Γ-convergence for pattern forming systems with competing interactions

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joint work with Dorian Goldman and Sylvia Serfaty

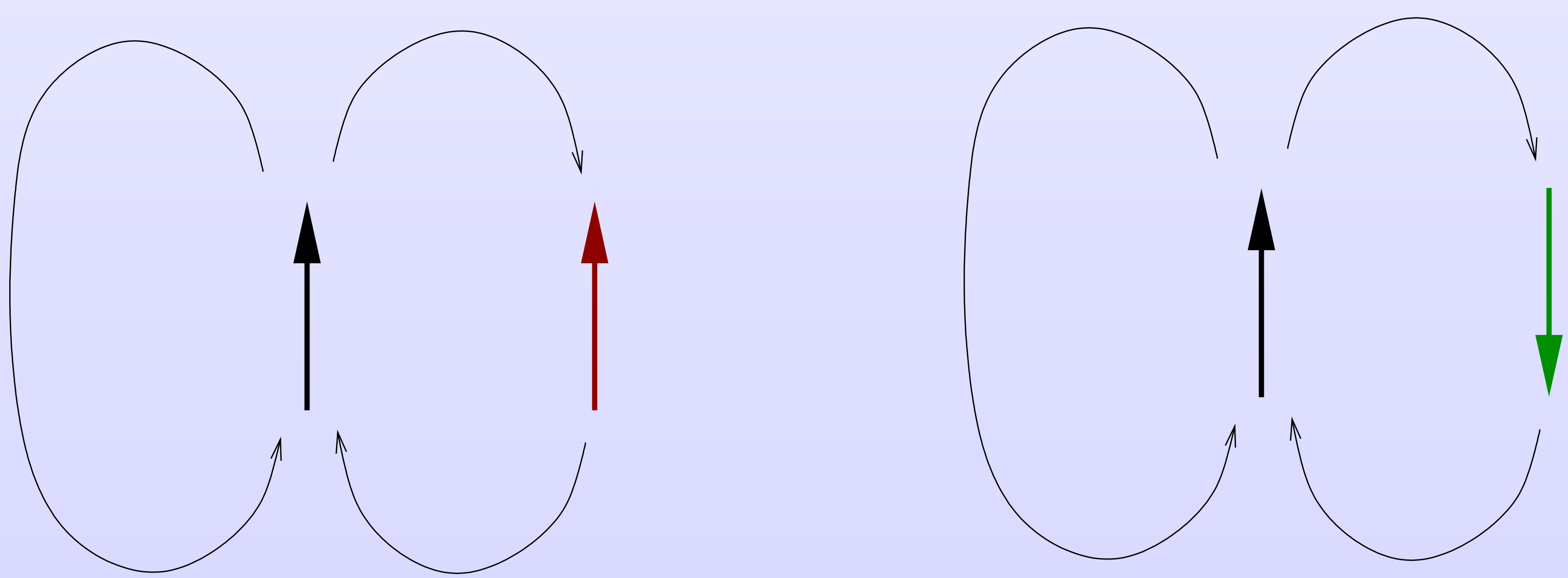
Competing interactions

Example: ferromagnetic materials

- short-range ordering of spins by exchange interactions

- long-range forces frustrate magnetic ordering

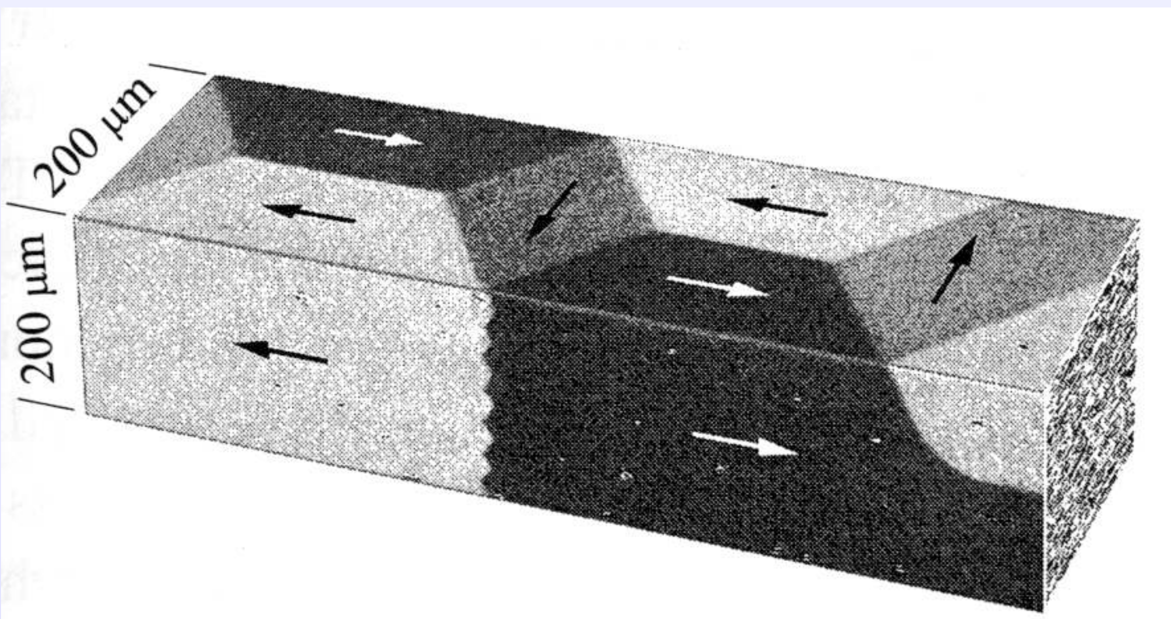
