Nonlocal PDEs & Non-Gaussian Dynamics

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Outline

Recall —-

Lin, Gao, Duan & Ervin, 2000:

Asymptotic Dynamical Difference between the Nonlocal and Local Swift-Hohenberg Models

Macroscopic – Macroscopic

Macroscopic $\to$ Microscopic

Dynamical systems

• **Deterministic dyn systems:** Geometric views on solutions
Ordinary diff eqns (ODEs)
Partial diff eqns (PDEs)

• **Stochastic dyn systems**: Geometric views on solutions
Stochastic diff eqns (SDEs)
Stochastic partial diff eqns (SPDEs)

Stochastic dynamical systems

1970s: Stochastic differential equations (SDEs)
Ikeda-Watanabe, Arnold, Friedman

1980s: Stochastic flows, cocycles
Elworthy, Baxendale, Bismut, Ikeda, Kunita, ...

1990s: Dynamical systems approaches for SDEs
L. Arnold, ...

Related development:

- Ergodic theory
- Statistical mechanics

Stochastic Dynamical Systems:

Dynamical systems with noise!

- Environmental or intrinsic fluctuations! (seen in observations)
- Gaussian noise or non-Gaussian noise
- Brownian motion or $\alpha-$stable Lévy motions

Scientific observations: Many independent measurements and then “averaging"

Where do Gaussian random variables come from?

$X_1, X_2, \cdots, X_n$ are independent, identically distributed (iid) random variables with finite mean μ and variance σ^2

Central limit theorem: Gaussian random variables come from “averaging"

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = X \ \sim \mathcal{N}(0,1) \quad \text{in distribution}$$

Namely,

$$\lim_{n\to\infty} \mathbb{P}(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

Scientific observations: Many independent measurements and then “averaging"

Where do stable random variables come from?

$X_1, X_2, \cdots, X_n$ are independent, identically distributed (iid) random variables (whose mean and variance may be infinite)

Definition:

X is a stable random variable if it is a limit (in distribution) of an averaging sequence of X_i's:

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - b_n}{a_n} = X \quad \text{in distribution}$$

for some constants a_n, b_n ($a_n \neq 0$)

Notation: $X \sim S_\alpha(\sigma, \beta, \mu)$

Gaussian vs. Non-Gaussian random variables

Gaussian random variable $X(\omega)$:

Probability density function (PDF) $\frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

Non-Gaussian random variable $X(\omega)$:

Probability density function (PDF) $f(x) \geq 0, \quad \int_{\mathbb{R}} f(x)dx = 1.$

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

Gaussian process & non-Gaussian process

Prob density function for a Gaussian random variable

Also called a normal random variable: $X \sim \mathcal{N}(\mu, \sigma^2)$

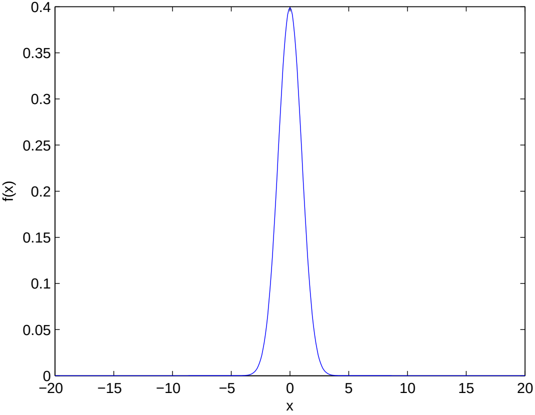
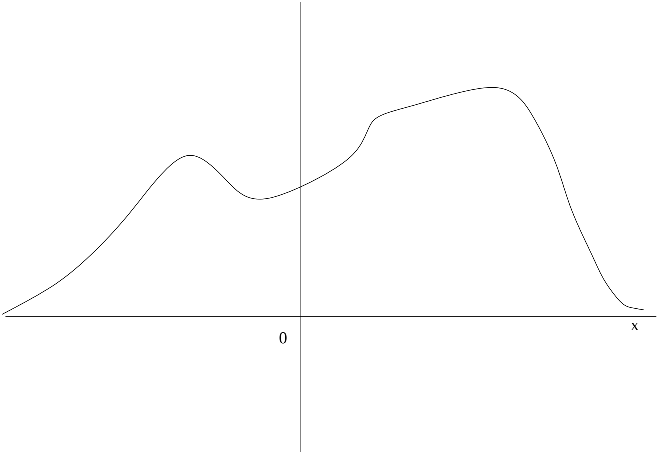


Figure: “Bell shape"

Prob density function for a non-Gaussian random variable



- Browniam motion is defined in terms of Gaussian r. v.
- $\alpha-$stable Lévy motion is defined in terms of stable r. v.

