Theory and simulations of hydrodynamic instabilities in inertial fusion

R. Betti
Fusion Science Center, Laboratory for Laser Energetics, University of Rochester

Acknowledgments

Special thanks to:

Radha Bahukutumbi (LLE)

Ryan Nora (UR-Physics Dept)
Outline

→ Theory of the Rayleigh-Taylor instability in inertial fusion
  • Ablation fronts in laser-driven targets
  • Classical linear Rayleigh-Taylor instability
  • Ablative linear Rayleigh-Taylor instability
  • Single-Mode Nonlinear Theory
  • Multimode ablative RT

→ Hydrodynamic simulations
  • Eulerian and Lagrangian hydrodynamics
  • Two-fluids, nonlocal heat conduction, radiation transport, laser absorption
  • Single mode and multimode simulations of hydrodynamic instabilities
Theory of the Rayleigh-Taylor instability in inertial fusion
The laser energy deposition generates a thermal conduction zone and an ablative flow.

- **Sonic point M=1**
- **Conduction zone**
  - Steady state
  - Subsonic flow $M < 1$
- **Mach number** $M = V/C_s$
- **Ablation velocity** $U_a$
  - Heat front velocity in dense cold target
- **Blow-off velocity** $U_b$
- **Heat flows by conduction**
- **Ablated plasma**
  - Light and Hot
- **Critical surface** $N_e \approx 10^{22}$ electrons/cc
- **X-axis**
- **a=acceleration in the lab frame**

Corona
Isothermal expansion
Time-dependent supersonic flow $M > 1$

- Laser energy deposited near critical surface
The outer surface of an imploding capsule separates a dense fluid supported by a lighter one.

\[ \mathbf{a} = \text{acceleration in the lab frame} \]

\[ \mathbf{g} = -\mathbf{a} = \text{acceleration in the shell frame} \]

In the shell frame:
- Dense → heavy
- Low density → light

Imploding dense shell

Outer shell surface

Low density ablated expanding plasma

Laser

Laser
The outer surface of an imploding capsule is unstable to the Rayleigh-Taylor instability.

**EQUILIBRIUM CONDITIONS**

\[ \rho_h \rightarrow \rho \rightarrow \rho_l \]

\[ \text{heavy} \rightarrow \text{light} \]

Pressure gradient is opposite to density gradient.

\[ \frac{dP}{dx} = \rho_h g \]

\[ \frac{dP}{dx} = \rho_l g \]
Derive the classical R-T from Newton’s law

\[ F = ma \rightarrow S \rho_h g \tilde{\eta} = \rho_h \lambda S \ddot{\eta} \rightarrow \ddot{\eta} = kg\tilde{\eta} \rightarrow \tilde{\eta} \sim e^{\gamma t} \rightarrow \gamma = \sqrt{\text{kg}} \]

The ABLATIVE R-T is just Newton’s law at work again but with a restoring force: the dynamic pressure.

\[ S[P_h - (P_\ell + \rho_\ell u_b^2)] = \rho_h \lambda S \ddot{\eta} \]

- Perturbed dynamic pressure:
  \[ \rho_\ell u_b^2 \sim \rho_\ell U_b \tilde{u}_b \]

Ablation rate = \( \dot{m}_a \)

\[ u_b = U_b + \tilde{u}_b \]
The perturbed dynamic pressure is stabilizing

\[ k(\rho_h g \tilde{\eta} - \dot{m}_a \tilde{u}_b) = \rho_h \gamma^2 \tilde{\eta} \]

Energy flow balance (see Appendix)

\[ \frac{5}{2} p u_b \approx q_{heat} \Rightarrow \tilde{u}_b \approx k U_b \tilde{\eta} \]

\[ \gamma = \sqrt{k g - k^2 U_a U_b} \]

Ablation introduces a cutoff (wave number) in the unstable spectrum

J. Sanz, Phys. Rev. Lett. 73, 2700 (1994)
Appendix: Perturbed blow-off velocity

\[ u_b = u_b^0 + \tilde{u}_b \]
\[ T = T^0 + \tilde{T} \]

