Swarm dynamics and equilibria for a nonlocal aggregation model

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Self-organizing animal aggregations

- Animal groups with a high structural order
- The behaviour of individuals is so coordinated, that the group moves as a single coherent entity
- Examples of self-organizing biological groups
  - schooling fish
  - herds of ungulates
  - swarming insects
  - zigzaging flocks of birds

Mathematical models

- The existing models fall into 2 categories: **Lagrangian** and **Eulerian**

- Lagrangian models: trajectories of all individuals of a species are tracked according to a set of interaction and decision rules
  - a large set of coupled ODE's
  - a large set of coupled difference equations (discrete time)

- Eulerian models: the problem is cast as an evolution equation for the population density field
  - parabolic
  - hyperbolic

A nonlocal Eulerian PDE swarming model

- We study the PDE aggregation model in $\mathbb{R}^n$:
  - continuity equation for the density ρ:

    $$\rho_t + \nabla \cdot (\rho v) = 0$$

  - the velocity v is assumed to have a functional dependence on the density

    $$v = -\nabla K * \rho$$

  - the potential K incorporates social interactions: attraction and repulsion

- The model was first suggested by Mogilner and Keshet, *J. Math. Biol.* [1999]

- Literature on this model has been very rich in recent years

Lagrangian description

N individuals; $X_i(t)$ = spatial location of the i-th individual at time t

$$\frac{dX_i}{dt} = -\frac{1}{N} \sum_{\substack{j=1\dots N \\ j\neq i}} \nabla_i K(X_i - X_j), \qquad i = 1 \dots N$$

PDE: continuum approximation, as $N \to \infty$

Assumption: social interactions depend only on the relative distance between the individuals

$$K(x) = K(|x|)$$

Notation: $F(r) = -K^{'}(r)$

$$\frac{dX_i}{dt} = \frac{1}{N} \sum_{\substack{j=1\dots N \\ j\neq i}} F(|X_i - X_j|) \frac{X_i - X_j}{|X_i - X_j|}, \qquad i = 1 \dots N$$

$F(|X_i - X_j|)$ = magnitude of the force that the individual X_j exerts on the individual X_i, along $X_i - X_j$.

Repulsion ($F(r) > 0$) acts at short ranges, attraction ($F(r) < 0$) at long ranges.

Motivation for this work

- Equilibria of the model should have biologically relevant features:
  - finite densities
  - sharp boundaries
  - relatively constant internal population

- The main motivation for this work is to
  - design interaction potentials K which lead to such equilibria
  - investigate analytically and numerically the well-posedness and long time behaviour of solutions

Interaction potential K

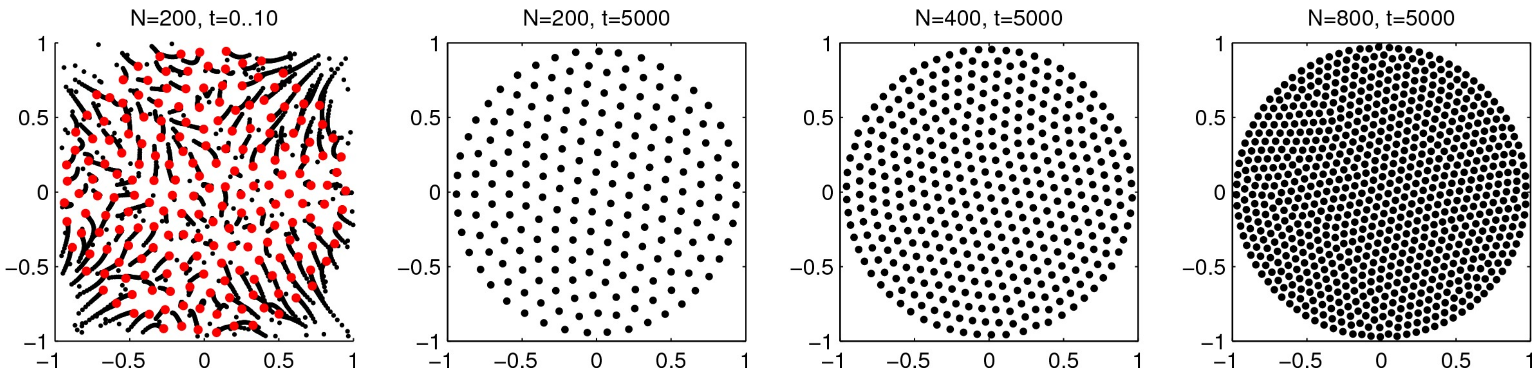
$$K(x) = K_r + K_a$$
$$= \phi(x) + \frac{1}{q}|x|^q, \qquad q > 2 - n$$

$\phi(x) =$ the free-space Green's function for $-\Delta$:

$$\phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2 \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

