, 1991)
- Multi-Dimensional (Y. Y. Lau, Luginsland, 2002)
- Short Pulses (Y. Y. Lau, Valfells 2002)

Time-Dependent Boundary Conditions

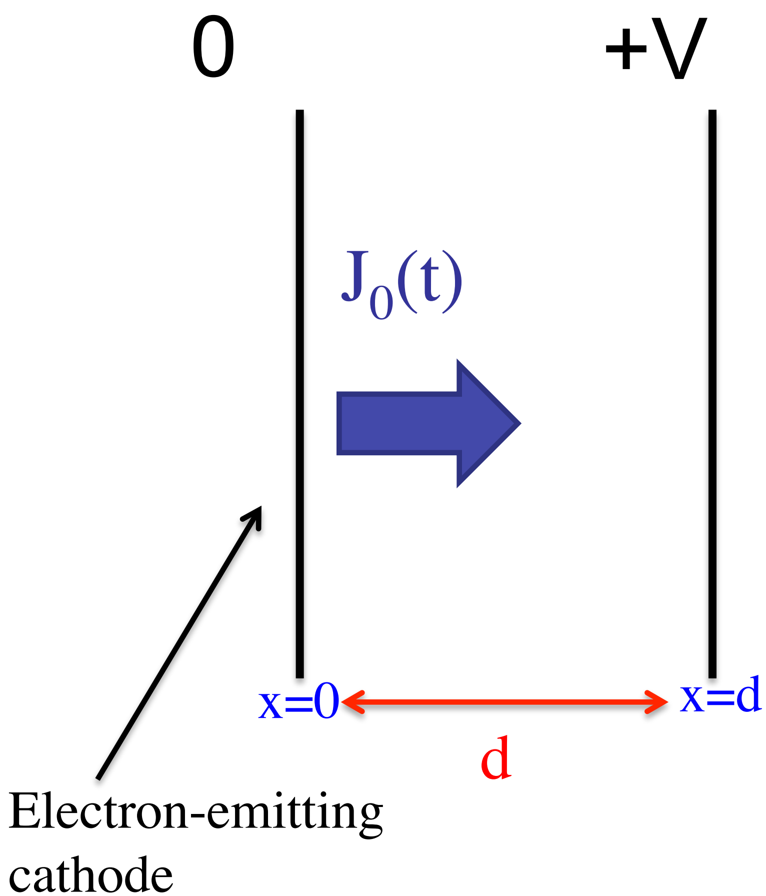


Instantaneous current leaving the diode can exceed the steady-state limit.

But what about the average current over a long period of time?

(Unremarkable) Upper Bound for Time-Averaged Current Density

Griswold, Fisch and Wurtele (2010)



$$J_{CL} \leq \frac{Q_{\max}}{\tau_{\min}} \leq 2.45 \cdot J_{CL}$$

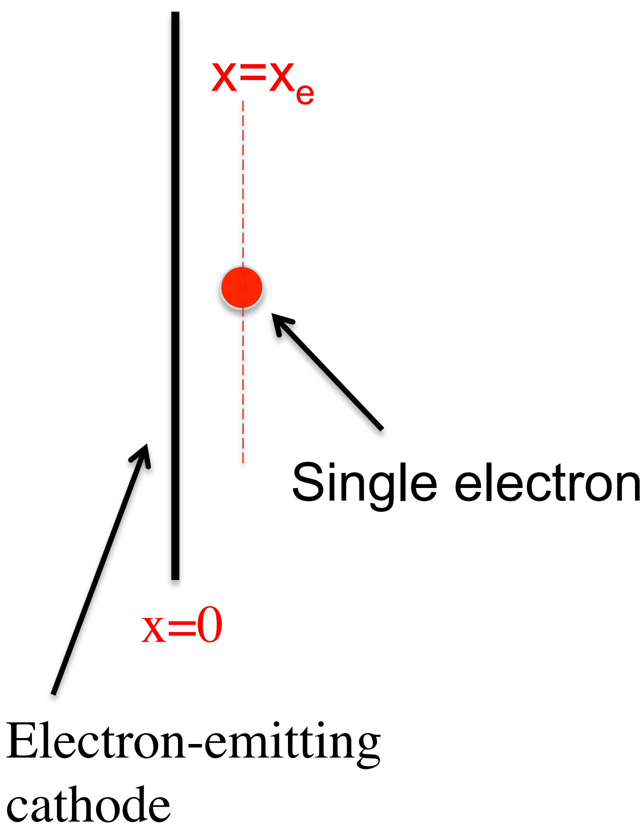
τ_{min} is the minimum transit time of an electron across the diode.

Q_{max} is the maximum charge (per unit area) that can be injected into the diode without violating the boundary condition at the cathode $qE(x=0)\geq 0$.

PIC simulations led us to conjecture that the time dependent limit is equal to the steady-state limit.

Exception:
J_{CL} exceeded by 13% in few Electron "Coulomb Blockade" Regime

Zhu and Ang (2011)



Here electron does not get pushed back into the cathode:

$$qE(x_e) \geq 0$$

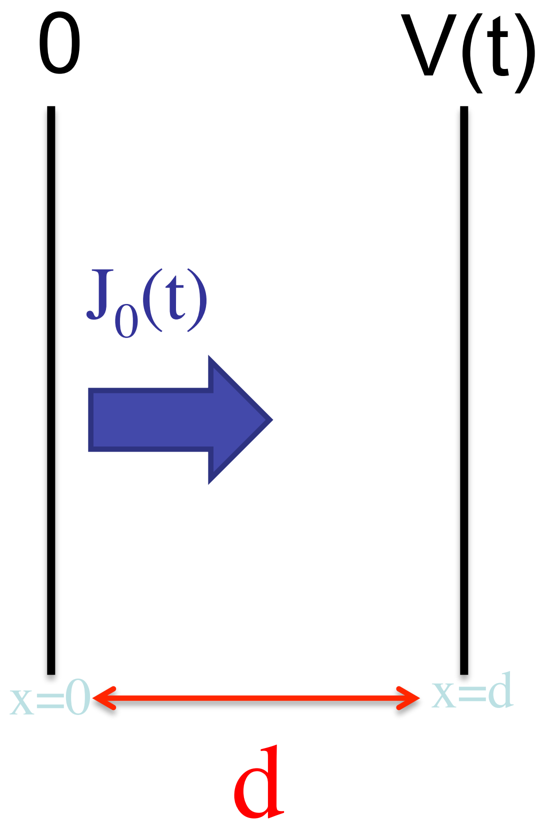
discontinuity in the electric field at the electron means the field at the cathode can fall below zero:

$$qE(x=0) \geq \frac{-q^2}{2\varepsilon_0 A}$$

Griswold, Fisch and Wurtele (2012).

Time Dependent Child-Langmuir Limit with Time Dependent Flux and Voltage

Caflisch and Rosin (2011) arXiv: 1110.2840v1



What is the proper limit to use in this case?

$$J_{CL} = \frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{d^2} \qquad \text{(steady-state)}$$

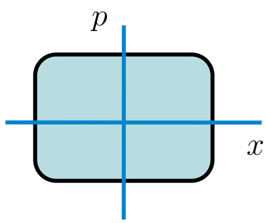
Caflisch and Rosin showed that it is possible to exceed the adiabatic average of the limit:

$$\bar{J}_{\max} \sim \frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\left\langle V^{3/2}(t)\right\rangle$$

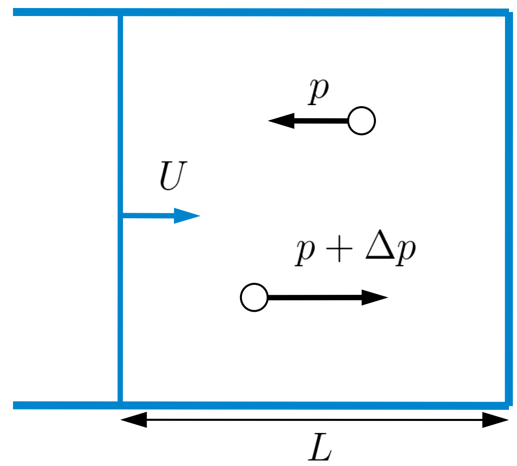
We use the limit defined by the "maximal" boundary conditions

Adiabatic Compression of Waves

$$\Delta p = 2mU = -2m\frac{\Delta L}{\Delta t} = -\frac{\Delta L}{L}p$$



$$\oint p\, dx = \text{inv}$$



$$\mathcal{E} = Np^2/(2m) \propto L^{-2} \propto V^{-2}$$

- Wave as a number of quanta:

$$\mathcal{E}/\omega = \hbar N = I$$

$$J = I/\mathcal{V}$$

$$\partial_t J + \nabla \cdot (\mathbf{v}_g J) = 0$$

$$E = \hbar\omega \qquad p = \hbar k$$

$$\oint p\, dx = \text{inv} \quad \Rightarrow \quad kL = \text{inv}$$

$$\omega = kc$$

$$\mathcal{E} = N\hbar\omega = N\hbar ck \propto L^{-1} \propto V^{-1}$$

What happens to imbedded waves as plasma is compressed?