Brownian motion B_t: A Gaussian process

- Independent increments: $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ independent
- Stationary increments with $B_t - B_s \sim N(0, t - s)$
  In particular, $B_t \sim N(0, t)$
- Continuous sample paths, but nowhere differentiable

Reference:
I. Karatzas and S. E. Shreve,
Brownian Motion and Stochastic Calculus

A sample path for Brownian motion $B_t(\omega)$

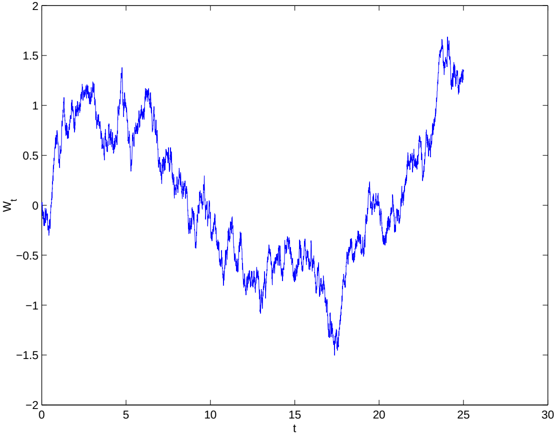


Figure: Continuous path, but nowhere differentiable

α-stable Lévy Motion L_t: A non-Gaussian process

Definition: α-stable Lévy motion $L_t(\omega)$ with $0 < \alpha < 2$:
(1) $L_0 = 0,\ a.s.$;
(2) L_t has independent increments;
(3) Stationary increments $L_t - L_s \sim S_\alpha(|t-s|^{\frac{1}{\alpha}}, \beta, 0)$;

Note 1. Paths are stoch. continuous (right continuous with left limit; countable jumps): $L_t \to L_s$ in prob. as $t \to s$
Note 2. $\alpha = 2$: Brownian motion B_t and
$B_t - B_s \sim S_2(|t-s|^{\frac{1}{2}}, 0, 0)$
Note 3. When $\beta = 0$, Lévy jump measure: $\nu_\alpha(du) = \frac{C_\alpha}{|u|^{1+\alpha}}(du)$, α-stable symmetric Lévy Motion. ($C_\alpha = \frac{\alpha\Gamma((d+\alpha)/2)}{2^{1-\alpha}\sqrt{\pi}\ \Gamma(1-\alpha/2)}$)

α-stable symmetric Lévy Motion L_t

Lévy jump measure: $\nu_\alpha(du) = c_\alpha \frac{1}{|u|^{1+\alpha}}(du)$
$0 < \alpha < 2$: Lévy motion L_t
$\alpha = 2$: Brownian motion B_t

Heavy tail for $0 < \alpha < 2$: Power law (non-Gaussian)

$$\mathbb{P}(|L_t| > u) \sim \frac{1}{u^\alpha}$$

Light tail for $\alpha = 2$: Exponential law (Guassian)