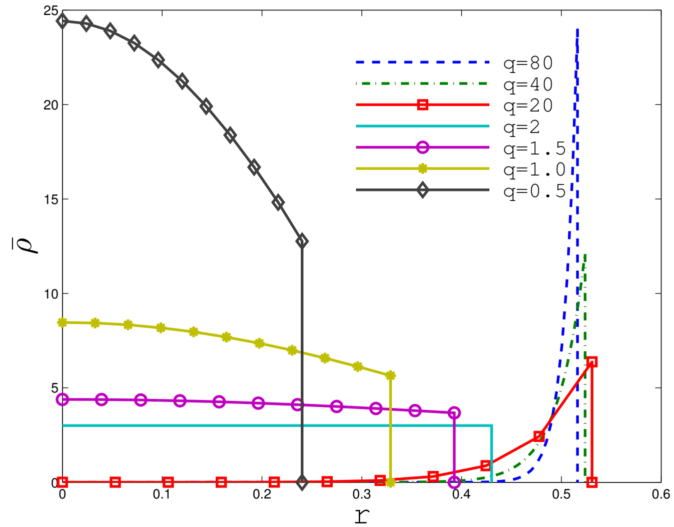
Example a. $n = 2, q = 2$: $\quad K(x) = -\ln|x| + \frac{1}{2}|x|^2; \qquad F(r) = \underbrace{\frac{1}{r}}_{repulsion} - \underbrace{r}_{attraction}$

Random initial conditions inside the unit square. The solution approaches a constant density in the unit disk.



Example b. $n = 3$, various q

$$K(x) = \frac{1}{4\pi|x|} + \frac{1}{q}|x|^q, \qquad q > -1$$



- The equilibria are monotone in the radial coordinate: decreasing about the origin for $2 - n < q < 2$, increasing for $q > 2$, and constant for $q = 2$.

- As $q \to \infty$, the radii of the equilibria approach a constant, and mass aggregates toward the edge of the swarm.

- As $q \searrow 2 - n$, the radii of equilibria approach 0 and mass concentrates at the origin.

- Numerics suggests that all these equilibria are global attractors for the dynamics.

Benefits of the Newtonian repulsion

Our model:

$$\rho_t + \nabla \cdot (\rho v) = 0, \qquad v = -\nabla K * \rho$$

$$K(x) = \phi(x) + \frac{1}{q}|x|^q, \qquad q > 2 - n$$

Expand $\nabla\cdot$ in the equation: $\quad \rho_t + v \cdot \nabla \rho = -\rho \, \mathrm{div}\, v$

Calculate $\mathrm{div}\, v$:

$$\begin{aligned} \mathrm{div}\, v &= \mathrm{div}(-\nabla K * \rho) \\ &= -\Delta K * \rho \\ &= \rho - \Delta\Big(\frac{1}{q}|x|^q\Big) * \rho \end{aligned}$$

The repulsion term has become local!

Lagrangian approach

Characteristic curves: $\frac{d}{dt}X(\alpha,t) = v(X(\alpha,t),t), \qquad X(\alpha,0) = \alpha$

Evolution equation for $\rho(X(\alpha,t),t)$:

$$\frac{D\rho}{Dt} = -\rho^2 + \rho\,\Delta\Big(\frac{1}{q}|x|^q\Big) * \rho$$

Special case $q = 2$: explicit calculations

$$\Delta\Big(\frac{1}{2}|x|^2\Big) = n, \quad \Delta\Big(\frac{1}{2}|x|^2\Big) * \rho = n \underbrace{\int_{\mathbb{R}^n} \rho(y)dy}_{=M}$$

ODE along characteristics: $\frac{D\rho}{Dt} = -\rho(\rho - nM)$

Exact solution: $\rho(X(\alpha,t),t) = \frac{nM}{1+\left(\frac{nM}{\rho_0(\alpha)}-1\right)e^{-nMt}}$

Asymptotic behaviour as $t \to \infty$?

Asymptotic behaviour $t \to \infty$

Density: $\rho(X,t) \to nM$, as $t \to \infty$, along particle paths with $\rho_0(\alpha) \neq 0$

Asymptotic behaviour of trajectories: $R_\alpha = \lim_{t\to\infty} |X(\alpha,t)|$

For radial solutions, it can be proved that trajectories are mapped into the ball of $\mathbb{R}^n$ of radius $R_\alpha = \frac{1}{(n\omega_n)^{\frac{1}{n}}}$.

Numerics suggest that *all* solutions have this asymptotic behaviour.

Global attractor: constant, compactly supported density:

$$\bar{\rho}(x) = \begin{cases} nM & \text{if } |x| < \frac{1}{(n\omega_n)^{\frac{1}{n}}} \\ 0 & \text{otherwise} \end{cases}$$

Global existence of particle paths

$$v(x) = \int_{\mathbb{R}^n} k(x-y)\rho(y)\,dy - Mx, \qquad (1)$$

where $$k(x) = \frac{1}{n\omega_n}\frac{x}{|x|^n}$$

The convolution kernel k is singular, homogeneous of degree $1-n$.

Equation (1) is analogous to Biot-Savart law, where vorticity ω is now replaced by density ρ.

Existence and uniqueness of particle paths follow similarly to that for incompressible Euler equations.

Extension to global existence: Beale-Kato-Majda criterion

$$\int_0^t \|\rho(\cdot,s)\|_{L^\infty} ds < \infty, \text{ for all finite times } t$$

Analysis extends to general exponent $q > 2-n$.

Case $q \neq 2$: Non-constant steady states

Assume (based on numerics) that the model admits a steady state supported on a ball $B(0,R)$.