Regime of adiabatic compression: $\frac{1}{\nu} < \tau_{comp} < \frac{1}{\omega}$

Action conservation: $\frac{VE^2}{\omega} \sim const$

Example:
Plasma Waves

$\omega \sim n^{1/2} \sim V^{-1/2} \longrightarrow E \sim V^{-3/4} \sim n^{3/4}$

$P_{pw} = \frac{E^2}{16\pi} \longrightarrow P_{pw} V^{3/2} = const$

compare: $PV^{\gamma} = const$

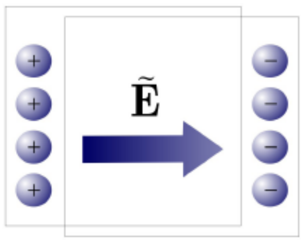
3D: $\gamma = 5/3$

1D: $\gamma = 3$

$\gamma = \frac{m+2}{m}$

Langmuir Wave Compression: Fluid Approach

$\omega_p^2 = 4\pi N e^2/m_e$



- Models vary in EOS, or the expression for $\hat{\mathbf{P}}_e$

$$\partial_t N_e + \nabla \cdot (N_e \mathbf{V}_e) = 0$$

$$\partial_t \mathbf{V}_e + (\mathbf{V}_e \cdot \nabla)\mathbf{V}_e = -(e/m_e)\,\nabla\varphi - \nabla \cdot \hat{\mathbf{P}}_e/(N_e m_e)$$

- Don't *assume* EOS; instead, *derive* it from

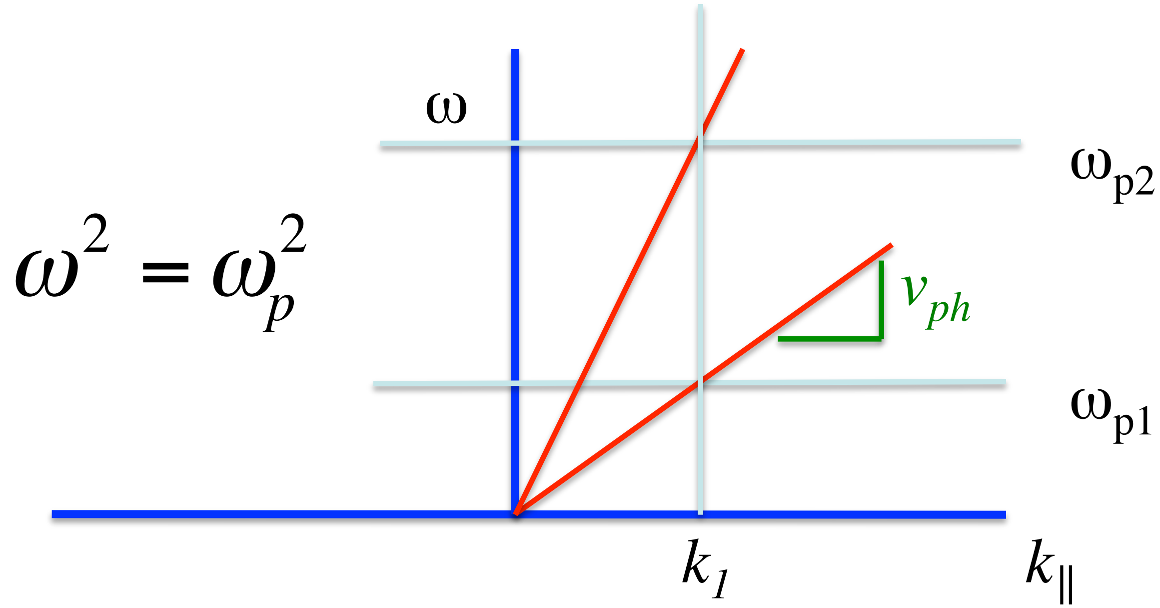
$$\partial_t \hat{\mathbf{P}}_e + (\mathbf{V}_e \cdot \nabla)\hat{\mathbf{P}}_e + \hat{\mathbf{P}}_e(\nabla \cdot \mathbf{V}_e) + [(\hat{\mathbf{P}}_e \nabla)\mathbf{V}_e] + [(\hat{\mathbf{P}}_e \nabla)\mathbf{V}_e]^{\mathsf{T}} = 0$$

$$\frac{\partial'^2 n}{\partial t^2} + \omega_p^2 n - C_{j\ell}\,\frac{\partial^2 n}{\partial x_j\,\partial x_\ell} + + 2\,\frac{\partial' n}{\partial t}\left(\frac{\Omega}{\omega_p}\,\frac{\partial \omega_p}{\partial x_\ell} + k_j W_{j\ell}\right)\frac{k_\ell}{k^2} - \left(\delta_{js} + \frac{k_j k_s}{k^2}\right)\frac{\partial C_{s\ell}}{\partial x_j}\,\frac{\partial n}{\partial x_\ell} = 0$$

$$\omega^2 = \omega_p^2 + 3k^2 v_{Te}^2 \approx \omega_p^2 \qquad\qquad \mathcal{E} = |E|^2/(8\pi) \propto N^{3/2}$$

Dodin, Geyko, & Fisch, POP, 2009

Compression Perpendicular to *k*



$\omega_{p1} \rightarrow \omega_{p2}$

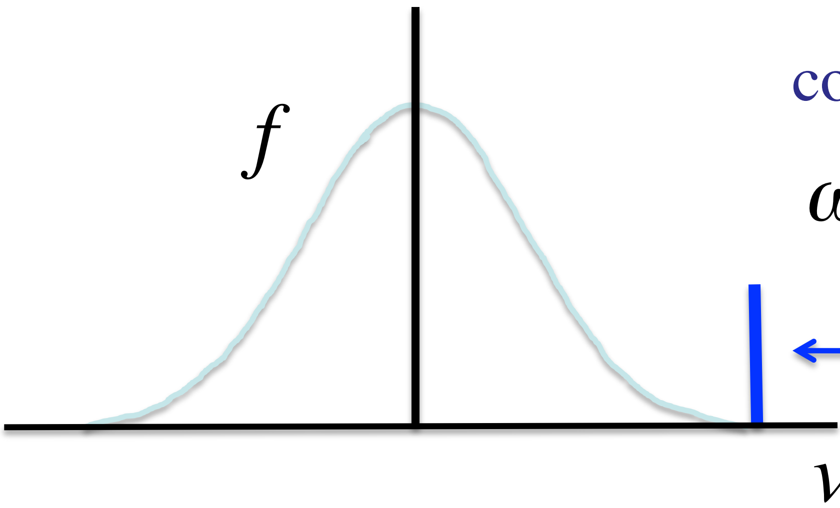
$k_1 \rightarrow k_1$

Under compression: Less damping,

if collisionless $\rightarrow$ $T_\perp > T_\parallel$

Under expansion: More damping, $T_\perp < T_\parallel$

Current Drive and Heating

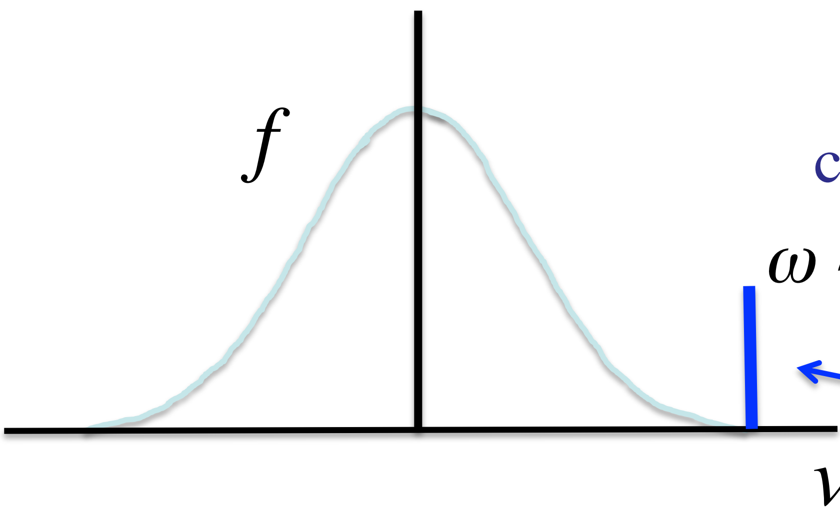


compression perpendicular to k

$\omega \sim n^{1/2}$ $\quad k \sim v_{\parallel} \sim const$

$$\omega / k v_{\parallel} \sim n^{1/2}$$

Note: under expansion, $T_{\perp} < T_{\parallel}$



compression parallel to k

$\omega \sim n^{1/2}$ $\quad k \sim 1/L \sim n$ $\quad v_{\parallel} \sim 1/L \sim n$