high energy



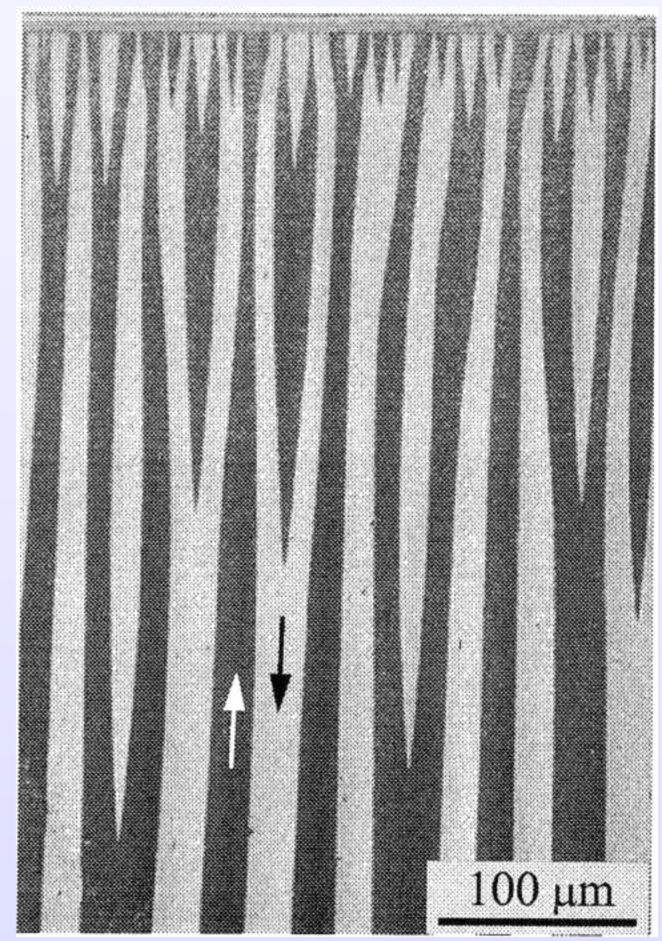
low energy

Magnetization patterns

Some examples:



iron whiskers



thick cobalt films

(from Hubert and Schafer: *Magnetic domains*)

Energetics of competing short-range and long-range interactions

Energy functional:

$$\mathcal{E}[u] = \int \left(\frac{1}{2} |\nabla u|^2 + f(u) \right) dx + \frac{\alpha}{2} \iint g[u(x)] G_0(x,y) g[u(y)] \, dx \, dy$$