$$\mathbb{P}(|B_t| > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi}u}$$

Reference:
D. Applebaum — **Lévy Processes and Stochastic Calculus**

A sample path for $\alpha-$stable Lévy motion $L_t(\omega)$: Jumps

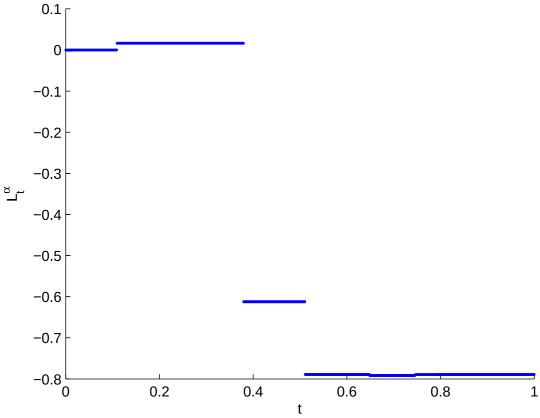


Figure: $\alpha-$stable Lévy motion: $\alpha = 0.25$

A sample path for $\alpha-$stable Lévy motion $L_t(\omega)$: Jumps

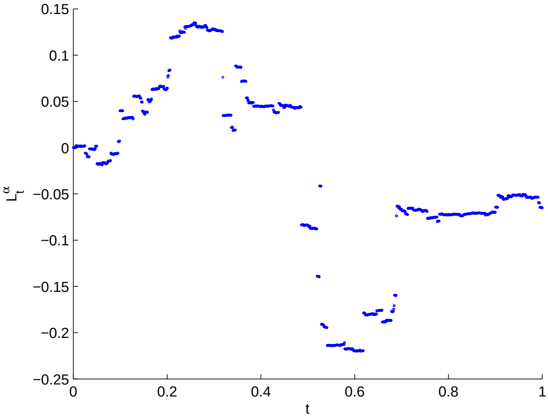


Figure: $\alpha-$stable Lévy motion: $\alpha = 0.75$

A sample path for $\alpha-$stable Lévy motion $L_t(\omega)$: Jumps

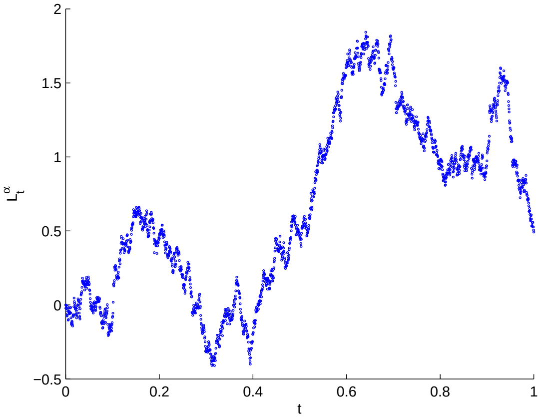


Figure: $\alpha-$stable Lévy motion: $\alpha = 1.9$

Summary on comparison of BM and $\alpha-$stable LM

Brownian Motion	$\alpha-$stable Levy Motion
Independent increment	Independent increment
Stationary increment $B_t - B_s \sim S_2(\|t-s\|^{\frac{1}{2}}, 0, 0)$	Stationary increment $L_t - L_s \sim S_\alpha(\|t-s\|^{\frac{1}{\alpha}}, \beta, 0)$
Continuous sample paths	Stoch continuous paths (“jumps")
Triplet $(a, d, 0)$	Triplet $(a, 0, \nu_\alpha)$

Properties of paths of $\alpha-$stable Lévy motion L_t

(i) Countable and dense jumps in time

(ii) Right continuous with left limit at each jump time

Why Lévy motion $L_t(\omega)$: Jumps or flights

- Abrupt climate change such as Dansgaard-Oeschger events.

  Ditlevsen 1999: Ice record for temperature
- Diffusion of tracers in rotating annular flows: *Pauses* near coherent structures & *jumps or "flights"* in between
  PDF for flight times: Power laws
  Swinney et al. 1995
- Data in biology and other areas
  Shlesinger et al.: *Lévy Flights and Related Topics in Physics*, 1995
- Woyczynski, Lévy processes in the physical science, 2001
- Financial data

What is noise?

Stationary process $X(t)$: mean $\mathbb{E}X(t)$ is a constant and the autocorrelation $\mathbb{E}(X(t_1)X(t_2))$ depends only on the time lag $t_2 - t_1$.

Noise: η_t

- A stationary stochastic process
- Mean $\mathbb{E}\eta_t = 0$
- Covariance $\mathbb{E}(\eta_t \eta_s) = K\ c(t-s)$ for all t and s, K is a constant

White noise: $c(t-s) = \delta(t-s)$ Dirac Delta function

Colored noise: Non-white noise

Gaussian noise: $\frac{d}{dt}B_t$

Brownian motion B_t is a Gaussian process with stationary (and also independent) increments, together with mean zero $\mathbb{E}B_t = 0$ and covariance $\mathbb{E}(B_t B_s) = t \wedge s = \min\{t, s\}$.

Increments: $B_{t+\Delta t} - B_t \approx \Delta t \, \dot{B}_t$ are stationary
Mean: $\mathbb{E}\dot{B}_t \approx \mathbb{E}\frac{B_{t+\Delta t} - B_t}{\Delta t} = \frac{0}{\Delta t} = 0$
Covariance:
By the formal formula $\mathbb{E}(\dot{X}_t \dot{X}_s) = \partial^2 \mathbb{E}(X_t X_s)/\partial t \partial s$, we see that

$$\mathbb{E}(\dot{B}_t \dot{B}_s) = \partial^2 \mathbb{E}(B_t B_s)/\partial t \partial s = \partial^2 (t \wedge s)/\partial t \partial s = \delta(t - s).$$

Made rigorous: Theory of Generalized Functions

Why is it called white noise?

The spectral density function for $\dot{B}_t$, i.e., the Fourier transform $\mathcal{F}$ for its covariance function $\mathbb{E}(\dot{B}_t \dot{B}_s)$, is constant

$$\mathcal{F}(\mathbb{E}(\dot{B}_t \dot{B}_s)) = \mathcal{F}(\delta(t-s)) = \frac{1}{2\pi}.$$

Thus $\eta_t = \dot{B}_t$ is taken as a mathematical model for white noise.

Analogy: White light in **Optics**

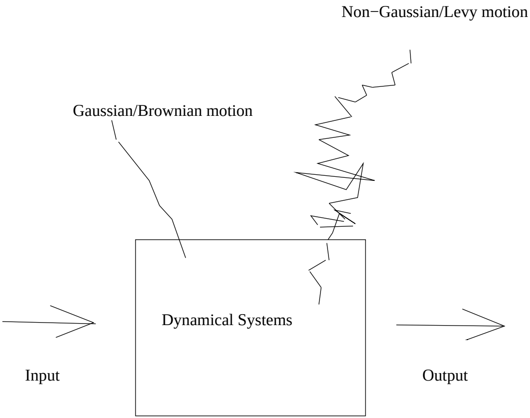
Non-Gaussian noise: $\frac{d}{dt}L_t$

Non-Gaussian noise: $\frac{d}{dt}L_t$
Lee & Shih 2008

Gaussian & Non-Gaussian Noise

- Gaussian noise: $\frac{d}{dt}B_t$
- Non-Gaussian noise: $\frac{d}{dt}L_t$

Dynamics under Noise



Impact of Gaussian noise on solutions

$$dX_t = (-X_t + X_t^3)dt + \varepsilon dB_t, \quad X_0 = 0.5.$$

A solution path:

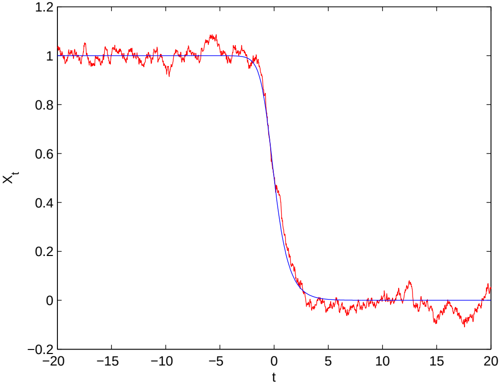


Figure: A solution path with "wigglings" when $\varepsilon = 0.05$, on top of the

Impact of non-Gaussian noise on solutions

$$dX_t = (-X_t + X_t^3)dt + \varepsilon dL_t, \quad X_0 = 0.5.$$

A solution path: $\alpha = 0.85$

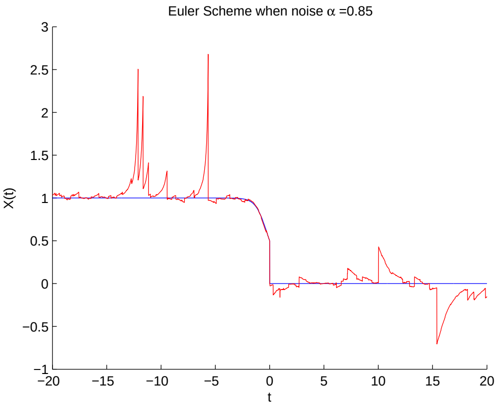


Figure: A solution path with “wigglings" when $\varepsilon = 0.05$, on top of the

Impact of non-Gaussian noise on solutions

$$dX_t = (-X_t + X_t^3)dt + \varepsilon dL_t, \quad X_0 = 0.5.$$

A solution path: $\alpha = 1.5$

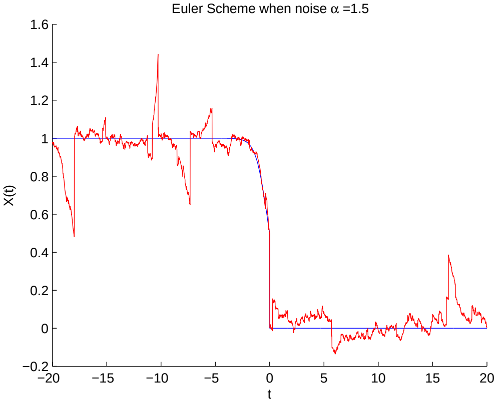


Figure: A solution path with “wigglings" when $\varepsilon = 0.05$, on top of the

Impact of noise on pathwise uniqueness of solutions

$\dot{x} = f(x)$: a sufficient condition for uniqueness of solutions is the local Lipschitz continuity condition for vector field f. Without this condition, the uniqueness is often **violated**.

However, for $\dot{x} = f(x) + \dot{L}_t$ with f being Borel measurable and bounded (no Lipschitz condition), the uniqueness **holds**.

Priola 2010: Nonlocal PDE argument!

What is dynamics?

- **Look at** solutions collectively, as time goes on
- **Examine** solution mappings, as time goes on
- **Discover** structures as stepping stones for understanding

Deterministic dynamical system

Linear system:

$$x' = Ax \quad , x(0) = x_0$$

Solution mapping (**Matrix exponential**): $\varphi(t, x_0) \triangleq e^{At} x_0$

“Flow" property: $\varphi(t+s, x_0) = e^{A(t+s)} x_0 = \varphi(t, \varphi(s, x_0))$

Nonlinear system:

$$x' = f(x) \quad , x(0) = x_0$$

Solution mapping: $\varphi(t, x_0)$

“Flow" property: $\varphi(t+s, x_0) = \varphi(t, \varphi(s, x_0))$

Stochastic dynamical systems

Langevin equation:

$$dx = -xdt + dB_t, \quad x(0) = x_0$$

Random solution mapping:
$\varphi(t, x_0, \omega) = x_0 e^{-t} + \int_0^t e^{-(t-s)} dB_s(\omega)$

Linear stochastic systems: **Random matrices**

How to understand stochastic dynamics?

- **Topological methods:** Invariants (e.g., Poincare index, Conley index)
- **Geometric methods:** Invariant structures (e.g., invariant manifolds)
- **Functional analytical methods:** Nonlocal PDEs!

Topological methods: Invariants

Still in infancy!

Conley index: Liu 2008; Chen-Duan-Fu 2010

Difficulty due to the very nature of random invariant sets!

A caveat!