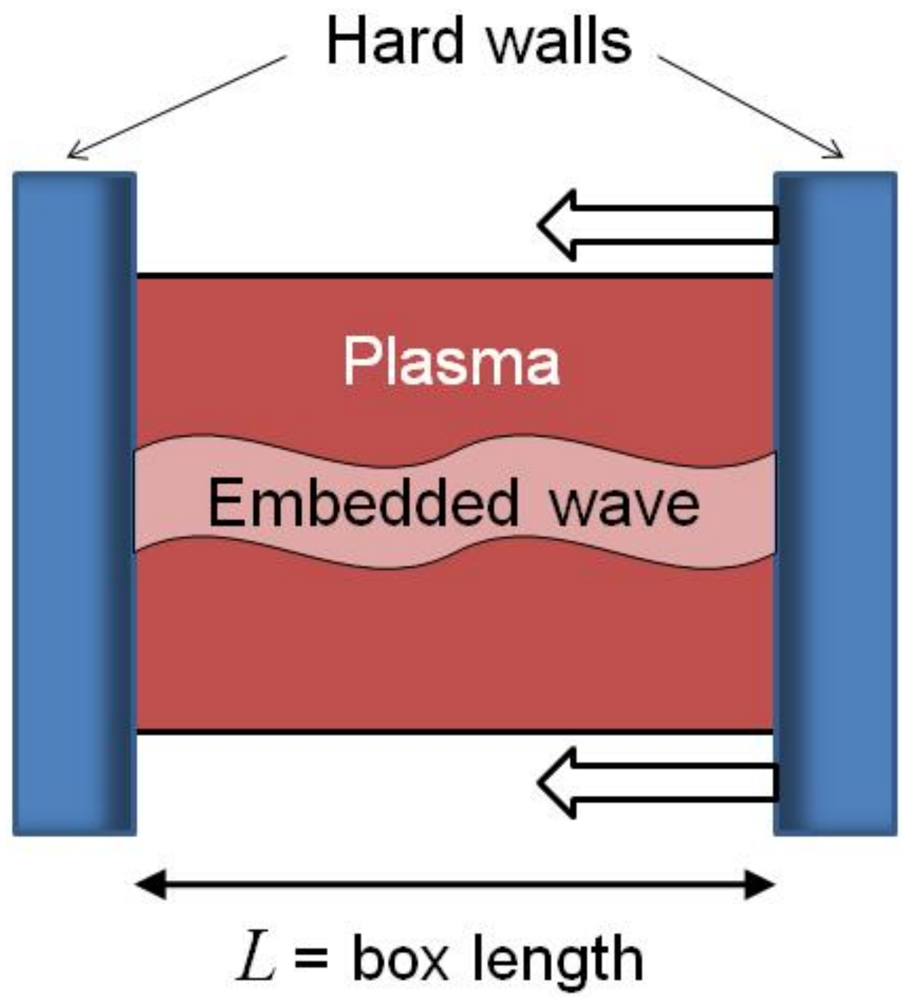
$$\omega / k v_{\parallel} \sim n^{-3/2}$$

Note: under compression, $T_{\perp} < T_{\parallel}$

In either case, extra wave energy can accentuate energy difference

Particle Simulations

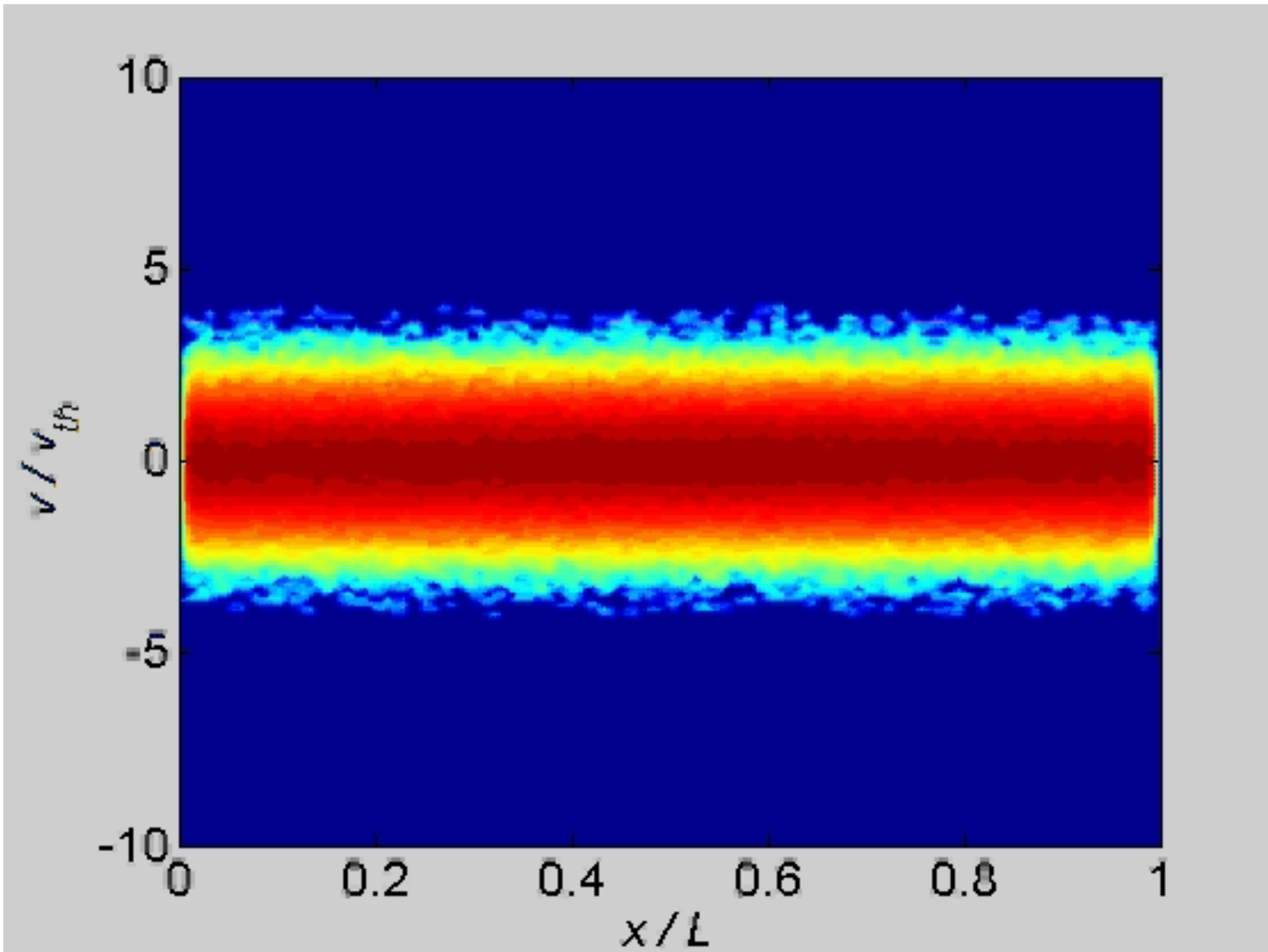
Schmit et al, 2010



PIC simulation schematic

Plasma wave compression

Longitudinally compressing Langmuir wave – $f(x,v)$

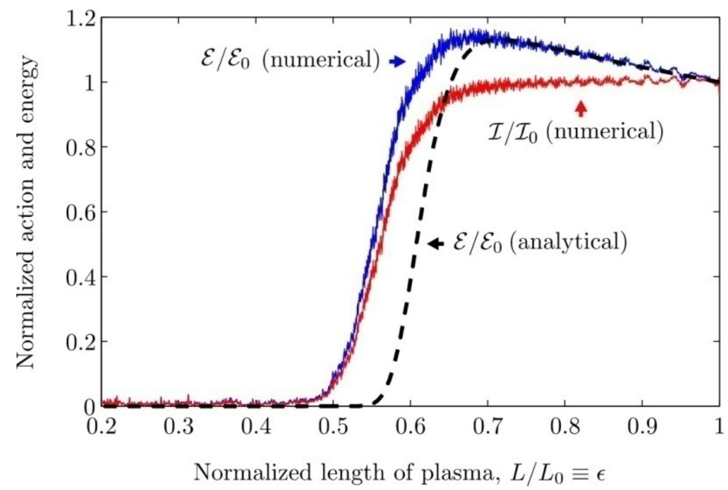


Langmuir Wave "Switch"

$$\omega^2 = \omega_p^2 + 3k^2 v_{Te}^2 \approx \omega_p^2(n)$$

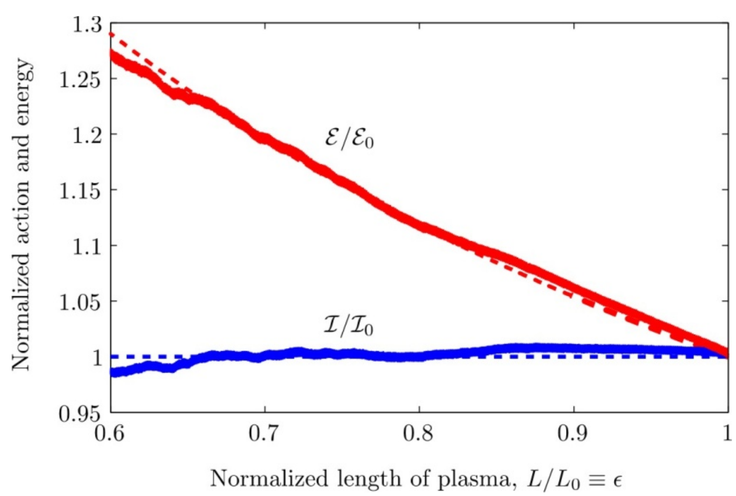
$$|E| \propto n^{3/4}$$

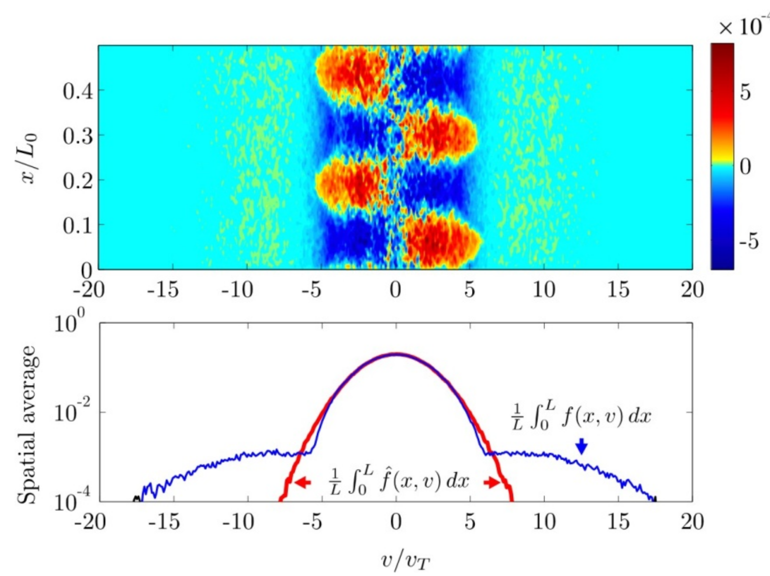
$$k\lambda_{\mathrm{D}} \propto L^{-3/2}$$