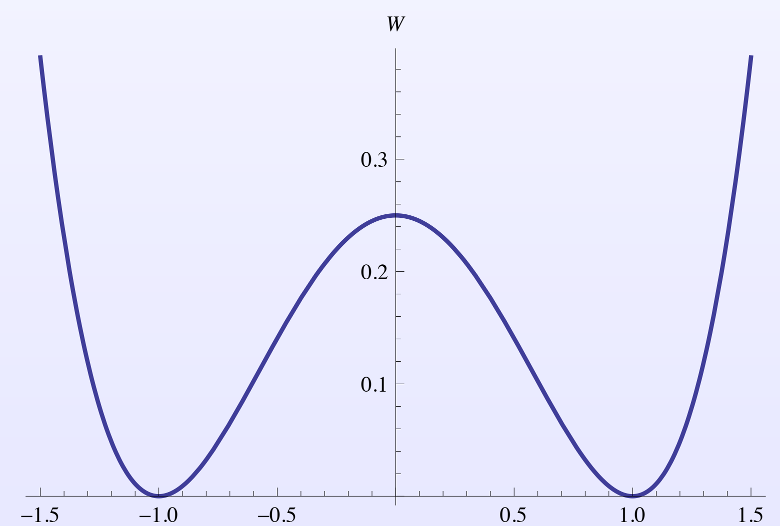
- local part favors phase segregation

- long-range kernel favors spatial homogeneity

- volume fraction of one phase fixed

Energetics of competing short-range and long-range interactions (cont.)

Ginzburg-Landau framework:



$$\mathcal{E}[u] = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx + \frac{1}{2} \int_\Omega \int_\Omega (u(x) - \bar{u}) G_0(x,y) (u(y) - \bar{u}) dx\, dy$$

$0 < \varepsilon \ll 1$ is the dimensionless interfacial thickness

of special physical interest is the *large domain* case

Canonical model

Ginzburg-Landau energy + squared negative Sobolev norm:

$$\begin{aligned}\mathcal{E}[u] &= \int_{\mathbb{T}_\ell^d} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx \\ &+ \frac{1}{2} \int_{\mathbb{T}_\ell^d} \int_{\mathbb{R}^d} \frac{(u(x) - \bar{u})(u(y) - \bar{u})}{|x - y|^\alpha} dx\, dy\end{aligned}$$

here:

$$\boxed{u \in H^1(\mathbb{T}_\ell^d) \qquad \mathbb{T}_\ell^d = [0, \ell)^d \qquad 0 < \alpha < d}$$

need "neutrality" condition: $\frac{1}{\ell^d} \int_{\mathbb{T}_\ell^d} u\, dx = \bar{u}$

Canonical model (cont.)

Ginzburg-Landau energy + squared negative Sobolev norm:

$$\begin{aligned} \mathcal{E}[u] &= \int_{\mathbb{T}_\ell^d} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx \\ &+ \frac{1}{2} \int_{\mathbb{T}_\ell^d} \int_{\mathbb{R}^d} \frac{(u(x) - \bar{u})(u(y) - \bar{u})}{|x - y|^\alpha} dx\, dy \end{aligned}$$

<u>physical cases:</u>

non-locality of Coulombic origin

α = 1, d = 3 - ceramic compounds, various polymer systems, etc.

α = 1, d = 2 - magnetic bubble materials, high-T_c sueperconductors, etc.

α = “0”, d = 2 - ordering during surface deposition, etc.

α = “3”, d = 2 - ultra-thin ferromagnetic films

Canonical model (cont.)

