Aggregation via the Newtonian Potential & Aggregation Patches

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March 7, 2012

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Multidimentional Aggregation Equation

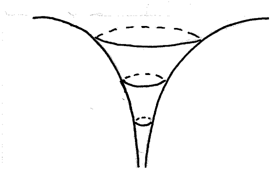
Continuum model for particles which interact via a pairwise interaction potential

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0 \\ \vec{v} = -\nabla N * \rho \end{cases}$$

$N : \mathbb{R}^d \to \mathbb{R}$

$\Delta N = \delta$

"interaction potential"



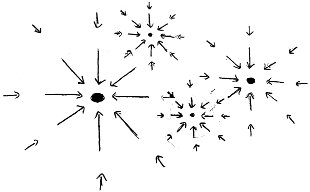
$\rho(x,t)$: density of particles

$\vec{v}(x,t)$: velocity of the particles located at x

$x \in \mathbb{R}^d$

Discrete model for N particles $X_1, \ldots, X_N$

$$\dot{X}_i = -\sum_{j=1}^{N} m_j \, \nabla N(X_i - X_j)$$



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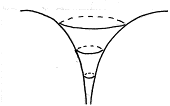
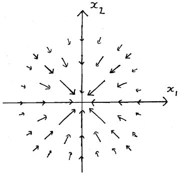


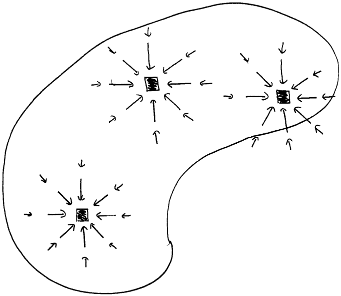
$-\nabla N : \mathbb{R}^d \to \mathbb{R}^d$

"attracting field"



$$\rho_t + \operatorname{div}(\rho \vec{v}) = 0$$
$$\vec{v} = -\nabla N * \rho$$

Every pieces of mass attracts one another according to the potential K.



Collapse!

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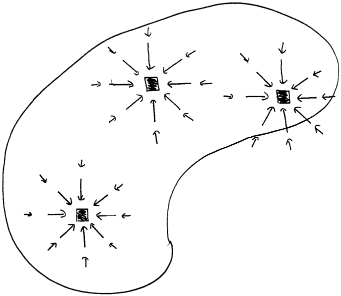


Collapse!

- Biology
- Evolution of vortex densities in superconductors,
- Simplified model for granular flow
- Materials sciences, ...

Relationship with 2D Euler

Vorticity-stream formulation of the 2D Euler Equation:

$$\begin{cases} \omega_t + \text{div}\,(\omega \vec{v}) = 0 \\ \vec{v} = -(\nabla N * \omega)^{\perp} \end{cases}$$

- $\vec{v}$ is divergence free
- ω is constant on particle path
- Global existence

Aggregation Equation in ND:

$$\begin{cases} \partial_t \rho + \text{div}\,(\rho \vec{v}) = 0 \\ \vec{v} = -\nabla N * \rho \end{cases}$$

- $\vec{v}$ concentrates the density
- ρ is grows along particle path
- Finite time blow up

Well-Posedness in $L^1 \cap L^\infty$

Particle path are well defined

Suppose $\rho(\cdot,t) \in L^1 \cap L^\infty$ and let

$$|||\rho(\cdot,t)||| = \|\rho(\cdot,t)\|_{L^1} + \|\rho(\cdot,t)\|_{L^\infty}$$

then

$$|v(x_1,t) - v(x_2,t)| \;\leq\; C\,|||\rho(\cdot,t)|||\;|x_1 - x_2|\;(1 - \log|x_1 - x_2|)$$

As long as the solution remains in $L^1 \cap L^\infty$ the velocity field is Log-Lipshitz and particle path are well defined