Start from flux balances
\[ \frac{5}{2} p u_b = -q_{\text{heat}} = \kappa \nabla T \]

\[ \frac{5}{2} p (u_b^0 + \tilde{u}_b) = \kappa \nabla T^0 + \kappa \nabla \tilde{T} \Rightarrow \frac{5}{2} p \tilde{u}_b = \kappa \nabla \tilde{T} \]

Use equilibrium fluxes are balanced
\[ \frac{5}{2} p \tilde{u}_b = \kappa k \tilde{T} \]

Find \( \tilde{T} \sim k \)
\[ \tilde{T} = T^0(\tilde{\eta}) - T^0 \text{ since } T(\tilde{\eta}) = T^0 + \frac{dT^0}{dx} \tilde{\eta} \Rightarrow \tilde{T} = \frac{dT^0}{dx} \tilde{\eta} \]

Define temperature perturbation
\[ \tilde{T} = T^0(\tilde{\eta}) - T^0 \]

Define
\[ \tilde{\eta} = \frac{5}{2} p u_b^0 k \tilde{\eta} \Rightarrow \tilde{u}_b = u_b^0 k \tilde{\eta} \]

Use equilibrium expansion
\[ \frac{5}{2} p \tilde{u}_b = \kappa k \frac{dT^0}{dx} \tilde{\eta} \Rightarrow \tilde{u}_b = \frac{5}{2} p u_b^0 k \tilde{\eta} \]

\[ \Rightarrow \text{Use equilibrium} \]
The ablative growth rate is significantly less than the classical value. Modes with $k > k_c$ are stable.

$$u_a = 3.5 \mu m/ns$$
$$g = 100 \mu m/ns^2$$
Nonlinear classical RT: the bubble velocity saturates when the bubble amplitude is \(~0.1\lambda\). The bubble amplitude does not saturate.

\[ \text{Buoyancy} \sim (\rho_h - \rho_\ell) S \lambda g \]
\[ \text{Drag} \sim \rho_h U^2 S \]

Saturation \(\Rightarrow\) buoyancy=drag

\[ U_{\text{bubble}}^{\text{sat}} \sim \sqrt{\lambda g \left(1 - \frac{\rho_\ell}{\rho_h}\right)} \]

\[ U_{\text{bubble}}^{\text{sat}(2D)} \approx \sqrt{\frac{g}{3k} \left(1 - \frac{\rho_\ell}{\rho_h}\right)} \]
\[ U_{\text{bubble}}^{\text{sat}(3D)} \approx \sqrt{\frac{g}{k} \left(1 - \frac{\rho_\ell}{\rho_h}\right)} \]

Transition to saturation:
linear bubble velocity = saturated velocity

\[ \tilde{\eta} = \tilde{\eta}(0)e^{\gamma t} \]
\[ \dot{\tilde{\eta}} = \gamma \tilde{\eta} \approx U_{\text{bubble}}^{\text{sat}} \]
\[ \tilde{\eta}_{\text{sat}}^{2D} \approx 0.1\lambda \]

What can we infer about the nonlinear ablative RT by simply looking at the linear spectrum?

Is this an isolated equilibrium? NO!
The theory predicts full nonlinear stability only for wave numbers exceeding a nonlinear cutoff beyond the linear cutoff.

\[ \text{Normalized threshold amplitude} \ (\alpha_{\text{th}}/\lambda) \]

- **Exponentially unstable**
- **Stable**

**Nonlinear cutoff for Fr = 5**
**Nonlinear cutoff for Fr = 10**
**Nonlinear cutoff for Fr = 20**

**Fr**
\[ Fr = \frac{V_a^2}{gL_0} \]

**Nonlinear cutoff**
\[ k_{\text{nonlinear cutoff}} = \frac{g}{3V_a^2} \left( \frac{2A}{1+A} \right) \]
A linearly stable perturbation \((k > k_{\text{cutoff}})\) becomes unstable for a sufficiently large initial amplitude (as predicted by theory).