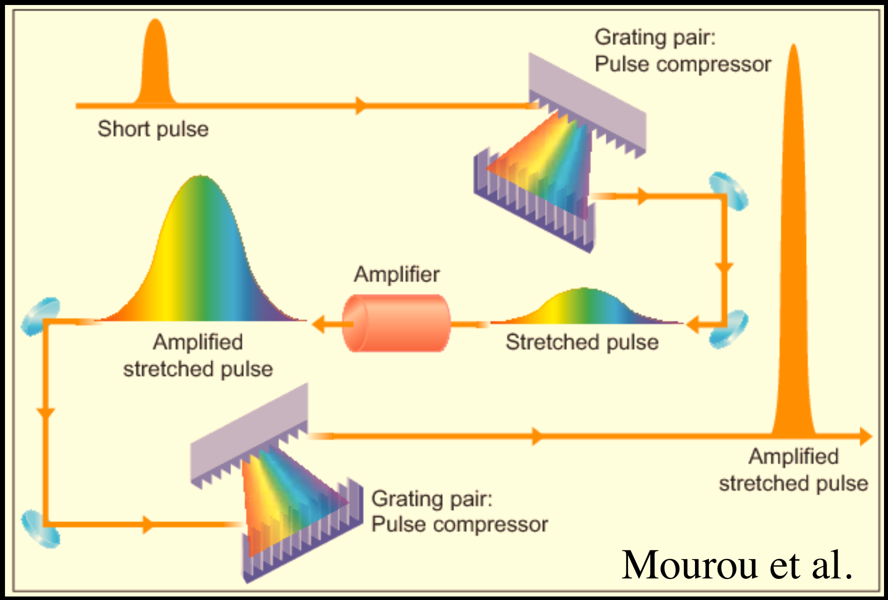
Dodin, Geyko, and Fisch, Phys. Plasmas (2010)

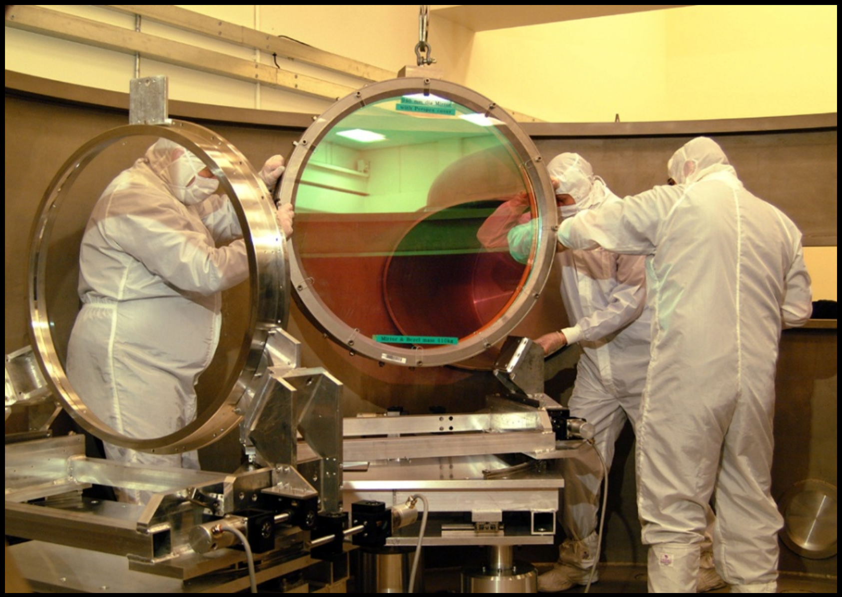
Schmit, Dodin, and Fisch, PRL (2010)





Chirped Pulse Amplification: stretch, amplify, then recompress





Gratings for Petawatt (10^{15}W) Laser

Limitations of CPA

Thermal damage to expensive gratings

Requires broad-bandwidth high-fluence amplifiers

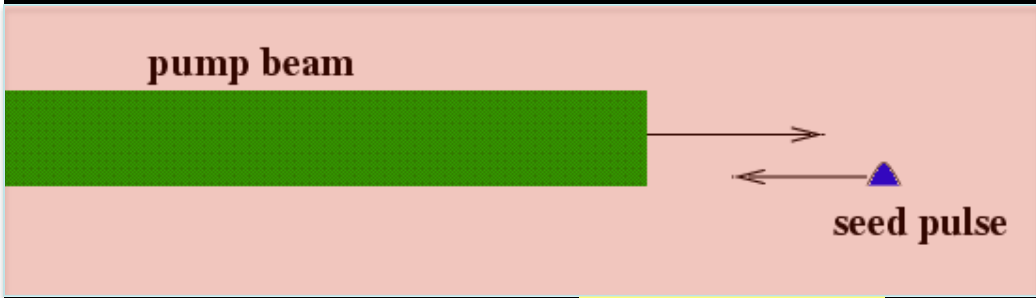
For PW

10^3 compression

< 10 ps

GW/cm² in amplifier → TW/cm² → 10^3 cm² gratings

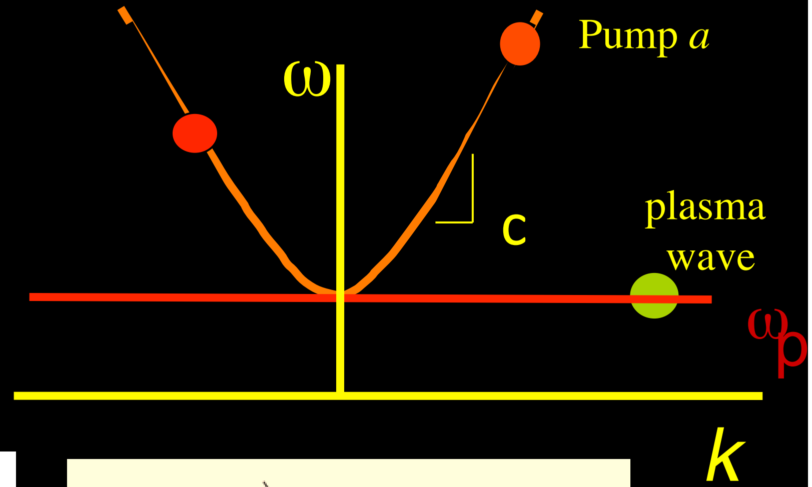
Resonant Raman Amplification and Compression