Topological: ***Brouwer fixed point theorem***
A continuous map from a convex compact subset K to K itself has a fixed point.

Does NOT hold for random mappings — **Some continuous random mappings on compact intervals do not have random invariant points**

Ochs and Oseledets 1999

How to understand stochastic dynamics?

- **Topological methods:** Invariants (e.g., Poincare index, Conley index)
- **Geometric methods:** Invariant structures (e.g., invariant manifolds)
- **Functional analytical methods:** Nonlocal PDEs!

Geometric methods: Invariant manifolds

Banach fixed point theorem
A contraction mapping has a unique fixed point.

Invariant manifolds (Liapunov-Perron method)

Still hold for random mappings — **contraction in mean**
Schmalfuss 1997

Basic definitions

- **Random invariant set $M(\omega)$:**

$$\varphi(t, \omega, M(\omega)) = M(\theta_t \omega) \;\; for \;\; t \in \mathbb{R}$$

- **Stable manifold:**
  If we can represent an invariant set as a graph of a Lipshitz mapping

$$h^s(\cdot, \omega) : H^s \to H^u$$

  such that

$$W^s(\omega) = \{\xi + h^s(\xi, \omega) \mid \xi \in H^s\}$$

  then $W^s(\omega)$ is called a Lipschitz stable manifold.

- **Unstable manifold:** Similar

Theorem: Impact of noise on stable manifold

- **Assumptions:** $\frac{du}{dt} = Au + F(u) + \epsilon\, u \circ \dot{L}_t$

  (i) Linear part A: exponential dichotomy (“saddle property")

  (ii) Nonlinear part $F(u)$: twice continuously Frechet differentiable with respect to u

  (iii) Gap condition: $K\ L_F\ (\frac{1}{\alpha} - \frac{1}{\beta}) < 1$

Theorem: Impact of noise on stable manifold

- **Random stable manifold:** $\tilde{W}^s(\omega) = \{\xi + h^s(\xi,\omega) \big| \xi \in H^s\}$
- **Approximating stable manifold:**
When ϵ is sufficiently small, W^s can be expressed as

$$h^s = h^{(d)}(\xi) + \epsilon h^{(1)}(\xi,\omega) + R_s$$

where $\|R_s\| \leq C(\omega)\epsilon^2$ with $C(\omega) < \infty$, $h^{(d)}$ is deterministic stable manifold, and

$$h^{(1)}(\xi,\omega) = \int_{\infty}^{0} e^{-As}\{[Z(\theta_r(\omega))dr$$
$$+Z(\theta_s(\omega))]P^u F(u_0) + P^u F_u^{u_0(s)}[u_1(s) - Z(\theta_s(\omega))u_0(s)]\}ds$$

Note: $Z(\omega)$ is the stationary solution of
$dZ(t) + Z(t)dt = dB(t)$

Example: What is impact of noise on invariant manifolds?

Consider a SDE system

$$\begin{cases} \dot{x} = -x + \epsilon x \circ \dot{B}_t, \\ \dot{y} = y + x^2 + \epsilon y \circ \dot{B}_t, \end{cases}$$

$0 < \epsilon \ll 1$

$\tilde{W}^s$: Stable manifold in a neighborhood around $(0, 0)$

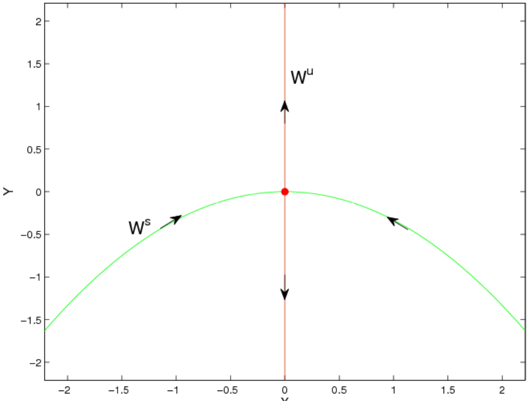
$\epsilon = 0$: Deterministic stable manifold is

$$W^s = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \Big| y = -\frac{x^2}{3} \right\}$$

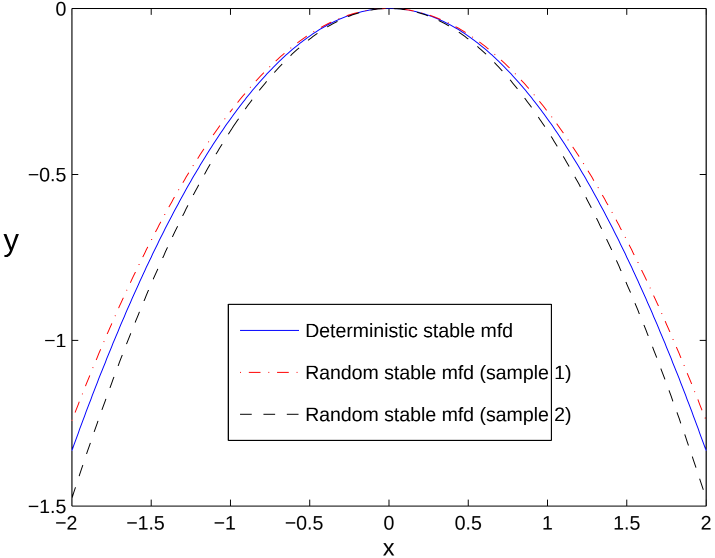
Deterministic stable manifold: $\varepsilon = 0$