Recall formula for div v: $\quad \operatorname{div} v = \rho - \Delta\left(\frac{1}{q}|x|^q\right) * \rho$

Equilibria supported on $B(0,R)$:

$$v = 0, \text{ hence } \operatorname{div} v = 0 \text{ in } B(0,R)$$

A steady state $\bar\rho$ satisfies:

$$\boxed{\bar\rho - (n+q-2)\int_{\mathbb{R}^n} |x-y|^{q-2}\bar\rho(y)dy = 0 \qquad \text{in } B(0,R)}$$

Define operator T_R: $\quad T_R\bar\rho(x) = (n+q-2)\int_{B(0,R)} |x-y|^{q-2}\bar\rho(y)dy$

Solutions $\bar\rho$ are eigenfunctions of T_R corresp. to eigenvalue 1:

$$\boxed{T_R\bar\rho = \bar\rho}$$

Existence and uniqueness of equilibria

Theorem. For every $q > 2 - n$ and $M > 0$, there exists a unique radius R (that depends on q and n only) and a unique steady state $\bar{\rho}$ that is supported on $B(0, R)$, has mass M and is continuous on its support.

Sketch of **proof**: Consider case $R = 1$ first.

$$T_1\bar{\rho}(x) = (n + q - 2)\int_{B(0,1)} |x - y|^{q-2}\bar{\rho}(y)dy$$

T_1 is a linear, strongly positive, compact operator that maps the space of continuous functions $C([0, 1], \mathbb{R})$ into itself.

Krein-Rutman theorem: there exists a *positive* eigenfunction $\bar{\rho}_1$ such that

$$T_1\bar{\rho}_1 = \lambda\bar{\rho}_1 \tag{2}$$

$\lambda(q, n)$ is the spectral radius of T_1; it is a simple eigenvalue and there is no other eigenvalue with a positive eigenvector.

Define, by rescaling: $\bar{\rho}(r) = \bar{\rho}_1(r/R)$, introduce in (2):

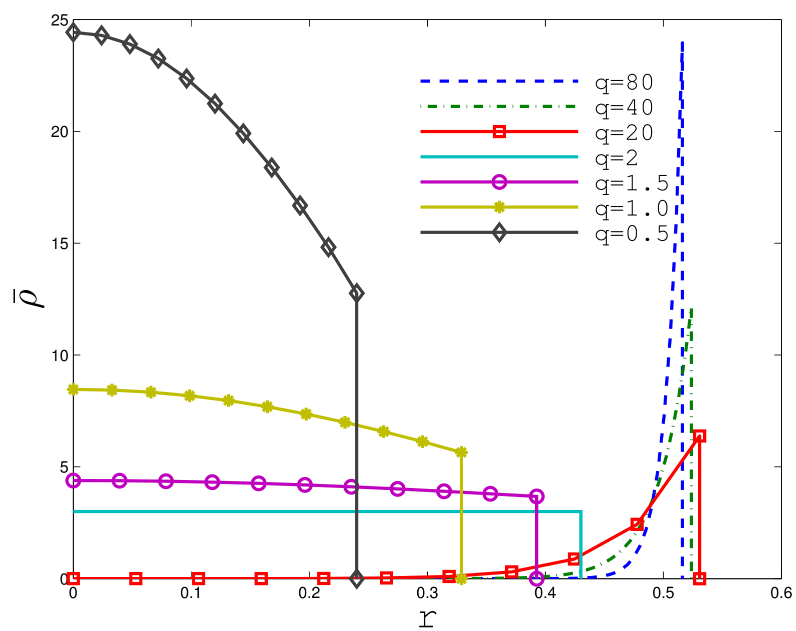
$$T_R\bar{\rho}(r) = \underbrace{R^{n+q-2}\lambda}_{=1}\bar{\rho}(r)$$

Find $R = \lambda^{-\frac{1}{n+q-2}}$.

Qualitative properties of equilibria

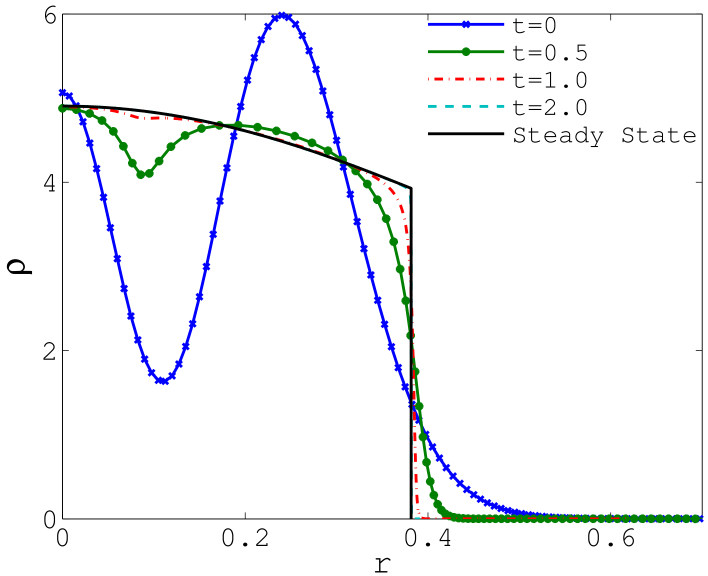
Theorem. Consider a bounded steady state $\bar{\rho}(x)$ that is supported in a ball $B(0, R)$ of $\mathbb{R}^n$. Then, $\bar{\rho}$ is radially symmetric and monotone about the origin. More specifically, we distinguish two cases: (i) $2 - n < q < 2$, when $\bar{\rho}$ is decreasing about the origin, and (ii) $q > 2$, when $\bar{\rho}$ is increasing.