Method of characteristics

$$\partial_t \rho + \text{div}\,(\rho \vec{v}) = 0$$

$$\partial_t \rho + \nabla \rho \cdot \vec{v} + \rho\, \text{div}\vec{v} = 0$$

But $\quad \vec{v} = -\nabla N * \rho \quad$ so $\quad \text{div}\vec{v} = -\Delta N * \rho = -\rho$

$$\partial_t \rho + \vec{v} \cdot \nabla \rho = \rho^2 \qquad \text{where} \qquad \vec{v} = -\nabla N * \rho$$

So along the characteristics the density ρ satisfies the ODE $\dot{y} = y^2$ (solution: $y(t) = \frac{1}{(1/y_0) - t}$), that is:

$$\rho(X(t), t) = \frac{1}{\frac{1}{\rho_0(X(0))} - t} \qquad \text{blows up at } t = \frac{1}{\rho_0(X(0))}$$

So the first blowup occurs at $t^* = \frac{1}{|\rho_0|_{max}}$.

Well-Posedness

Theorem

Suppose ρ_0 is bounded and compactly supported.
Let T be such that

$$0 < T < \frac{1}{|\rho_0|_{max}}.$$

Then there exists a unique bounded and compactly supported solution on [0,T].

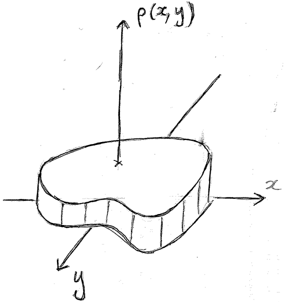
To be more precise, the solution belongs to the space

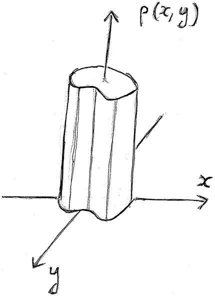
$$C([0,T], L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0,T))$$

Aggregation Patches
&
Collapse to skeleton

Aggregation Patches

$$\rho_0(x) = \frac{1}{|\Omega_0|}\,\chi_{\Omega_0}(x) \qquad\qquad \rho(x,t) = \frac{1}{|\Omega_t|}\,\chi_{\Omega_t}(x)$$



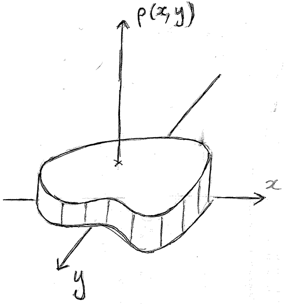


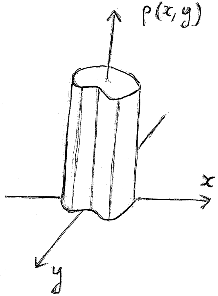
$$h'(t) = h^2, \qquad h(t) = \frac{1}{\frac{1}{h_0} - t} = \frac{1}{|\Omega_0| - t}$$

Aggregation Patches

$$\rho_0(x) = \frac{1}{|\Omega_0|}\,\chi_{\Omega_0}(x) \qquad\qquad \rho(x,t) = \frac{1}{|\Omega_t|}\,\chi_{\Omega_t}(x)$$

$$= \frac{1}{|\Omega_0| - t}\,\chi_{\Omega_t}(x).$$





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Aggregation Patches

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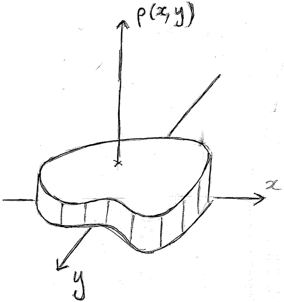
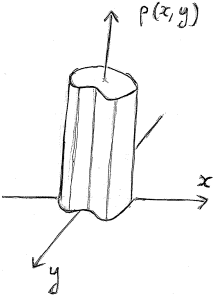
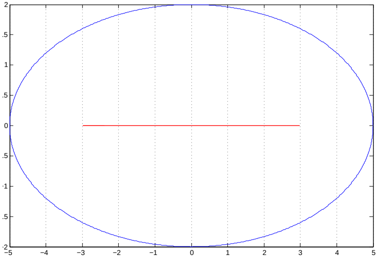




$$h'(t) = h^2, \qquad h(t) = \frac{1}{\frac{1}{h_0} - t} = \frac{1}{|\Omega_0| - t}$$

If ρ_0 = uniform distribution on a domain Ω_0
Then $\rho(\cdot,t)$ = uniform distribution on a time evolving domain Ω_t with $|\Omega_t| = |\Omega_0| - t$

Movies!