\(k > k_{\text{cutoff}}\): Linearly stable 10-\(\mu\)m perturbation

**Small initial amplitude**

**Larger initial amplitude**

In the deeply nonlinear phase, the vorticity accumulates inside the bubble raising the bubble terminal velocity.

Single-mode simulation of the deeply nonlinear ablative Rayleigh-Taylor instability

A weak vortex is used as initial perturbation.
Simulations show vorticity convection and accumulation

- **ρ (g/cc)**
  - t = 1 ns
  - Vortex center

- **ω (ns⁻¹)**
  - t = 1 ns
  - 8.1
  - 0.8

- **ρ (g/cc)**
  - t = 2.2 ns
  - 3.7

- **ω (ns⁻¹)**
  - t = 2.2 ns
  - 33
  - 23

FSC
A large vortex forms inside the bubble; the vortex generates a centrifugal force \( (F_C) \) pushing on the bubble tip.

\[ \omega \ (\text{ns}^{-1}) \]

\[ t = 2.8 \ \text{ns} \]

\[ 40 \ \mu\text{m} \]

\[ 10 \ \mu\text{m} \]

\[ \Omega \approx \omega/2 \]

\[ R \approx \lambda/2 \]
The asymptotic bubble velocity is higher than the classical value due to the vorticity accumulated inside the bubble.

Centrifugal force and buoyancy force add up

Centrifugal force:
\[ \rho_{\text{bubble}} R \Omega^2 = \rho_{\text{bubble}} \frac{R \omega^2}{4} \]

Buoyancy force:
\[ (\rho_{\text{dense}} - \rho_{\text{bubble}}) g \]

Bubble-velocity enhancement

\[ V_{\text{bubble}}^{\text{vort}} = \sqrt{\frac{g}{3k}} (1 - r_d) + r_d \frac{\omega^2}{4k^2} \]
The bubble accelerates to final velocities well above the classical value and in agreement with the theory.

\[ \frac{v_{\text{bubble}}}{v_{\text{class 2-D bubble}}} \]

\[ \sqrt{\frac{g}{3k}}(1 - r_d) + r_d \frac{\omega(t)^2}{4k^2} \quad - \quad \sqrt{\frac{g}{3k}}(1 - r_d) + \frac{V_a^2}{r_d} \]

Multimode nonlinear interaction leads to an envelope growth of the bubble front $h = \beta gt^2$ with $\beta$ dependent on the box size.

$h_{\text{bubbles}}^{\text{classical}} \approx \beta_{\text{cl}} gt^2 \Rightarrow \beta_{\text{cl}} \sim 0.05$

$h_{\text{bubbles}}^{\text{ablative}} \approx \beta_{\text{abl}} gt^2$

$$\beta_{\text{abl}} = \beta_{\text{cl}} \left(1 - \sqrt{\frac{k_{\text{box}}}{k_{\text{nonl-cutoff}}}}\right)^2$$

This conclusion does not include the effect of bubble acceleration.

Hydrodynamic Simulations

Excellent tutorial at http://hedpschool.lle.rochester.edu/1000_proc2011.php by Radha Bahukutumbi
Hydrodynamic codes use a combination of Eulerian and Lagrangian grids (ALE = arbitrary Lagrangian-Eulerian)

In the frame of the box

Eulerian form of fluid equations

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} &= 0 \\
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (P + \rho u^2)}{\partial x} &= 0 \\
\frac{\partial (\rho \varepsilon)}{\partial t} + \frac{\partial ([P + \rho \varepsilon] u)}{\partial x} &= 0
\end{align*}
\]

\[P = P(\rho, \varepsilon)\]

Equation-of-State (EOS)

Replace to get flow equations in the fluid frame (Lagrangian equations)

\[
\begin{align*}
\frac{d \rho}{dt} + \rho \frac{\partial u}{\partial x} &= 0 \\
\frac{d u}{dt} &= -\frac{\partial P}{\partial x} \\
\frac{d \varepsilon}{dt} &= \frac{P}{\rho^2} \frac{d \rho}{dt} \\
P &= P(\rho, \varepsilon)
\end{align*}
\]
Shocks are treated by adding an artificial viscous term to the ion pressure