Proof uses the method of moving planes.

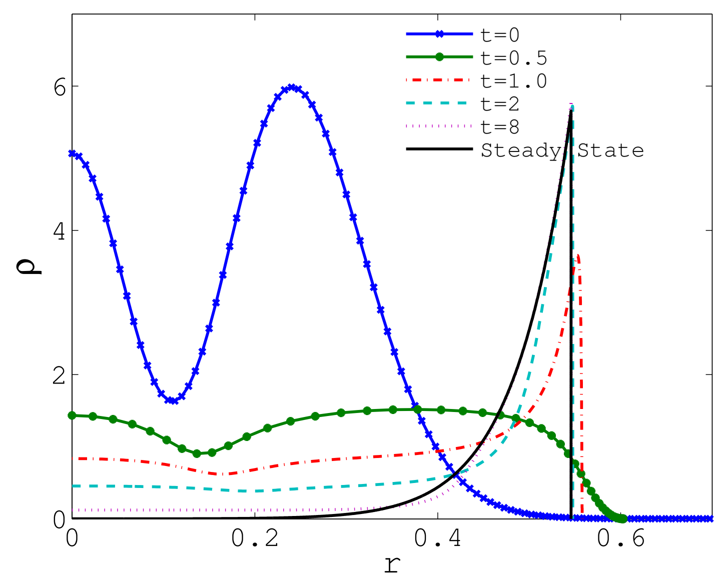


Dynamic evolution to equilibria

Numerical results in $n = 3$ dimensions:



(a) $q = 1.5$



(b) $q = 20$

The solutions approach asymptotically the steady states studied and shown in previous slides.

Numerics with a variety of other initial conditions suggests that these equilibria are global attractors for the dynamics.

Asymptotic behavior of equilibria: $q \to \infty$ and $q \searrow 2-n$

First consider case $R = 1$ and the eigenvalue problem

$$T_1\bar{\rho}_1(x) = \lambda\bar{\rho}_1(x), \qquad \text{for } x \in B(0,1),$$

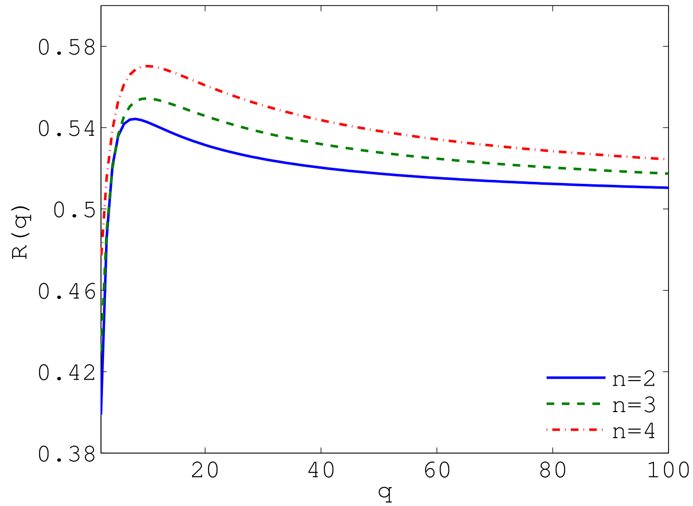
with

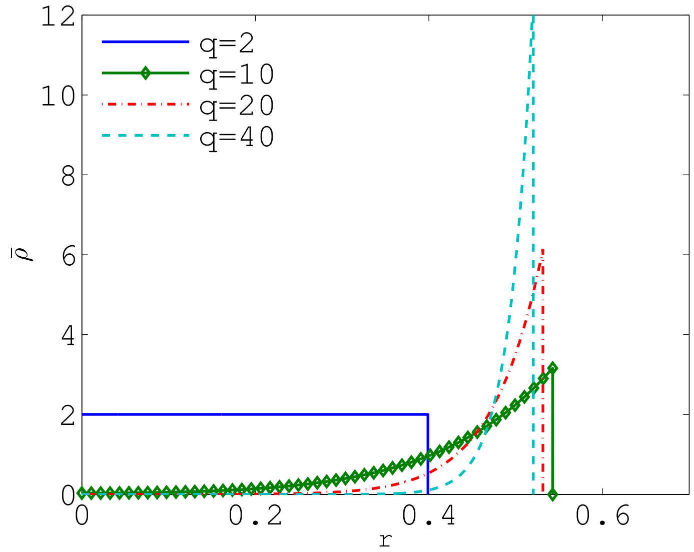
$$T_1\bar{\rho}_1(x) = (n+q-2)\int_{B(0,1)} |x-y|^{q-2}\bar{\rho}_1(y)dy$$

Goal: Use perturbation methods to find approximations for λ (spectral radius of T_1) and $\bar{\rho}_1$ (corresponding eigenvector).

Then, consider general R and find approximations to equilibrium solution $\bar{\rho}(r) = \bar{\rho}_1(r/R)$ and its radius of support $R = \lambda^{-\frac{1}{n+q-2}}$.