Theorem (Elliptical patch)

Let ρ_0 be the uniform distribution on the ellipse

$$\frac{x^2}{a_0^2} + \frac{y^2}{b_0^2} = 1 \qquad \textit{and let} \quad T^* = |\Omega_0| = \pi a_0 b_0.$$

As $t \to T^*$, $\rho(t)$ converges weakly-$*$ to the probability measure supported on the segment $[-r_0, r_0]$ with mass distribution

$$f(x) = \frac{2}{\pi r_0^2}\sqrt{r_0^2 - x^2}$$

where $r_0 = a_0 - b_0$.

3D Movies!

Numerics

$$\vec{v}(x,t) = -\frac{1}{|\Omega_0| - t}(\nabla N * \chi_{\Omega_t})(x)$$

Integrating by part we find

$$v(x,t) = \frac{1}{|\Omega_0| - t}\int_{\partial\Omega_t} N(x-y)n(y)d\sigma(y)$$

Curve Evolution

In 2D, letting $\partial\Omega_t = \{z(\alpha,t) \in \mathbb{R}^2 : \alpha \in [0, 2\pi)\}$

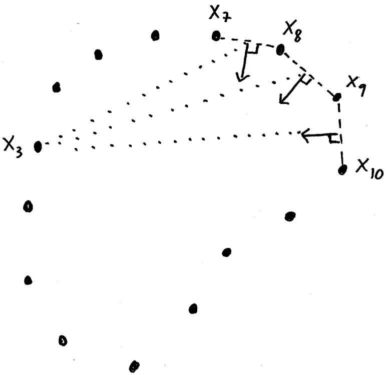
we get: $\frac{\partial z}{\partial t}(\alpha,t) = \vec{v}(z(\alpha,t),t)$

and therefore:

$$\frac{\partial z}{\partial t}(\alpha,t) = \frac{1}{|\Omega_0| - t}\,\frac{1}{2\pi}\int_0^{2\pi} \ln\left|z(\alpha,t) - z(\alpha',t)\right| \left[\frac{\partial z}{\partial \alpha}(\alpha',t)\right]^{\perp} d\alpha'$$

change of variable $\quad s = \ln\left(\frac{|\Omega_0|}{|\Omega_0| - t}\right)$

$$\frac{\partial z}{\partial s}(\alpha, s) = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|z(\alpha, s) - z(\alpha', s)\right| \left[\frac{\partial z}{\partial \alpha}(\alpha', s)\right]^{\perp} d\alpha'$$



The spreading case
&
Convergence to self-similar circular patch

Multidimentional Aggregation Equation

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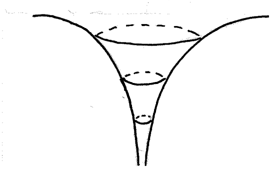
$N : \mathbb{R}^d \to \mathbb{R}$

"interaction potential"

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0 \\ \vec{v} = +\nabla N * \rho \end{cases}$$



$\rho(x,t)$: density of particles

$\vec{v}(x,t)$: velocity of the particles located at x

$x \in \mathbb{R}^d$

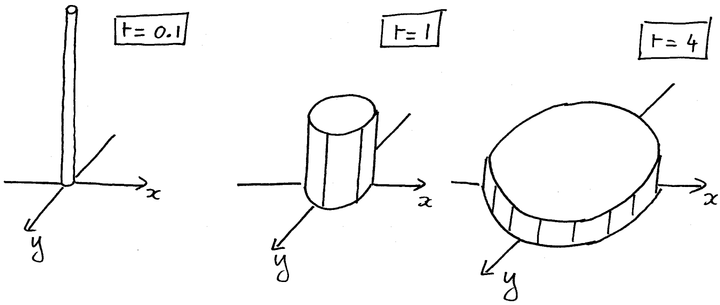
Method of characteristics

$$\partial_t \rho + \vec{v} \cdot \nabla \rho = -\rho^2 \qquad \text{where} \qquad \vec{v} = -\nabla N * \rho$$

So along the characteristics the density ρ satisfies the ODE $\dot{y} = -y^2$.