Alternative rescaling: $\boxed{\ell \gg 1}$

$$\begin{aligned}\mathcal{E}[u] &= \int_{\mathbb{T}_\ell^d} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx \\ &+ \frac{\varepsilon^{d-\alpha}}{2} \int_{\mathbb{T}_\ell^d} \int_{\mathbb{R}^d} \frac{(u(x) - \bar{u})(u(y) - \bar{u})}{|x-y|^\alpha} dx\, dy\end{aligned}$$

$\Rightarrow \varepsilon$ is the relative strength of long-range forces

need $\varepsilon \lesssim 1$: if $\varepsilon \gg 1$, then the functional is convex

bifurcation at $\varepsilon = \varepsilon_c = O(1)$

far from bifurcation $\Rightarrow$ $\boxed{\varepsilon \ll 1}$

Long-range Coulomb repulsion

u - charge density on a torus in $\mathbb{R}^3$ or $\mathbb{R}^2$

G_0 - Green's function of the Laplace's equation

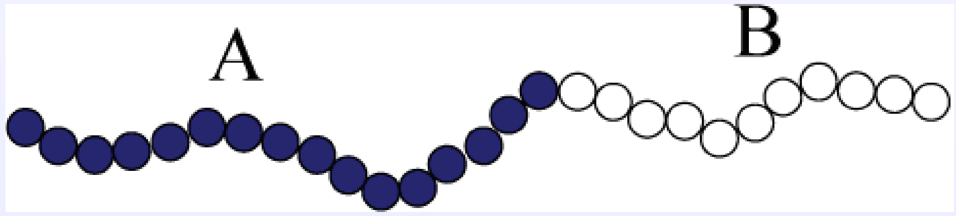
$$-\Delta G_0(x,y) = \delta(x-y) - \frac{1}{\ell^d}, \qquad \int_{\mathbb{T}_\ell^d} G_0(x,y)\,dx = 0$$

charge neutrality condition: $\displaystyle \frac{1}{\ell^d}\int_{\mathbb{T}_\ell^d} u\,dx = \bar{u}$

Ohta-Kawasaki model (diblock copolymers)

Ohta-Kawasaki energy

diblock-copolymer melts



$$E \propto \int \left(\frac{1}{2} |\nabla \phi|^2 - \frac{\xi^{-2}}{2} \phi^2 + \frac{g}{4} \phi^4 \right) d^3\mathbf{r}$$
$$+ \frac{\alpha}{2} \iint \frac{(\phi(\mathbf{r}) - \bar{\phi})(\phi(\mathbf{r}') - \bar{\phi})}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}\, d^3\mathbf{r}'$$

$$\alpha = \frac{12}{N^2 f(1-f)}$$

qualitative model for mesophases under strong segretation

Long-range forces of *entropic* origin

(Leibler'80; Stillinger'83; Ohta, Kawasaki'86; Choksi, Ren'03)