Asymptotic behavior $q \to \infty$





Numerics suggests: the radius of the support R approaches a constant, as $q \to \infty$ (left).

As q increases, mass aggregates toward the edge of the swarm, creating an increasingly void region in the centre (right).

$$\lambda \bar{\rho}_1(x) = (n+q-2) \int_{B(0,1)} |x-y|^{q-2} \bar{\rho}_1(y) dy$$
$$\approx (n+q-2) \bar{\rho}_1(1) \int_{B(0,1)} |x-y|^{q-2} dy.$$

Evaluate at $x = 1$, cancel $\bar{\rho}_1(1)$ and find a *coarse* approximation to λ.

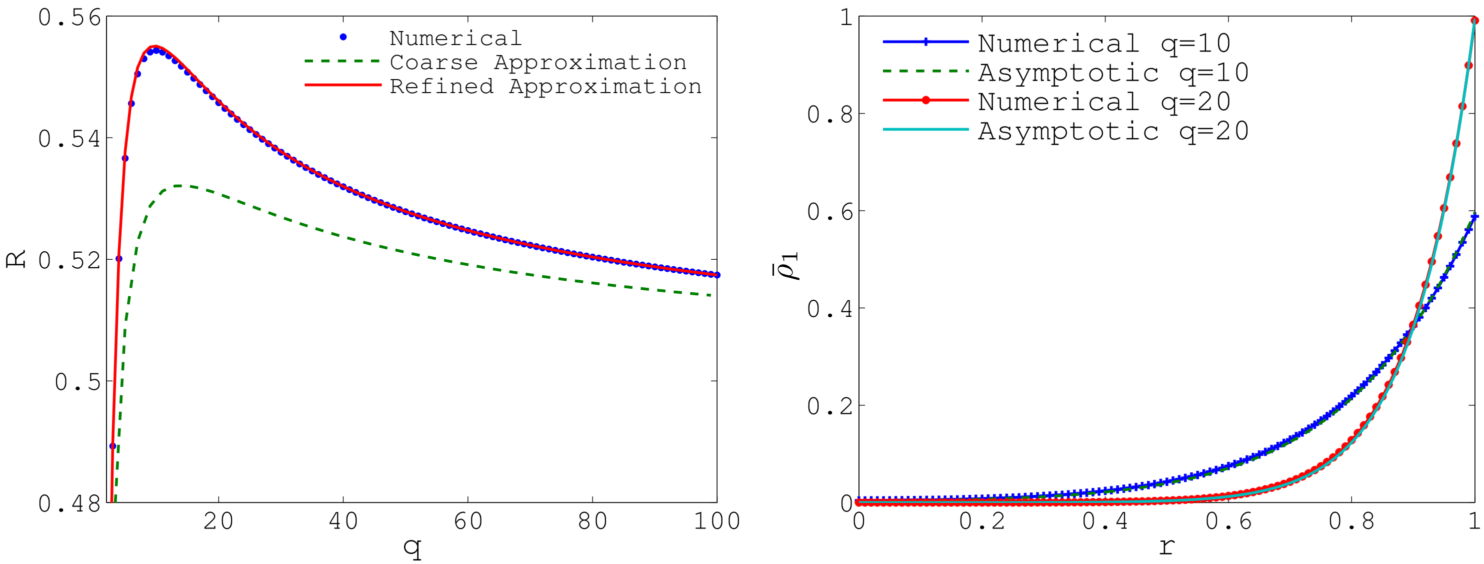
$$\lambda \approx (n-1)\omega_{n-1} 2^{n+q-3} \Gamma(\frac{n-1}{2}) \Gamma(\frac{n+q-1}{2}) / \Gamma(n-1+\frac{q}{2})$$

Approximate $\bar{\rho}_1$:

$$\bar{\rho}_1(r) \approx C \int_0^{\pi} \sin^{n-2}\theta (\sqrt{1 - r^2 \sin^2\theta} - r\cos\theta)^{n+q-2} d\theta,$$

Iterate the procedure to find a *refined* approximation for λ.

Find approximations for R.



$n = 3$: Numerical and asymptotic solutions, as $q \to \infty$. There is excellent agreement between the two solutions.

Asymptotic behavior $q \searrow 2-n$

Denote $\quad q = 2-n+\epsilon, \quad \epsilon > 0$

Asymptotic study in the small ϵ regime

$$\lambda_\epsilon \bar{\rho}_1^\epsilon(x) = \epsilon \int_{B(0,1)} \frac{1}{|x-y|^{n-\epsilon}} \bar{\rho}_1^\epsilon(y)dy, \qquad \epsilon = q-1. \qquad (3)$$

Asymptotic expansion (suggested by numerical simulations)

$$\lambda_\epsilon = \lambda_0 + \lambda_1\epsilon + \lambda_2\epsilon^2 + \cdots$$

$$\bar{\rho}_1^\epsilon(x) = \bar{\rho}^{(0)}(x) + \epsilon\bar{\rho}^{(1)}(x) + \epsilon^2\bar{\rho}^{(2)}(x) + \cdots$$

The kernel $\frac{1}{|x-y|^{n-\epsilon}}$ is not integrable for $\epsilon = 0$, so we can not substitute the formal expansions into (3) directly.