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= y + x^2 \end{aligned}$$

Stable manifold W^s : $\quad y = -\frac{1}{3}x^2$



Random stable manifold: $\varepsilon = 0.01$



Random stable manifold: $0 < \varepsilon \ll 1$

Random stable manifold:

$$\tilde{W}^s = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| y = -\frac{x^2}{3} - \epsilon \frac{x^2}{3} \left(\int_0^\infty e^{-3\tau} \, dB_\tau \right) + O(\epsilon^2) \right\}$$

Remarks:
With accuracy of $O(\epsilon^2)$,

(i) **Mean** of random stable manifold is just the deterministic stable manifold

(ii) **Variance** of the random stable manifold is $\frac{1}{54}x^4$

Sun, Duan & Li: 2010
Open problem: Non-Gaussian Lévy noise

How to understand stochastic dynamics?

- **Topological methods: Invariants** (e.g., Poincare index, Conley index)
- **Geometric methods: Invariant structures** (e.g., invariant manifolds)
- **Functional analytical methods:** Nonlocal PDEs!

Functional analytical methods: Nonlocal PDEs!

Main Ideas:

(i) Solutions of stochastic systems are Markov processes

(ii) Markov processes $\rightarrow$ Semigroups

(iii) Semigroups' generators A: Nonlocal operators!

(iv) Non-Gaussian Lévy noise $\sim$ Nonlocal operators!

Generators of Markov process X_t: $X_0 = x$

Semigroup: For observable φ

$P_t\varphi(x) \triangleq \mathbb{E}\varphi(X_t)$
$P_{t+s} = P_t P_s$

Macroscopic description

Generator of semigroup: Derivative of semigroup at time 0

$A\varphi(x) \triangleq \frac{d}{dt}|_{t=0} P_t\varphi(X_t)$

Example: Generators of B_t & L_t

Brownian motion B_t: Generator is $\frac{1}{2}\,\Delta$

$\alpha-$stable Lévy motion: Generator is $-K_\alpha\,(-\Delta)^{\frac{\alpha}{2}}$

Nonlocal operator:

$\int_{\mathbb{R}^d\setminus\{0\}}[u(x+y)-u(x)-I_{\{|y|<1\}}\,yu'(x)]\,\nu_\alpha(\mathrm{d}y)=-K_\alpha\,(-\Delta)^{\frac{\alpha}{2}}$

Applebaum 2009

Functional analytical methods: Nonlocal PDEs!

- Escape probability
- Mean exit time
- Bifurcation

Escape probability from a domain D

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

$p(x)$: Likelihood that a “particle" first escapes D and lands in a domain U

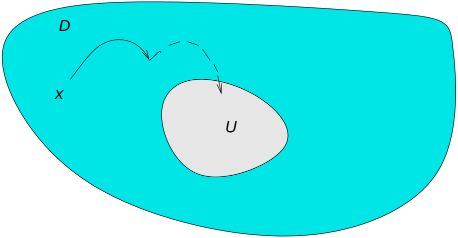


Figure: Escape probability for SDEs driven by Lévy motions: an open annular domain D, with its inner part U (which is in D^c) as a target domain

Escape probability from a domain D

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

$p(x)$: Likelihood that a “particle" first escapes D and lands in a domain U

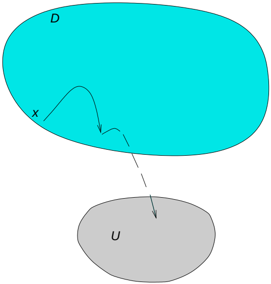


Figure: Escape probability for SDEs driven by Lévy motions: a general open domain D, with a target domain U in D^c

Recall: Classical harmonic functions

$\Delta h(x) = 0$

But Δ is the generator of Brownian motion B_t

$h(x)$: Harmonic function with respect to Brownian motion

What is a harmonic function with respect to $\alpha-$stable Lévy motion?

Brownian motion B_t: Generator is $\frac{1}{2}\Delta$

$\alpha-$stable Lévy motion: Generator is $-K_\alpha\,(-\Delta)^{\frac{\alpha}{2}}$

Harmonic function with respect to $\alpha-$stable Lévy motion:
$-(-\Delta)^{\frac{\alpha}{2}}h(x) = 0$

General harmonic functions

Harmonic function with respect to a Markov process with generator A:

$Ah(x) = 0$

"Probability" feedbacks to "Analysis"!