Block copolymer morphologies

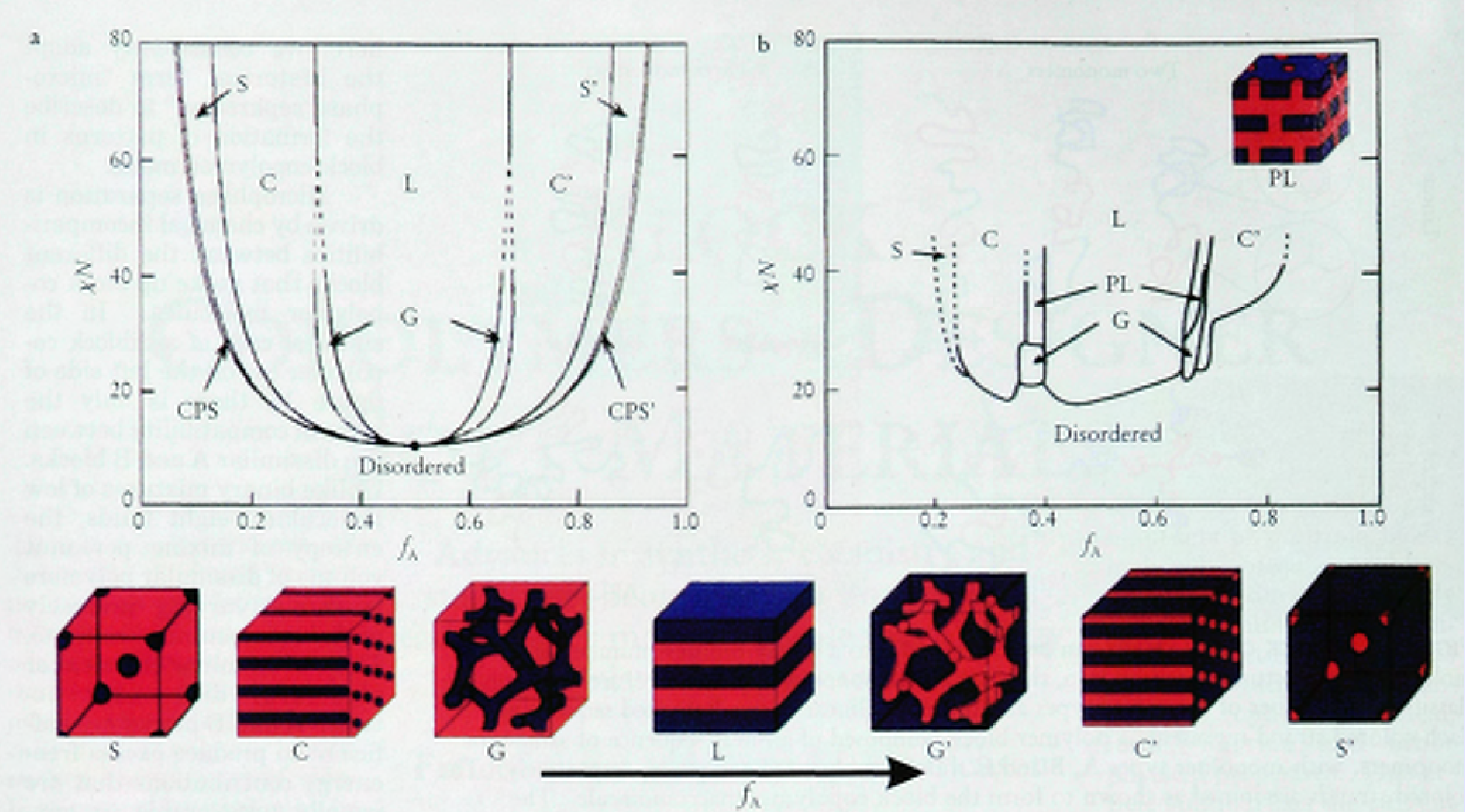


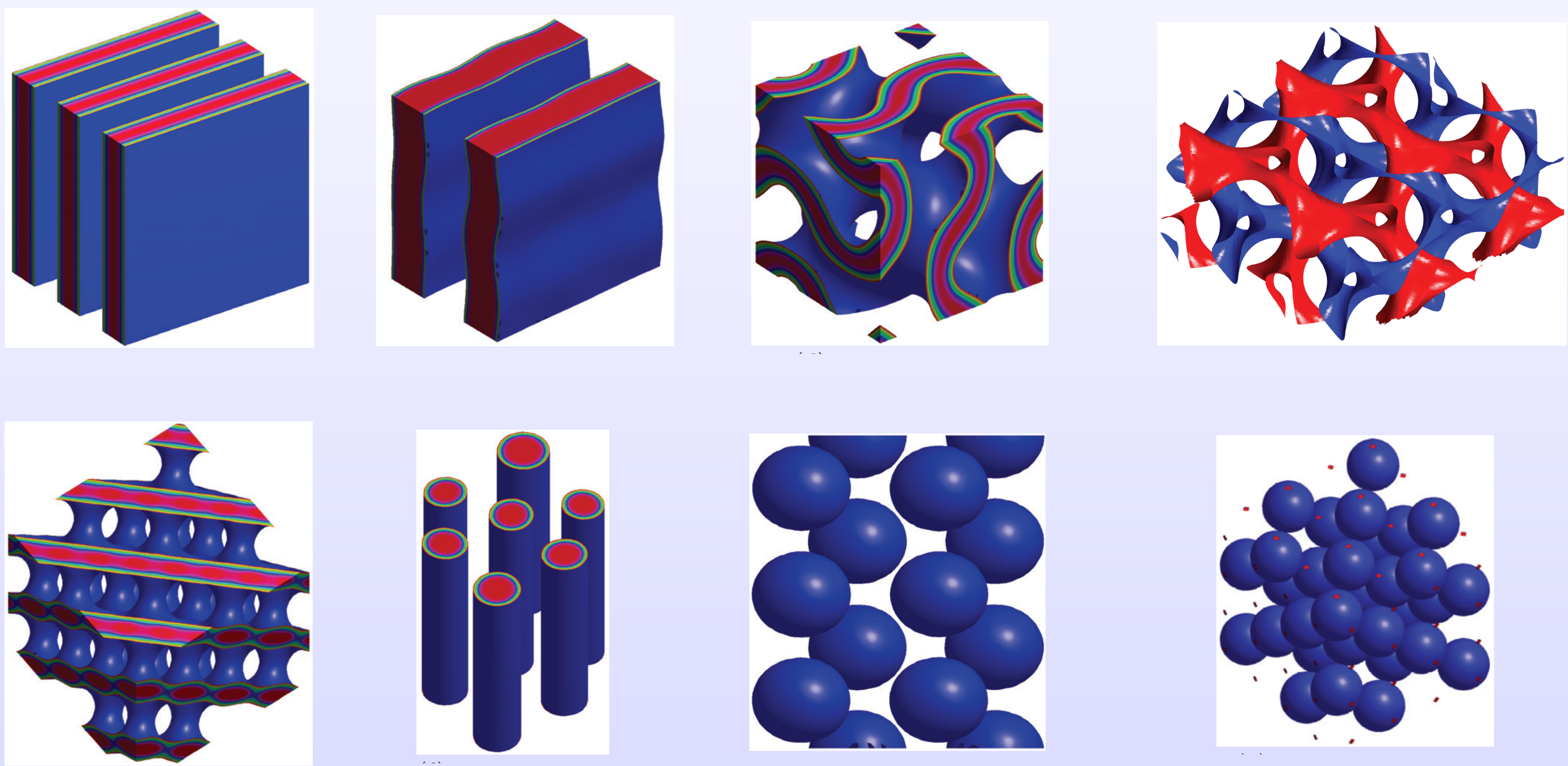
FIGURE 3. PHASE DIAGRAM for linear AB diblock copolymers, comparing theory and experiment. **a:** Self-consistent mean-field theory[8] predicts four equilibrium morphologies: spherical (S), cylindrical (C), gyroid (G) and lamellar (L), depending on the composition f and combination parameter χN. Here, χ is the segment-segment interaction energy (proportional to the heat of mixing A and B segments) and N is the degree of polymerization (number of monomers of all types per macromolecule). **b:** Experimental phase portrait for poly(isoprene-styrene) diblock copolymers.[9] The resemblance to the theoretical diagram is remarkable, though there are important differences, as discussed in the text. One difference is the observed PL phase, which is actually metastable. Shown at the bottom of the figure is a representation of the