Subtract on both sides $k_\epsilon(|x|)\bar{\rho}_1^\epsilon(x)$, where

$$k_\epsilon(|x|) = \epsilon \int_{B(0,1)} \frac{1}{|y-x|^{\epsilon-n}} dy$$

$$(\lambda_\epsilon - k_\epsilon(|x|))\bar{\rho}_1^\epsilon(x) = \epsilon \int_{B(0,1)} \frac{\bar{\rho}_1^\epsilon(y) - \bar{\rho}_1^\epsilon(x)}{|y-x|^{n-\epsilon}} dy, \qquad r = |x|,$$

The integral in RHS is now $O(1)$ for $\epsilon = 0$, provided $\bar{\rho}^{(0)}$ is Hölder continuous

Expand: $\quad k_\epsilon(r) = n\omega_n + \frac{n\omega_n}{2} \ln(1-r^2)\epsilon + k^{(2)}(r)\epsilon^2 + O(\epsilon^3)$

Match powers of ϵ on both sides:

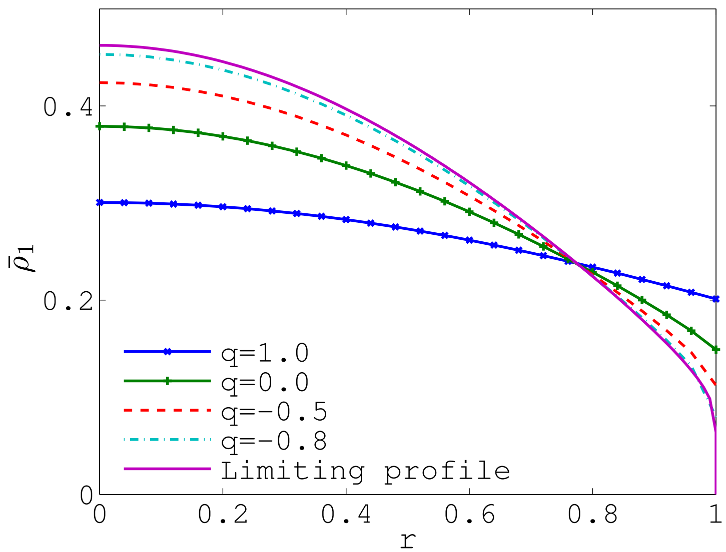
$$O(1): \qquad \lambda_0 - n\omega_n = 0$$

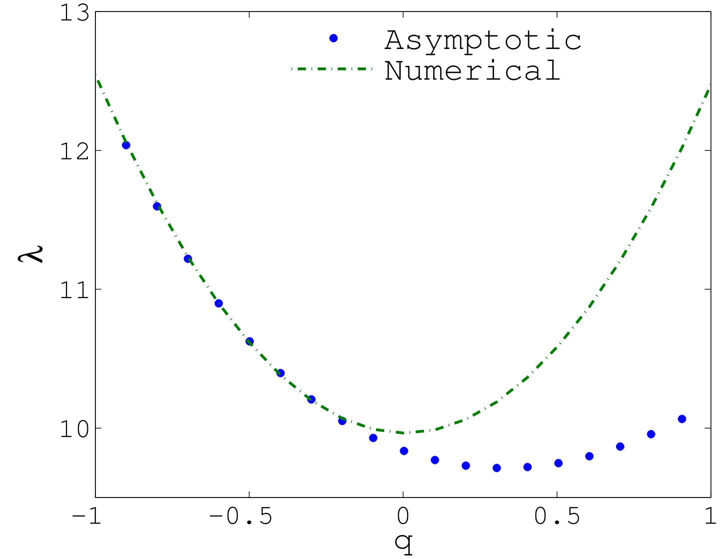
$$O(\epsilon): \qquad \int_{B(0,1)} \frac{\bar{\rho}^{(0)}(x) - \bar{\rho}^{(0)}(y)}{|y-x|^n} dy - (\lambda_1 - \frac{1}{2} n\omega_n \ln(1-|x|^2))\bar{\rho}^{(0)}(x) = 0$$

Leading order: $\lambda_0 = const.$ and limiting profile $\bar{\rho}^{(0)}$ (solved by inverse iteration)

$q \searrow 2-n$: Numerics $\sim$ asymptotics

$n = 3$ (critical value $q = -1$)





Left: Equlibria $\bar{\rho}_1^\epsilon$ for $\epsilon = 2, 1, 0.5, 0.2$ ($q = 1, 0, -0.5, -0.8$).

The plain solid line is the limiting profile $\bar{\rho}^{(0)}$ found from asymptotics. As $\epsilon \to 0$, $\bar{\rho}_1^\epsilon$ approaches the limiting profile, confirming the asymptotic expansion.

Right: Eigenvalues obtained numerically (dots) and from the asymptotic expansion valid at order $O(\epsilon^2)$. Excellent agreement for small ϵ ($q \approx -1$).

Remark. As $q \searrow 2-n$, radius $R = \lambda^{-1/(n+q-2)} = \lambda^{-1/\epsilon}$ of support approaches 0 exponentially fast and $\bar{\rho}(r) = \bar{\rho}_1(r/R)$ converges to a Dirac delta.