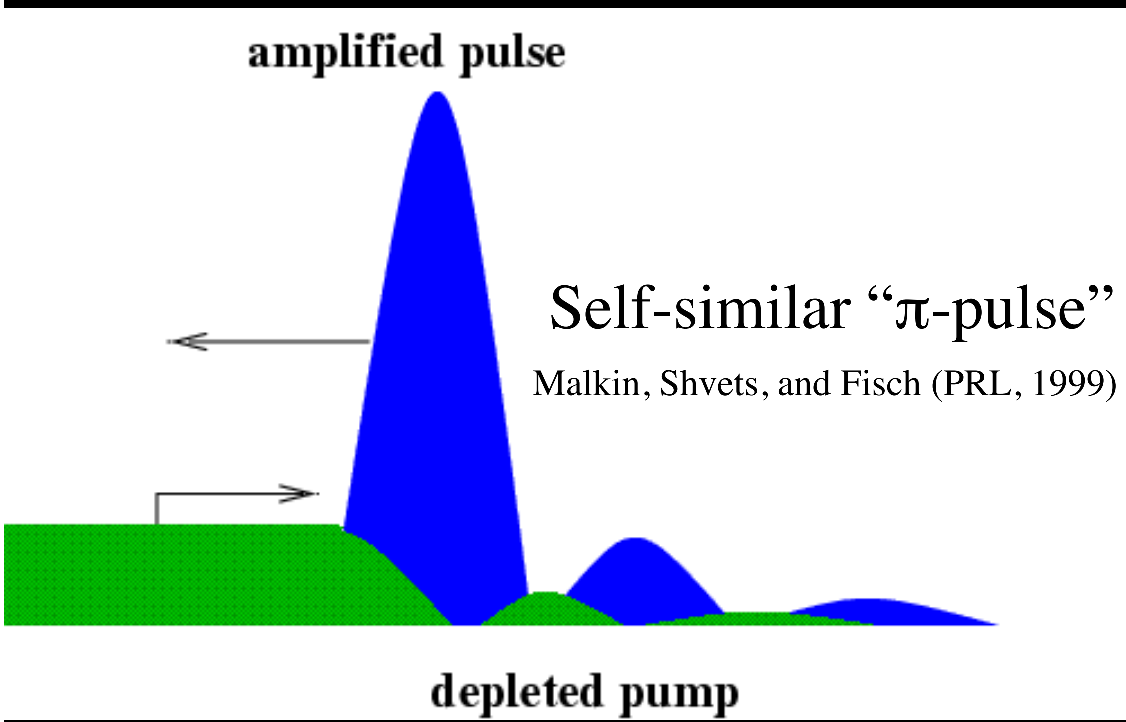


resonance condition

$$\omega_a - \omega_b = \omega_p$$

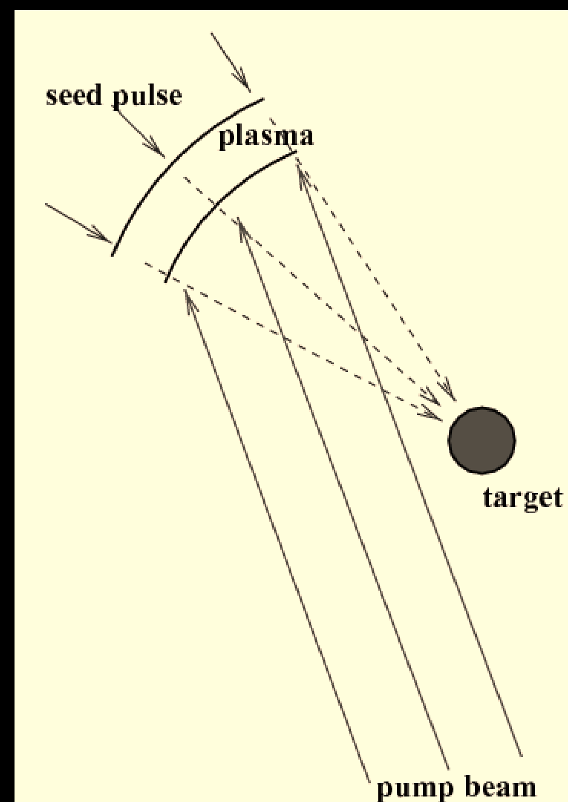
$$\vec{k}_a - \vec{k}_b = \vec{k}_p$$



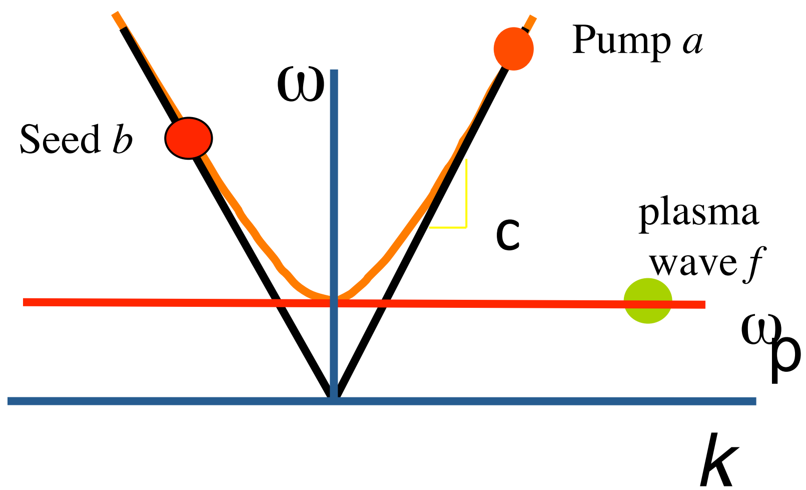


Self-similar "π-pulse"

Malkin, Shvets, and Fisch (PRL, 1999)



Moderately under-critical plasma



$$a_t + c_a a_z = V_3 f b$$

$$b_t - c_b b_z = -V_3 a f^* - i\kappa b_{tt} + iR\,|b|^2\,b$$

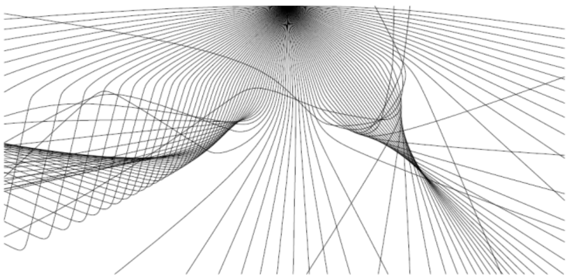
$$f_t = -V_3 a b^*$$

Method of Dodin – Generalized Lagrangian Approach

Modeling wave propagation in a dispersive medium: Linear waves

- Asymptotic geometrical-optics, or eikonal methods of modeling wave propagation:

e.g., Runborg (2007), Kravtsov and Orlov (1990)…



Ray tracing

$$\mathfrak{D}(\omega, \mathbf{k}; t, \mathbf{x}) = 0$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_g, \quad \frac{d\mathbf{k}}{dt} = -\nabla\omega$$

Hamilton-Jacobi methods

$$\mathfrak{D}(-\partial_t S,\ \nabla S;\ t, \mathbf{x}) = 0$$

Wave kinetic equation

$$\mathfrak{D}(\omega, \mathbf{k}; t, \mathbf{x}) = 0$$

$$\frac{\partial f}{\partial t} + \mathbf{v}_g \cdot \frac{\partial f}{\partial \mathbf{x}} - \nabla\omega \cdot \frac{\partial f}{\partial \mathbf{k}} = 0$$

↓

Method of momenta

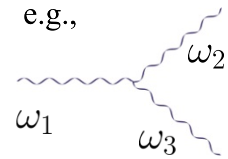
$$\frac{\partial}{\partial t}\left(\frac{\mathcal{E}}{\omega}\right) + \nabla \cdot \left(\mathbf{v}_g \frac{\mathcal{E}}{\omega}\right) = 0$$

action conservation theorem =
continuity equation for the photon density

Modeling wave propagation in a dispersive medium: Nonlinear waves

- A physical model must be conservative:
  - Single wave still conserves its action, or quanta
  - Resonant interactions conserve Manley-Rowe integrals

e.g.,



$$N_1 + N_2 = \mathrm{const}$$
$$N_1 + N_3 = \mathrm{const}$$
$$N_2 - N_3 = \mathrm{const}$$

NLSE

$$i(\partial_t \psi + \mathbf{v}_{g0} \cdot \nabla \psi) + \frac{1}{2} \mathbf{v}'_{g0} : \nabla^2 \psi - \omega_{\mathrm{NL}} \psi = C(\boldsymbol{\psi})$$

↓

Wigner-Moyal

$$\partial_t f + (\mathbf{v}_{g0} + \mathbf{v}'_{g0} \cdot \mathbf{p}) \cdot \partial_{\mathbf{x}} f - \omega_{\mathrm{NL}} \sin\left(\frac{1}{2} \overleftarrow{\partial}_{\mathbf{x}} \overrightarrow{\partial}_{\mathbf{p}}\right) f = C(f)$$

↓

Method of momenta

$$\partial_t \mathcal{I} + \nabla \cdot (\mathbf{v}_g \mathcal{I}) = C(\boldsymbol{\mathcal{I}})$$
$$\partial_t \mathbf{k} + \nabla \omega = 0$$

$$f \approx \sum \mathcal{I}(t, \mathbf{x})\, \delta(\mathbf{p} - \Delta \mathbf{k})$$

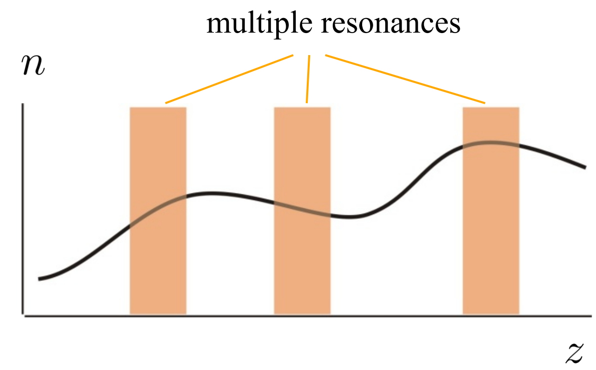
$$\mathbf{v}_g \equiv \mathbf{v}_{g0} + \mathbf{v}'_{g0} \cdot \Delta \mathbf{k}$$

$$\omega = \omega_0 + \partial_{\mathbf{k}} \omega_0 \cdot \Delta \mathbf{k} + \omega_{\mathrm{NL}}$$

- But… These *nonlinear* GO envelope equations assume
  - that the nonlinearity is adequately modeled by the NLSE
  - that the underlying medium is homogeneous and stationary

Weakly nonlinear GO waves in inhomogeneous nonstationary medium

- Field-theoretical Lagrangian approach yields equations that are conservative in *general* GO medium, at all *z* and *t*



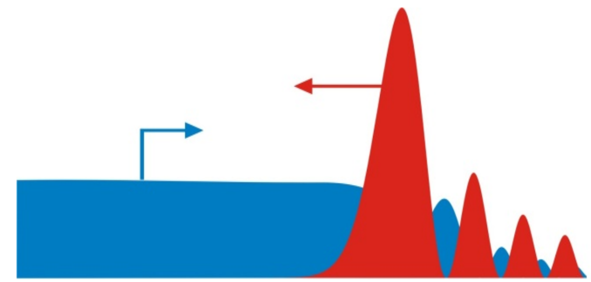
Single wave

Whitham (1965), Bretherton and Garrett (1968)...