Escape probability vs. harmonic functions

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

For

$$\varphi(x) = \begin{cases} 1, & x \in U, \\ 0, & x \in D^c \setminus U, \end{cases}$$

$$\begin{aligned} \mathbb{E}[\varphi(X_{\tau_{D^c}}(x))] &= \int_{\{\omega : X_{\tau_{D^c}}(x) \in U\}} \varphi(X_{\tau_{D^c}}(x))dP(\omega) \\ &\quad + \int_{\{\omega : X_{\tau_{D^c}}(x) \in D^c \setminus U\}} \varphi(X_{\tau_{D^c}}(x))dP(\omega) \\ &= \mathbb{P}\{\omega : X_{\tau_{D^c}}(x) \in U\} \\ &= p(x). \end{aligned}$$

Macroscopic description

Left hand side is a harmonic function: Liao 1989

Escape probability from a domain D

Liao 1989; Kan, Qiao & Duan, 2011

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

$p(x)$: Likelihood that a “particle" first escapes D and lands in a domain U

Theorem

Escape probability is solution of Balayage-Dirichlet problem

$$\begin{cases} Ap = 0, \\ p|_U = 1, \\ p|_{D^c \setminus U} = 0, \end{cases}$$

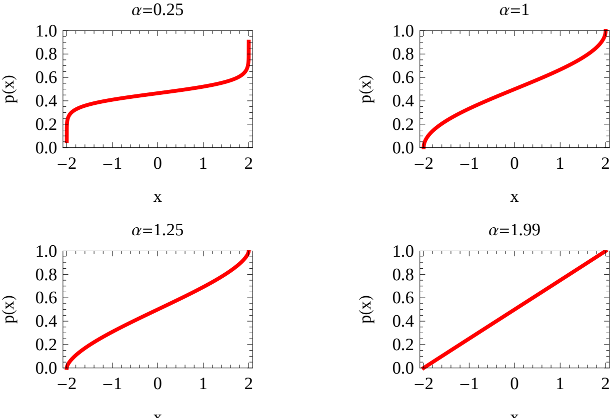
where A is the generator for this system.

Example: Pure jump system

Blumenthal, Getoor & Ray 1961

$$dX_t = 0dt + dL_t, \quad X_0 = x$$

$p(x)$: Likelihood that a “particle" first escapes $D = (-2, 2)$ and lands in a domain $U = [2, \infty)$.

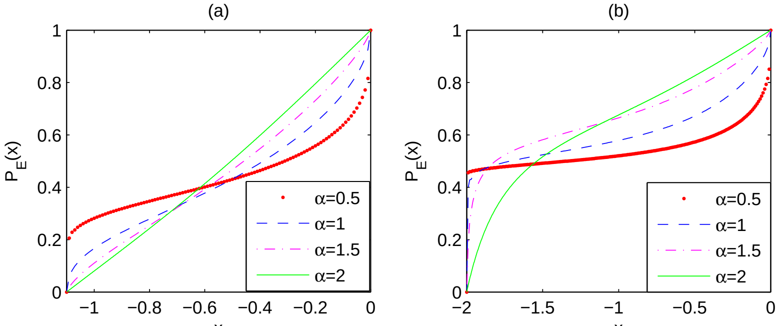


Another example: Double-well system

Gao, Duan, Li & Song 2011: A cross-over phenomenon

$$dX_t = (X_t - X_t^3)dt + dL_t, \quad X_0 = x$$

$p(x)$: Likelihood that a "particle" first escapes $D = (-1.1, 0)$ and lands in a domain $U = [0, \infty)$; i.e., transition from stable state $\{-1\}$ to another stable state $\{1\}$.

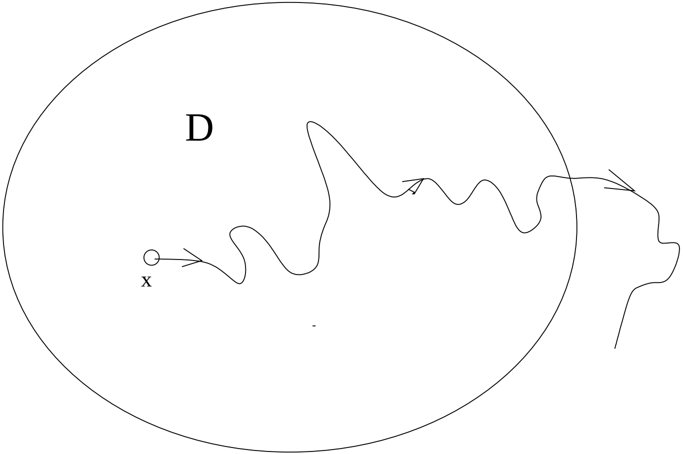


Functional analytical methods: Nonlocal PDEs!

- Escape probability
- Mean exit time
- Bifurcation

Mean exit time from a domain

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x$$



How to quantify mean exit time?

$$dX_t = f(X_t)dt + g(X_t)dL_t, \qquad X_0 = x \in D$$

$f(\cdot)$: **Deterministic vector field**

Exit time from a domain D: $\sigma_x(\omega) = \inf\{t : \ X_t \in D^c\}$

Mean exit time (**for a 'particle' starting at** x) from a domain D:
$u(x) = \mathbb{E}\sigma_x(\omega)$

Theorem: Mean exit time $u(x)$ satisfies
$Au = -1$ for $x \in D$, $u|_{D^c} = 0$,
where A is the generator for this system.