The area of a patch satisfies $|\Omega_t| = |\Omega_0| + t$

$\Phi(\cdot, t)$ is the circular patch of area t and mass 1.



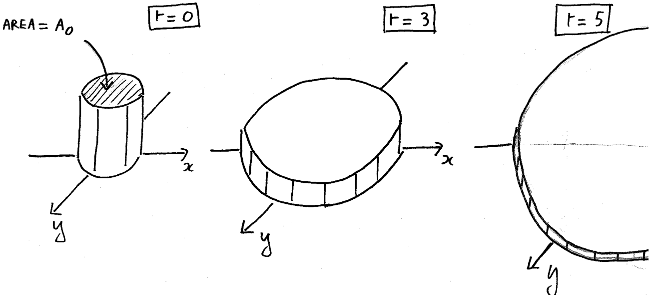
Theorem

Let $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$ be compactly supported and bounded.
Let $\rho(x, t)$ be the solution. Then

$$\|\rho(\cdot, t) - \Phi(\cdot, t)\|_{L^1} \leq \frac{C}{t^\lambda} \qquad \lambda = \frac{1}{2^{d-1}}$$

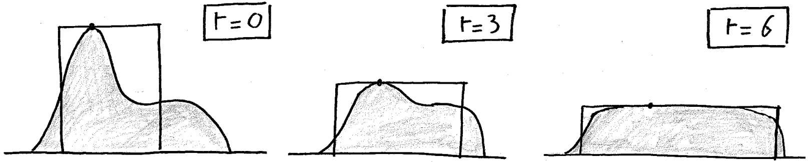
In $\mathbb{R}^2$ the rate of convergence is $\frac{1}{\sqrt{t}}$ and it is sharp.

$\Phi_{A_0}(x,t)$ = circular patch of area $A_0 + t$



Remark: $\|\Phi_{A_0}(\cdot,t) - \Phi(\cdot,t)\|_{L^1} = 2\frac{A_0}{A_0+t}$

Prove convergence to the fundamental solution which has same height than ρ_0 at time zero.



At all time $\rho(x,t)$ and the fundamental solution have same height

Change of variable

Go to the reference frame of this fundamental solution:

$$\tilde{x} = \frac{x}{R(t)} \qquad \tilde{t} = \ln\left(\frac{A_0 + t}{A_0}\right) \qquad \tilde{\rho} = \frac{1}{\omega_d}\,\frac{\rho}{h(t)}$$

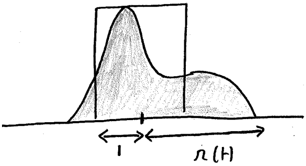
Φ_{A_0} is now a stationary circular patch of radius 1, height $1/\omega_d$ and mass 1.

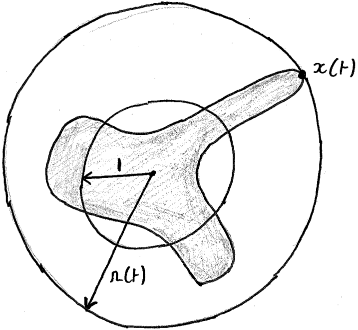
In these new variable ρ satisfies the PDE:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$
$$v = \omega_d \nabla N * \rho - \frac{x}{d}$$

and we have $\rho(t) \leq \frac{1}{\omega_d}$ for all t.

Movies!