Even q: polynomial steady states

Integral equation for $\bar{\rho}$ in radial coordinates:

$$\bar{\rho}(r) = c(q,n) \int_0^R (r')^{n-1} \bar{\rho}(r') I(r,r') dr', \qquad 0 \le r < R$$

$$I(r,r') = \int_0^\pi (r^2 + (r')^2 - 2rr' \cos\theta)^{q/2-1} \sin^{n-2}\theta d\theta.$$

Kernel $I(r,r')$ is separable when q is even.

Define the i-th order moments of the density ($m_0 = M$):

$$m_i = n\omega_n \int_0^R r^{n+i-1} \bar{\rho}(r) dr. \tag{4}$$

Example: $q = 4$

$$I(r,r') = (r^2 + (r')^2) \int_0^\pi \sin^{n-2}\theta d\theta$$

$$\bar{\rho}(r) = (n+2) m_0 r^2 + (n+2) m_2 \tag{5}$$

Plug (5) into (4): find a linear system to solve for m_0 and m_2

General q even: $\bar{\rho}(r)$ is a polynomial of even powers, of degree $q-2$

Inverse problem: custom designed potentials

Inverse problem: given a density $\bar{\rho}(x)$, can we find a potential K for which $\bar{\rho}(x)$ is a steady state of the model?

Answer: Yes, provided $\bar{\rho}(x)$ is radial and is a polynomial in $|x|$.

Theorem: In dimensions $n = 2$ or $n = 3$, consider a radially symmetric density $\bar{\rho}(x) = \bar{\rho}\,(|x|)$ of the form

$$\bar{\rho}(r) = \begin{cases} b_0 + b_2 r^2 + b_4 r^4 + \ldots + b_{2d} r^{2d} & |x| < R \\ 0 & \text{otherwise.} \end{cases}$$

Then $\bar{\rho}(r)$ is the steady state corresponding to the force

$$F(r) = \frac{1}{n\omega_n}\frac{1}{r^{n-1}} - \sum_{i=0}^{d} \frac{a_{2i}}{2i+n} r^{2i+1},$$

where the constants $a_0, a_2, \ldots, a_{2d}$, are computed uniquely from $b_0,\ b_2, \ldots, b_{2d}$ by solving a linear system.

Remark: Case $d = 0$ corresponds to family $q = 2$ from previous slides:

$$\bar{\rho}(r) = const., \qquad F(r) = \frac{1}{n\omega_n}\frac{1}{r^{n-1}} - Cr$$

Inverse problem: examples

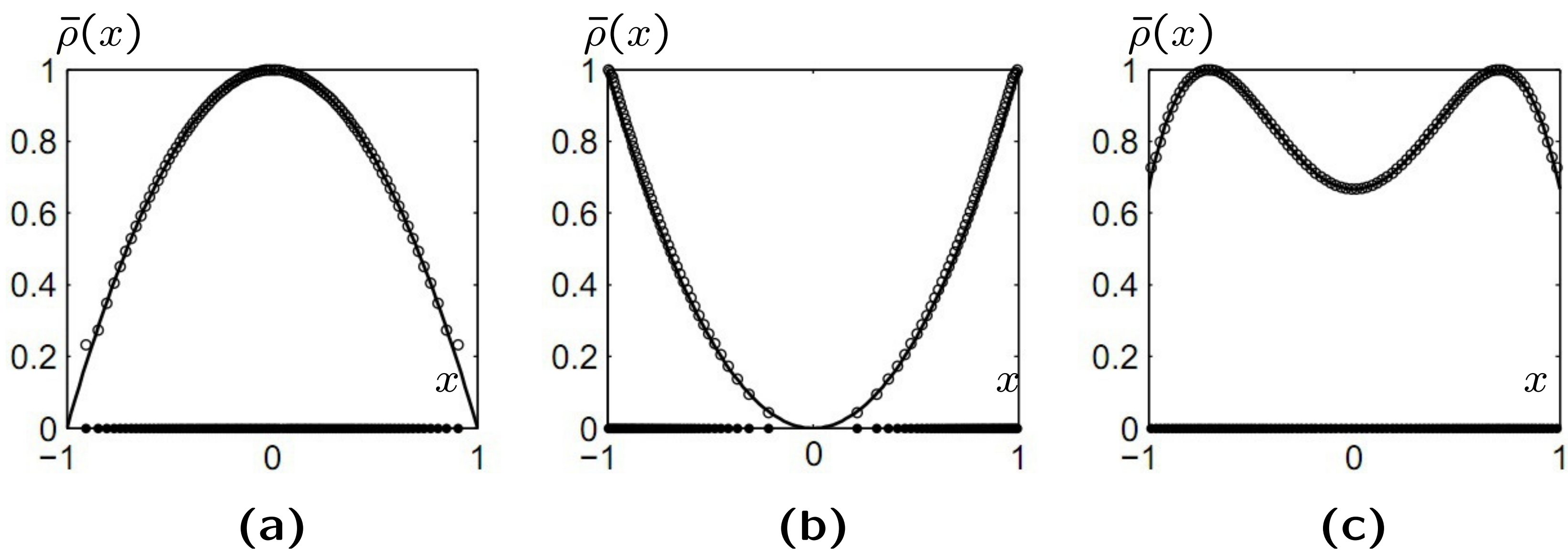
Examples: $n = 1$, $R = 1$

$$(a)\quad \bar\rho(x) = 1 - x^2; \qquad (b)\quad \bar\rho(x) = x^2; \qquad (c)\quad \bar\rho(x) = \frac{1}{2} + x^2 - x^4$$