One-dim system with $\alpha-$stable Lévy motion: $(0, d, \nu_\alpha)$

$$dX_t = f(X_t)dt + dL_t, \quad X_0 = x \in D$$

Generator:

$$\begin{aligned} Au &= f(x)u'(x) + \frac{d}{2}u''(x) \\ &+ \int_{\mathbb{R}\setminus\{0\}} [u(x+y) - u(x) - I_{\{|y|<1\}}\, yu'(x)]\ \nu_\alpha(\mathrm{d}y) \end{aligned}$$

$\nu_\alpha(\mathrm{d}x) = C_\alpha |x|^{-(1+\alpha)}\,\mathrm{d}x$ with $C_\alpha = \dfrac{\alpha}{2^{1-\alpha}\sqrt{\pi}} \dfrac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(1-\frac{\alpha}{2})}$

$u(x)$: Mean exit time for a “particle" first escaping D

$$Au = -1 \text{ for } x \in D, \quad u|_{D^c} = 0$$

Example: Mean exit time for $\alpha-$stable Lévy motion

$$dX_t = 0dt + dL_t, \qquad X_0 = x$$

Getoor 1961: Mean exit time from interval $D = (-r, r)$

$$u(x) = \frac{\sqrt{\pi}}{2^\alpha \Gamma(1+\frac{\alpha}{2})\Gamma(\frac{1}{2}+\frac{\alpha}{2})} (r^2 - x^2)^{\frac{\alpha}{2}}$$

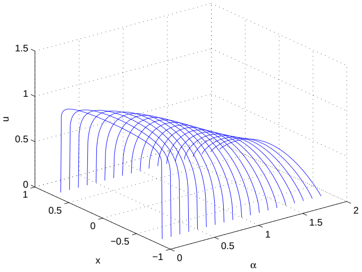
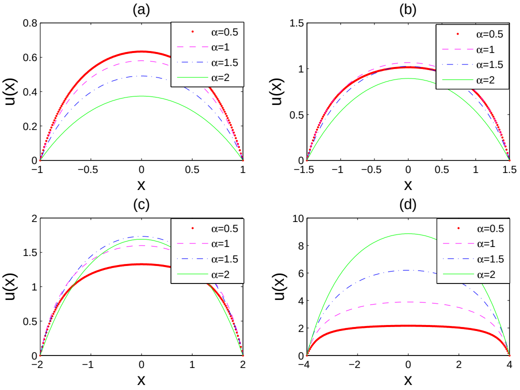


Figure: Mean exit time from the interval $(-0.75, 0.75)$

Example: Mean exit time for Ornstein-Uhlenbeck system with $\alpha-$stable Lévy motion

$$dX_t = -X_t dt + dL_t, \quad X_0 = x$$

L_t: Triplet $(0, 1, \nu_\alpha)$



Functional analytical methods: Nonlocal PDEs!

- Escape probability
- Mean exit time
- Bifurcation

Bifurcation under non-Gaussian noise

$$dX_t = f(b, X_t)dt + dL_t$$

Bifurcation: How does this system evolve as parameters b, α vary?

Space of paths: A big mess!

Space of probability measures: Order emerges out of orderless!

Bifurcation under non-Gaussian noise

$$dX_t = f(b, X_t)dt + \epsilon dL_t.$$

Fokker-Planck equation for the stationary probability density function $p(x)$:

$$-[f(b,x)p(x)]' + \epsilon \int_{\mathbb{R}\setminus\{0\}} [p(x+y) - p(x) - I_{\{|y|<1\}} y p'(x)] \frac{dy}{|y|^{1+\alpha}} = 0$$

under $p(x) \geq 0$, $\int_{\mathbb{R}} p(x)dx = 1$.

Chen, Duan & Zhang 2011

Example: Bifurcation under non-Gaussian noise

$dX_t = (bX_t - X_t^3)dt + dL_t$ for $b \in \mathbb{R}^1$

Result: Stationary prob. density function $b < 0$ unimodal $\rightarrow$ $b > 0$ bimodal

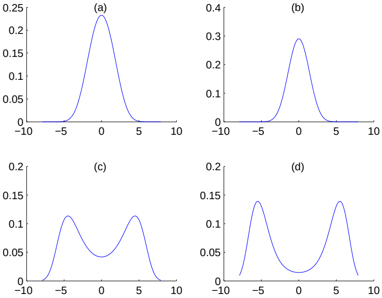


Figure: $\alpha = 1.99$. (a) $b = -30$; (b) $b = -50$; (c) $b = 20$; (d) $b = 30$

Conclusions

Nonlocal PDEs & Non-Gaussian Dynamics

- Escape probability
- Mean exit time
- Bifurcation

Computation!