Lagrangian density: $\mathfrak{L}(a, \omega, \mathbf{k})$

dispersion relation: $\mathfrak{L}_a = 0$

envelope equation: $\partial_t \mathfrak{L}_\omega - \nabla \cdot \mathfrak{L}_\mathbf{k} = 0$



Multiple resonant waves

e.g., $\eta = \theta_1 - \theta_2 - \theta_3$

$$\mathfrak{L}_{a_j} = 0$$

$$\partial_t \mathbf{k}_j = -\nabla \omega_j$$

$$\partial_t \mathfrak{L}_{\omega_j} - \nabla \cdot \mathfrak{L}_{\mathbf{k}_j} = \pm \mathfrak{L}_\eta$$

$$\partial_t \eta = \omega_1 - \omega_2 - \omega_3$$

- Included beat phases, $\mathfrak{L} = \mathfrak{L}(a_j, \omega_j, \mathbf{k}_j, \eta)$
  - Closed set of slow-motion PDEs
  - Any resonances (linear and nonlinear) and wave self-action included
  - Wave action/Manley-Rowe integrals are manifestly conserved
  - …and, in plasma, we also know the nonlinear Lagrangian explicitly!

cf. Brizard and Kaufman (1995)

Method of Dodin -- Master Lagrangian: Raman scattering as an example

- The wave Lagrangian can be expressed through ensemble-averaged oscillation-center energies

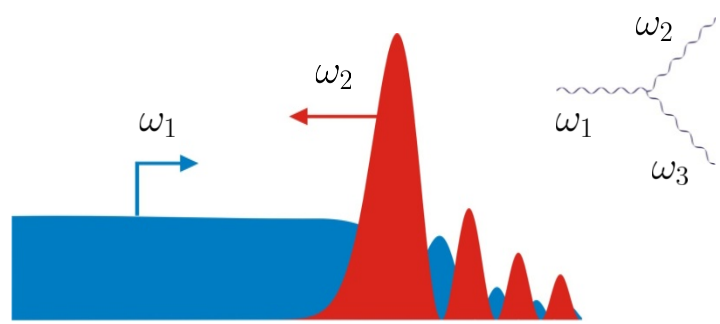
$$\mathfrak{L}(a, \omega, \mathbf{k}) = \frac{\langle E^2 - B^2 \rangle}{8\pi} - \sum_s n_s \langle \mathcal{H}_s \rangle_f$$

- Making approximation *in the Lagrangian* does not affect the conservative properties of the equations

- Example:

$$\mathcal{H} \approx \frac{P^2}{2m} + \frac{e^2 |\mathbf{E}|^2}{4m\omega^2}$$

$$|\mathbf{E}|^2 = |\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + \mathbf{E}_1 \cdot \mathbf{E}_2^* + \mathbf{E}_1^* \cdot \mathbf{E}_2$$



$$\mathfrak{L} = \mathfrak{L}_1 + \mathfrak{L}_2 + \mathfrak{L}_3 - \frac{e^2 E_1 E_2}{4m\omega_1\omega_2} \tilde{n} \cos\eta$$

$$\mathfrak{L}_{1,2} = \frac{E_{1,2}^2}{16\pi} \left[\epsilon_\perp(\omega) - \frac{k^2 c^2}{\omega^2} \right]_{1,2}$$

$$\mathfrak{L}_3 = \frac{\epsilon_\parallel(\omega_3, \mathbf{k}_3)}{16\pi} \left(\frac{4\pi e \tilde{n}}{k_3} \right)^2$$

- The nonlinear coupling affects the dispersion; e.g,

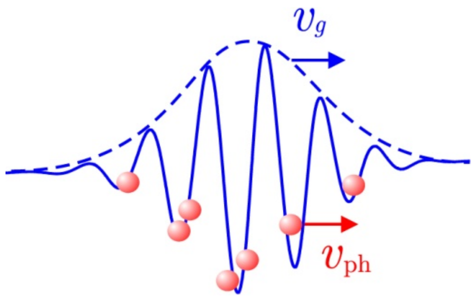
$$\frac{E_1}{16\pi} \left[\epsilon_\perp(\omega_1) - \frac{k_1^2 c^2}{\omega_1^2} \right] - \frac{e^2 E_2 \tilde{n}}{4m\omega_1\omega_2} \cos\eta = 0$$

- The nonlinear coupling affects the transport:

$$\frac{\partial}{\partial t} \left(\frac{\mathcal{E}_j}{\omega_j} + \Delta\mathfrak{L}_{\omega_j} \right) + \nabla \cdot \left(\mathbf{v}_g \frac{\mathcal{E}_j}{\omega_j} - \Delta\mathfrak{L}_{\mathbf{k}_j} \right) = \pm \mathfrak{L}_\eta$$

cf. Brizard and Kaufman (1995)

Effects due to trapped particle are *special*. The NLSE does not apply



- Trapped electrons contribute an E-independent term

$$\mathfrak{L}(\omega, k, E) = \epsilon(\omega, k)\,\frac{E^2}{16\pi} + \sigma e E + \frac{m\sigma\omega^2}{2k}$$

Dodin and Fisch, Phys. Plasmas (2012a,c)

- The only self-action *not* described by NLSE!

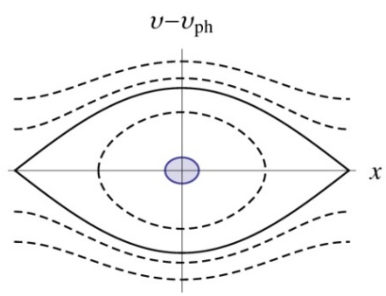
$$(\partial_t + v_{g0}\,\partial_x)\psi + \frac{1}{2}\,v'_{g0}\,\partial^2_{xx}\psi = -i\omega_{\rm NL}\psi$$

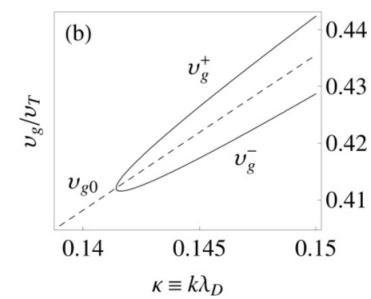
- TPMI theory must be revised

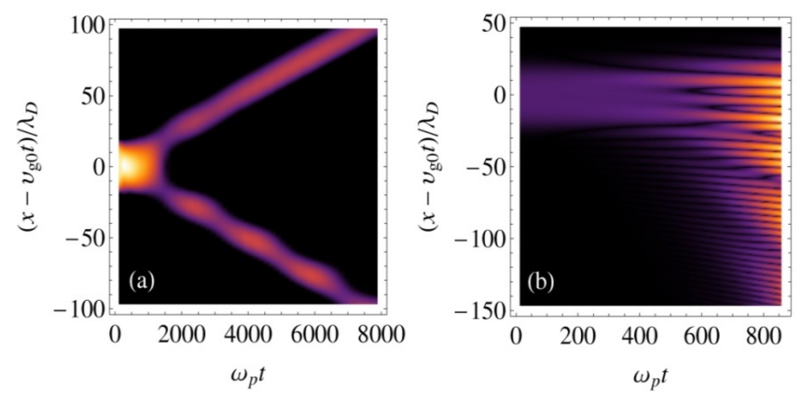
$$\gamma \approx \Delta k\, v_{g0}\Omega_E\sqrt{S\,(S-1/2)}$$

$$\Omega_E = \sqrt{\frac{eEk}{m\omega_p^2}} \qquad S = \frac{\text{trapped-}e\text{ energy flux}}{\text{passing-}e\text{ energy flux}}$$

cf. Dewar et al., 1972; Ikezi et al., 1978; Rose, 2005; Rose and Yin, 2008; Istomin and Karpman, 1972; Benisti et al., 2010...





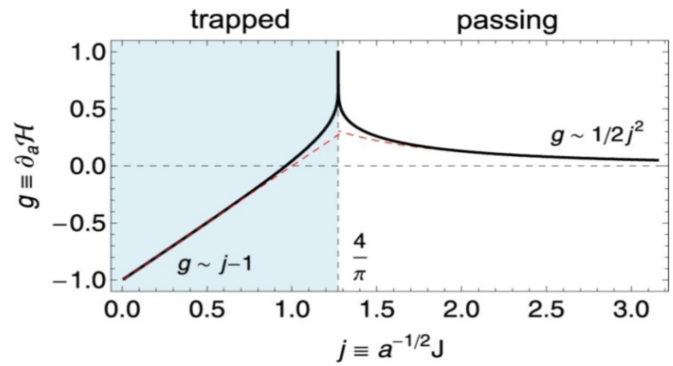


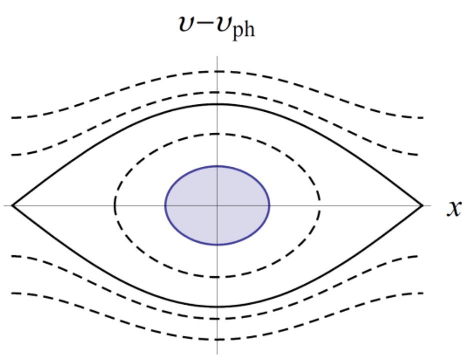
Nonlinear dispersion of waves with trapped electrons (e.g., BGK modes)

- For any distribution $F(\mathsf{J})$, $\mathfrak{L}_a = 0$ yields

$$\omega^2 = \omega_p^2 \, \frac{2}{a} \int_0^\infty g(j) F(\mathsf{J}) \, d\mathsf{J}$$

(similarly for other waves, e.g., whistlers)

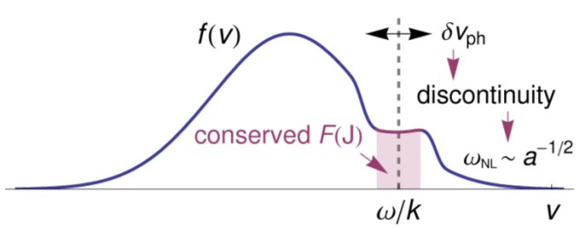




$$\epsilon(\omega, k) + \sqrt{a}\, \frac{\omega \varkappa}{\omega_p} \left[\frac{\omega_p^3 m^3}{2k^6} F''(0) \right] + \ln a \left[\frac{m^2 \omega_p^2}{2k^4} F'(0) \right] = 0$$

Dodin and Fisch, PRL (2011); PoP (2012b)

- δ-beam: $\omega^2 = \omega_L^2 - 2\omega_b^2 / a$
- flat beam: $\omega^2 = \omega_L^2 - [8/(3\pi)] a^{-1/2} \, \omega_p^2 F_b$



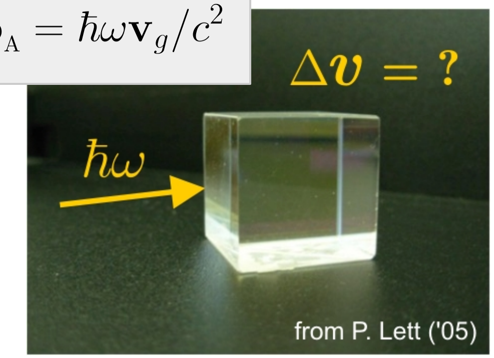
cf. Manheimer and Flynn, 1971; Dewar, 1972; Winjum et al., 2007; Khain and Friedland, 2007; Goldman and Berk, 1971; Krasovsky, 2007; Rose and Russell, 2001; Benisti and Gremillet, 2007; Lindberg et al., 2007...