The corresponding forces given by the Theorem are:

$$(a)\quad F(x) = \frac{1}{2} - \frac{9}{10}x + \frac{1}{4}x^3; \qquad (b)\quad F(x) = \frac{1}{2} + \frac{9}{10}x - \frac{1}{2}x^3;$$

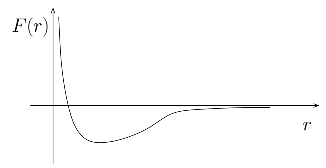
$$(c)\quad F(x) = \frac{1}{2} + \frac{209425}{672182}x - \frac{2075}{2527}x^3 + \frac{3}{19}x^5.$$

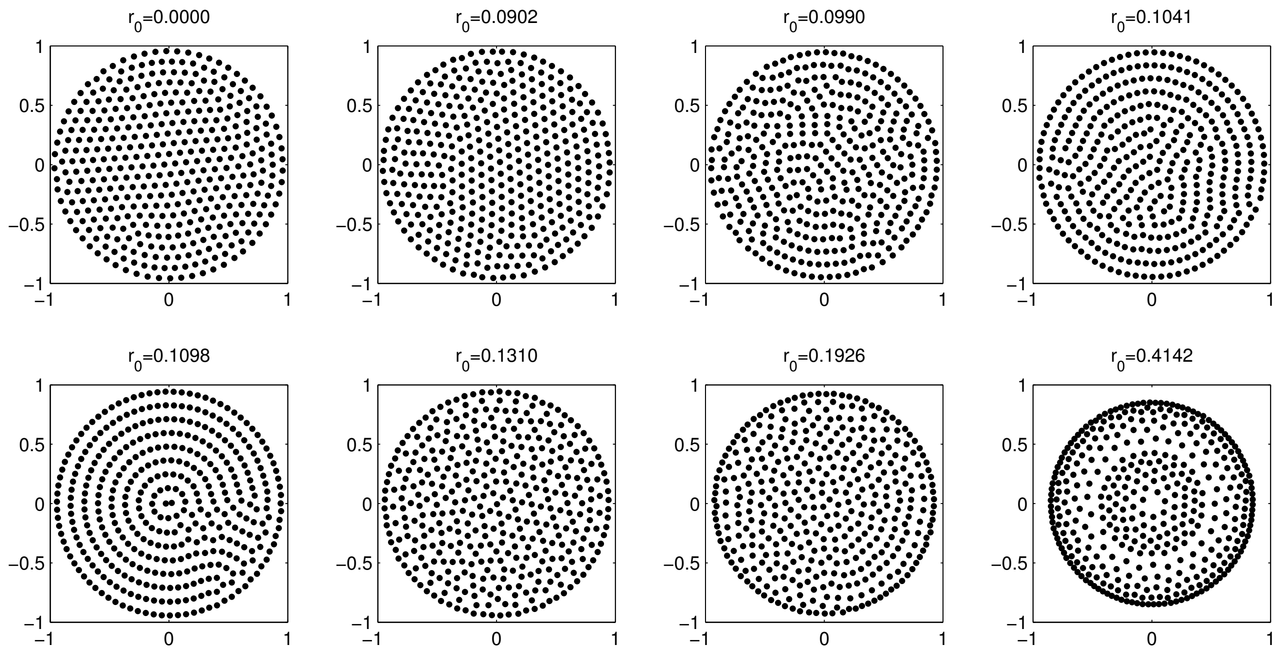


Filled circles along the x-axis: the steady states reached by numerical time evolution. Empty circles: density function as computed from the filled circles. Solid line: analytical expression for $\bar\rho$.

Regularized potentials

$$F(r) = \frac{1}{r} - r \quad \xrightarrow{\text{regularize}} \quad F(r) = \begin{cases} C_1, & 0 \le r < r_0 \\ \frac{1}{r} - r, & r_0 \le r \le 2 \\ -C_2 \exp(-r), & 2 < r \end{cases} \qquad (6)$$





Equilibrium states for the regularized interaction force (6). Initial conditions were chosen at random in the unit square. For $r_0 < 0.09$, the steady state is the same as for $r_0 = 0$ (uniform density in the unit circle).

Bibliography

1. R.C. Fetecau, Y. Huang and T. Kolokolnikov [2011]. Swarm dynamics and equilibria for a nonlocal aggregation model, *Nonlinearity*, Vol. 24, No. 10, pp. 2681-2716 (featured article)

2. R.C. Fetecau and Y. Huang [2012]. Equilibria of biological aggregations with nonlocal repulsive-attractive interactions, submitted

Future Directions

Energy considerations: local/ global minima

$$E[\rho] = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-y)\rho(x)\rho(y)dydx$$

The model is a gradient flow with respect to this energy:

$$\frac{d}{dt}E[\rho] = -\int_{\mathbb{R}^n} \rho(x)|\nabla K * \rho(x)|^2 dx \leq 0$$