$x(t)$: particle
the furthest away

$r(t) = |x(t)| =$
radius of the cloud of
particles

$1 =$ radius of the
steady state

Estimate of the velocity of the particle the furthest away

By Newton's Theorem: $\nabla N * \chi_{B(0,r)}(x) = (\text{mass of } \chi_{B(0,r)}) \;\; \nabla N(x) = \frac{x}{d}$

$$v(x) = \omega_d \nabla N * \rho - \frac{x}{d} = \omega_d \nabla N * \rho - \nabla N * \chi_{B(0,r)}$$

$$= -\omega_d \left[\nabla N * \left(\frac{1}{\omega_d} \chi_{B(0,r)} - \rho \right) \right](x)$$

$$v(x) \cdot \left(- \frac{x}{|x|} \right) = \omega_d \int \nabla N(x-y) \cdot \frac{x}{|x|} \left[\frac{1}{\omega_d} \chi_{B(0,r)} - \rho \right](y) dy$$

$$\geq \frac{1}{d\, \omega_d (2r)^{d-1}} \omega_d \int \left[\frac{1}{\omega_d} \chi_{B(0,r)} - \rho \right](y) dy$$

$$\geq \omega_d \left(\frac{1}{d\, \omega_d (2r)^{d-1}} \right) \left(r^d - 1 \right) \qquad \geq C \frac{r^d - 1}{r^{d-1}}$$

where we have used: $\nabla N(x-y) \cdot \frac{x}{|x|} \geq \frac{1}{d\, \omega_d (2r)^{d-1}} \qquad \forall y \in B(0,r)$

Estimate of the area of the support of the patch

$$r'(t) = v(x(t), t) \cdot \frac{x(t)}{|x(t)|} \leq -C\frac{r^d - 1}{r^{d-1}}$$

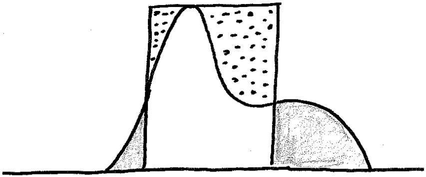
$$r^{d-1} r' \leq -C(r^d - 1)$$

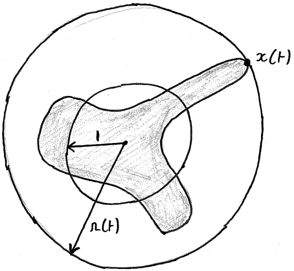
$$\frac{d}{dt}(r^d - 1) \leq -Cd(r^d - 1)$$

$$\frac{d}{dt}(\omega_d r^d - \omega_d) \leq -Cd(\omega_d r^d - \omega_d)$$

So the difference of area between the big disc and the small disc decays exponentially fast (in the rescaled variable).

Estimate of the L^1-difference between the patch and the steady state



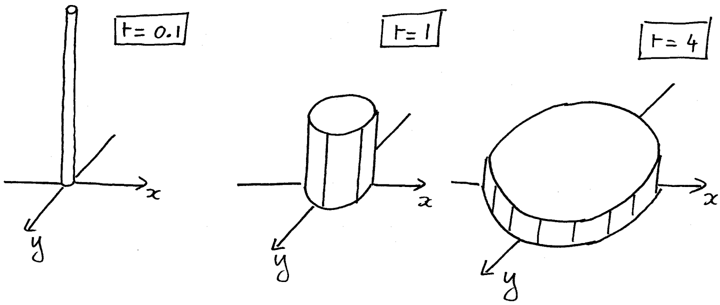


L^1-difference between the patch and the steady state

$$\leq 2 \times (\text{difference of area between the big disc and the small disc}) \times \frac{1}{\omega_d}$$

$$\leq Ae^{-Ct}$$

$\Phi(\cdot, t)$ is the circular patch of area t and mass 1.



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In $\mathbb{R}^2$ the rate of convergence is $\frac{1}{\sqrt{t}}$ and it is sharp.

Work in Progress

The boundary of the patch remains smooth
up to the collapse time $T^* = |\Omega_0|$

$$\text{if } \partial\Omega_0 \text{ is } C^{1,\gamma} \text{ for some } \gamma \in (0,1)$$

$$\text{then } \partial\Omega_t \text{ is } C^{1,\gamma} \text{ for all } t \in [0, T^*)$$

with A. Bertozzi and J. Garnett