Photon momentum in a dielectric. Resolving the Abraham-Minkowski controversy

- The same Lagrangian approach *actually* resolves the 100-year-old "dilemma" about the wave energy-momentum in dielectric

$$\mathbf{p}_{\mathrm{M}} = \hbar\mathbf{k}$$

$$\mathbf{p}_{\mathrm{A}} = \hbar\omega\mathbf{v}_g/c^2$$



Canonical, or Minkowski EMT

This part is known; cf. Sturrock (1961), Whitham (1965), Dougherty (1970)...

$$\left(\begin{array}{c|c} \mathcal{E} = \mathcal{I}\omega & \mathbf{Q} = \mathbf{v}_g\mathcal{E} \\ \hline \boldsymbol{\mathcal{P}} = \mathbf{k}\mathcal{E}/\omega & \hat{\Pi} = \boldsymbol{\mathcal{P}}\mathbf{v}_g \end{array}\right)$$

Barnett (2010); Kemp (2011) ; Milonni and Boyd (2011); Baxter and Loudon (2010); Pfeifer *et al* (2007)…

Physical, or Abraham EMT

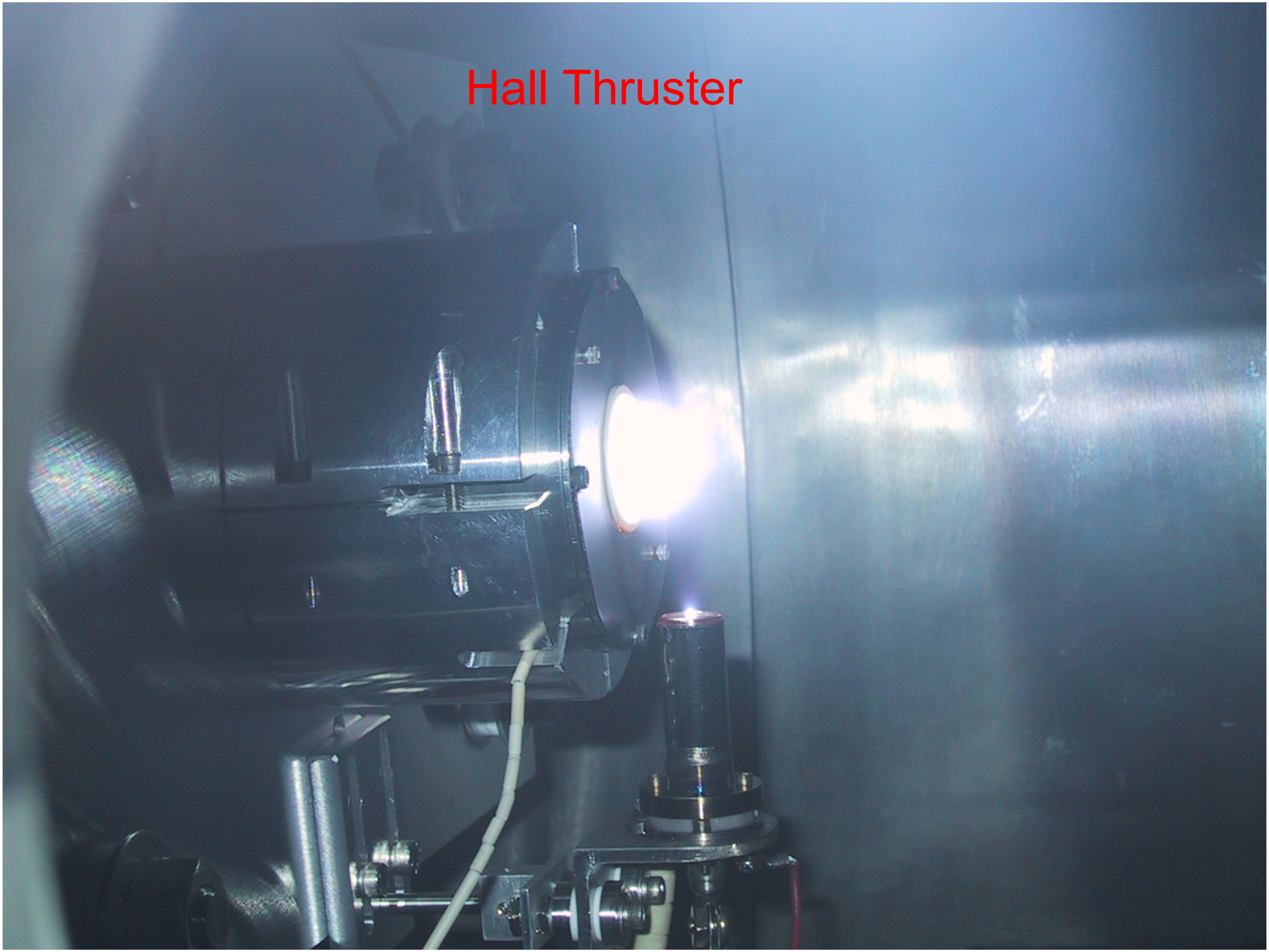
Dodin and Fisch, submitted to PRA; generalizes Dewar (1977)

- We can derive the physical EMT, *including* striction effects, without specifying the wave nature
  - Abraham's formula, $p = p_A$, holds only in resting fluid
  - Now we can calculate the *full* ponderomotive force

$$\tau^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\begin{pmatrix} \mathcal{E} & \mathcal{E}\mathbf{v}_g/c \\ \mathcal{E}\mathbf{v}_g/c & \mathcal{E}\,\mathbf{k}\mathbf{v}_g/\omega + \mathcal{U}\,\hat{\mathbf{1}} \end{pmatrix}'$$

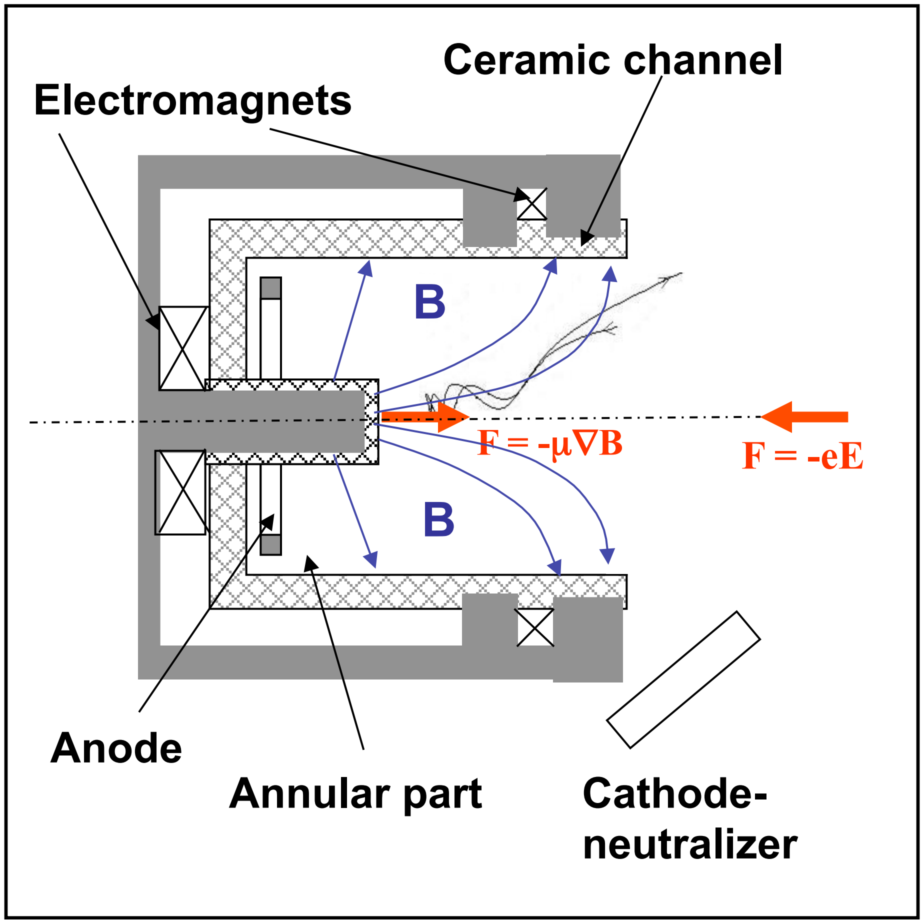
U is the ponderomotive energy density,
Λ is the matrix of Lorentz transformation