- Dissipative mechanisms like viscosity and heat conduction introduce a thin transition layer instead of a sharp discontinuity.
- Use an “artificial viscosity”
  
  \[ Q = a_o^2 \rho (\delta u)^2 \quad \delta u < 0 \]
  \[ = 0 \quad \delta u \geq 0 \]
  
  \[ P \rightarrow P + Q \] in fluid equations

- Originally proposed by Richtmyer and von Neumann

- No special internal boundary conditions required
- Shocks as approximate discontinuities in \( \rho, \varepsilon, P \)
- Obeys the basic conservation laws
The energy equation is simply extended to ions and electrons

Lagrangian form of energy equation written in terms of two temperatures

\[
\rho C_{ve} \frac{dT_e}{dt} = \frac{\partial}{\partial x} \left( \kappa_e \frac{\partial T_e}{\partial x} \right) - P_e \frac{\partial u}{\partial x} - \tau_{ei}^{-1} (T_e - T_i) + S_e
\]

\[
\rho C_{vi} \frac{dT_i}{dt} = \frac{\partial}{\partial x} \left( \kappa_i \frac{\partial T_i}{\partial x} \right) - P_i \frac{\partial u}{\partial x} + \tau_{ei}^{-1} (T_e - T_i) + S_i
\]

- **Electrons**
- **Ions**

**Specific heat**
- **Electron-ion relaxation time**
- **Source terms**
  - Ions – shock, alphas
  - Electrons – radiation, laser energy, alphas

**conductivity**
Deposition of laser energy into a low density plasma is typically modeled using a ray trace algorithm.

- Geometrical optics approximation is used with Inverse Bremsstrahlung as the mechanism for energy deposited in cell $j$.

\[
W_j = E_{j+1} \left( 1 - e^{-\int K_{IB} ds} \right)
\]

\[
K_{IB} \sim \frac{n_e^2 <Z^2> \ln \Lambda}{\omega_o^2 (m_e kT_e)^{3/2} <Z> \left(1 - \frac{\omega_p^2}{\omega_o^2}\right)^{-1/2}}
\]
Geometrical optics ignores various coronal laser-plasma interactions

- Typically thresholds for evaluated to identify if they are exceeded
- Ultimately empirical evidence is the best indicator of the adequacy of the raytrace approximation
Most fluid codes approximate radiative transfer using multi-group diffusion

- The frequency spectrum is divided into $G$ bins.
  \[ \nu_g < \nu < \nu_g + \Delta \nu_g, g = 1, 2, ... G \]

- For each group solve the diffusion equation.
  \[
  \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + \frac{\partial u}{\partial x} \right) U_v = c \rho \kappa_v \left( B_v - U_v \right) - \frac{\partial q_v}{\partial x} - P_v \frac{\partial u}{\partial x}
  \]

- The energy removed or added to each zone is summed over all energy groups and added as a source term to the heat equation.
Empirical studies indicate that heat flux inhibition is required in laser driven plasmas

\[ L_T = |T/\nabla T| = \text{temperature gradient scale length} \]

\[ \lambda_e = \text{electron mean free path} \]

\[ \lambda_e \ll L_T \rightarrow \text{diffusive heat transport SH} \]

\[ \lambda_e \geq L_T \rightarrow \text{nonlocal heat transport FS} \]

- \[ q_{SH} = - \kappa \nabla T \quad q_{FS} = nTV_T \]
- Sharp cutoff \[ q_{eff} = \min(q_{SH}, f q_{FS}) \]
- \[ 0.04 < f < 0.1 \]
Single mode simulations are carried out to assess the suitability of the ALE (arbitrary Lagrangian-Eulerian) grid
Multimode laser-imprinting simulations of CH+DT targets show Rayleigh-Taylor instability growth at the ablation front.
Imploding shells can develop multiple unstable interfaces depending on the material opacities.