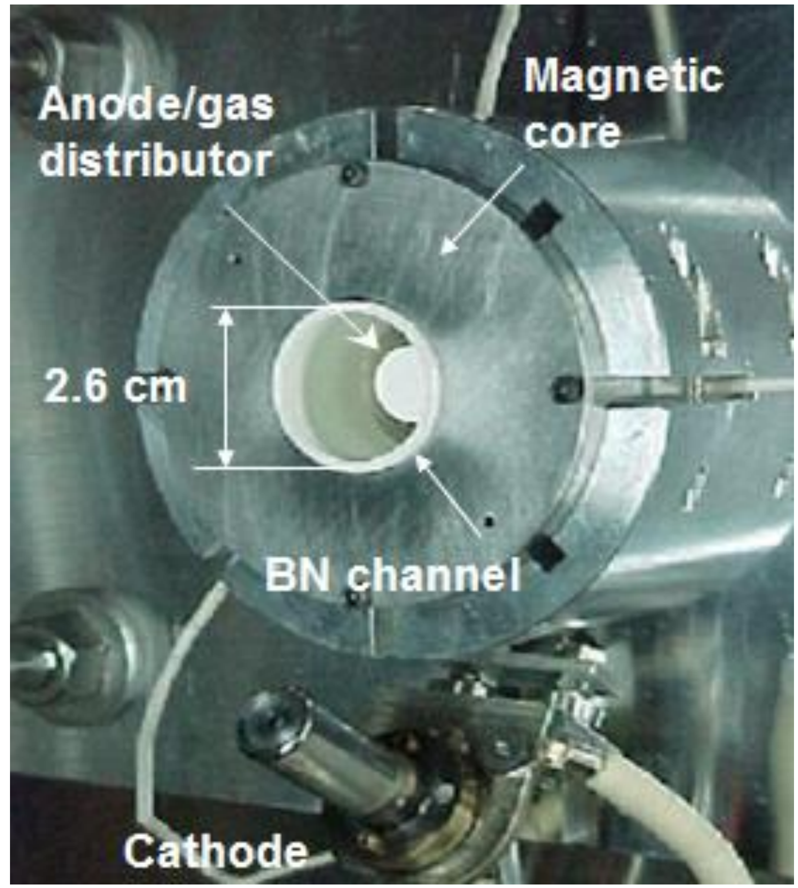
…could also include these average forces to model the bulk plasma dynamics…


Hall Thruster

Cylindrical Hall Thruster

Fundamentally different from conventional HT:
Electrons are confined in a hybrid magneto-electrostatic trap.

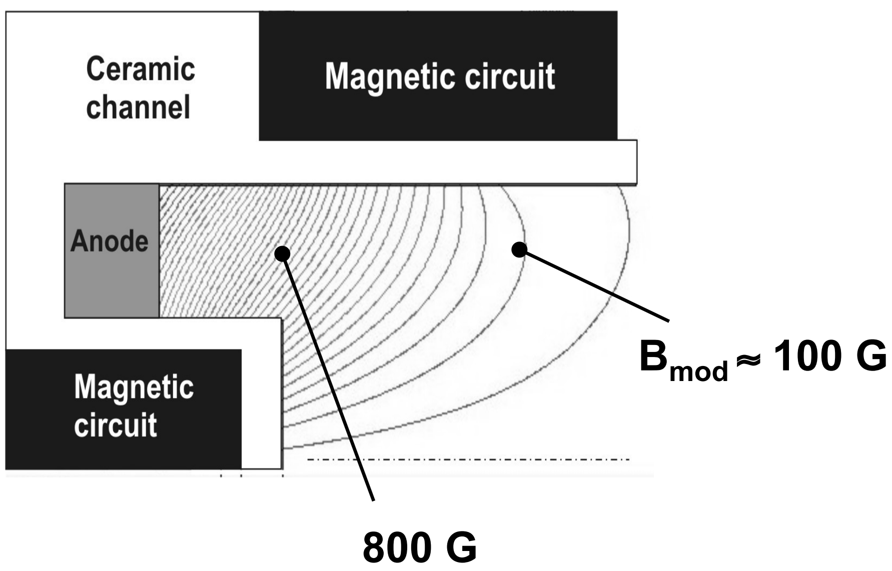




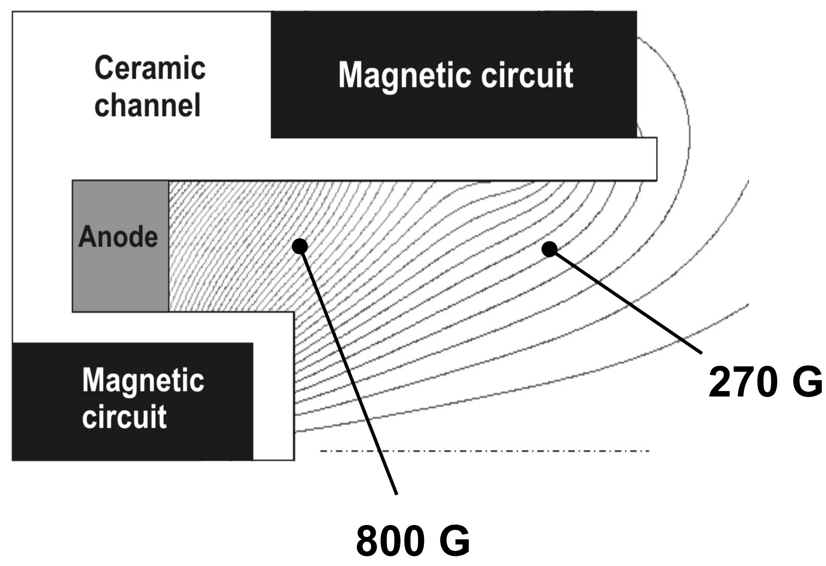
PPPL CHT: P = 50 - 300 W
OD = 2.6 cm
T = 2 - 12 mN

Y. Raitses and N. J. Fisch, Physics of Plasmas, 8, 2579 (2001).

Cylindrical Configurations



Cusp Geometry

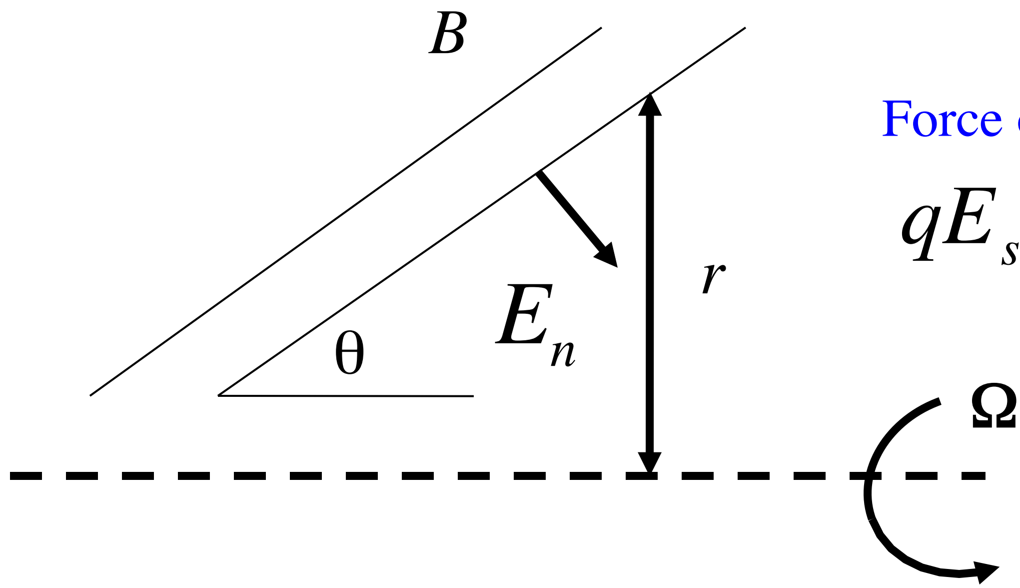


Direct Geometry

Cusp Geometry was thought important to produce axial thrust

A. Rotation of Force Vector by Supersonically Rotating Electrons

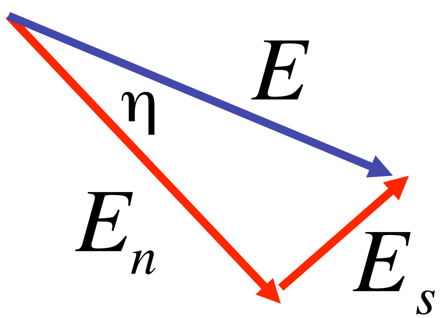
Fisch, Raitses, Fruchtman (2011)



Force on ions = centrifugal force on electrons

$$qE_s = -eE_s = m_e \Omega^2 r \cos\theta$$

$$\Omega_e \equiv eB/m$$



$$\eta \approx \sin\eta = E_s / E_n = \frac{\Omega}{\Omega_e}\cos\theta = \frac{\rho_L}{r}\left(\frac{E/B}{v_T}\right)\cos\theta$$

So rotate η by about 6 degrees for sonic rotation and more for supersonic rotation!

Example: T_e = 20 eV,

E_n = 200 V/cm, a=L=1 cm

$$\rho_L \cong T_{20}^{1/2} / B_{100}\ mm$$

$$r \sim 10\text{ mm}$$

or 12 degrees for r = 5mm

Summary:

1. Free Energy of Plasma under Wave Diffusion
2. Rigorous upper bound for space-charge limited current
3. Wave compression in plasma
4. Lagrangian description of wave propagation including trapped particles (method of Dodin)
5. Collimation of ions in magnetic fields (self-organization of supersonically rotating electrons)