**Diagram:**

- **Left Panel:**
  - DT ice + SiO$_2$
  - Double ablation front
  - Rad-front
  - E-front
  - SiO$_2$

- **Right Panel:**
  - DT ice + CH+CHSi(5%)+Si
  - Double ablation front + Classical interface
  - Rad-front
  - E-front
  - CH
  - CHSi
Multimode laser-imprinting simulations of SiO₂ +DT targets show Rayleigh-Taylor instability growth at the radiative/classical front.
Hydrodynamic instabilities in ICF are generally understood but challenges remain in developing an accurate predictive capability for 2D-3D multimode interactions.

- The linear theory of the ablative RT is fully developed and the physics is well understood. Ablation is stabilizing in the linear phase.

- The nonlinear single-mode evolution is well understood and ablation is destabilizing in the deeply nonlinear phase (bubble acceleration).

- The effect of the initial conditions on the nonlinear multimode ablation front dynamics is not well understood.

- An accurate evaluation of the instability seeds from laser non-uniformities (imprinting) is difficult.

- Three dimensional simulations are computational expensive and great difficulties remain in developing accurate nonlinear multimode simulations (this also applies to 2D simulations).
BACK UP SLIDES ON NONLINEAR ARTI
The density in the bubble is the same as predicted by the linear theory* and a significant fraction of the dense target density

$$\rho_{\text{bubble}}^{\text{linear}} \approx (0.1 k L_m)^{2/5} \rho_{\text{dense}}$$

$$\rho_{\text{dense}} \approx 4 \text{ g/cc}$$

$$L_m \approx 0.18 \ \mu\text{m} \quad \lambda \approx 10 \ \mu\text{m}$$

$$\rho_{\text{bubble}}^{\text{linear}} \approx 0.66 \text{ g/cc}$$

$$\rho_{\text{bubble}}^{\text{simulation}} \approx 0.65 \text{ g/cc}$$

$L_m = \text{the minimum density-gradient scale length}$

$k = \text{mode wave number}$

The bubble velocity is defined as the penetration velocity inside the overdense target.

![Graph showing bubble velocity and density as a function of distance. The graph includes two curves: one for velocity and one for density. The y-axis represents the distance in micrometers, and the x-axis represents the density in grams per cubic centimeter. The graph illustrates the behavior of the bubble and dense target regions.]
Mitigation techniques for the Rayleigh-Taylor instability in laser accelerated targets:
1. Reduce the seeds
2. Reduce the growth rates
Reducing the seeds for the RT (by making uniform laser beams) improve the integrity of the imploding shell

The growth rates can be reduced by shaping the entropy of the imploding shell

Stabilize the RT by increasing the ablation velocity

\[ \gamma \approx 0.94 \sqrt{k g} - 2.7 k u_a \]

\[ P_{app} \sim \alpha \rho^{3/5} \]

\[ \alpha = \text{measure of entropy} \]

\[ u_a = \frac{m_a}{\rho} \sim \frac{m_a}{P_{app}^{3/5}} \alpha^{3/5} \]

Increase the entropy (\( \alpha \)) at the ablation front while keeping \( \alpha \) low in the unablated material
Adiabat shaping reduces the RT growth without degrading the final compression.

Shell at start of acceleration

\[ \alpha \sim \frac{P}{\rho^{5/3}} \]

Adiabat shaping can be done by shaping \( P \) or \( \rho \) or both.

Adiabat shaping can be induced by a decaying pressure or by a relaxed density.

\[ \alpha \sim \frac{P}{\rho^{5/3}} \]

Shaping $P$ with a decaying shock:

- Laser
- Shock position at different times

Shaping $\rho$ by relaxation:

- Laser
- Shock position at different times
The adiabat is shaped by adding an intensity spike to the main laser pulse.

Decaying shock:
- Spike
- Main pulse
- Foot

Launches a decaying shock

Relaxation:
- Spike
- Shut off
- Main pulse
- Foot

Relaxes the shell density