equilibrium microdomain structures as f_A is increased for fixed χN, with type A and B monomers confined to blue and red regions, respectively.

M.W. Matsen, M. Schick, Phys. Rev. Lett. (1994)

A. K. Khandpur et al., Macromolecules (1995)

Ohta-Kawasaki model

many local minimizers:



(Choksi, Peletier and Williams'09)

Sharp interface energy

reduced energy $\int_{-1}^{1} \sqrt{2W(u)}\, du = 1.$

$$E[u] = \frac{\varepsilon}{2} \int_{\mathbb{T}_\ell^d} |\nabla u| \, dx + \frac{1}{2} \int_{\mathbb{T}_\ell^d} \int_{\mathbb{T}_\ell^d} (u(x) - \bar{u}) G(x-y) (u(y) - \bar{u}) \, dx \, dy$$

where $u \in BV(\Omega; \{-1, 1\})$ and (M'98; M'02)

$$-\Delta G(x) + \kappa^2 G(x) = \delta(x) \qquad \kappa = \frac{1}{\sqrt{W''(1)}}$$

G is a *screened* Coulomb kernel, *no* neutrality constraint

<u>Theorem:</u> if $\bar{u} \in (-1, 1)$ and $d = 2$, then

$$\frac{\min \mathcal{E}}{\min E} \to 1 \qquad \text{as} \qquad \varepsilon \to 0$$ (M'10)

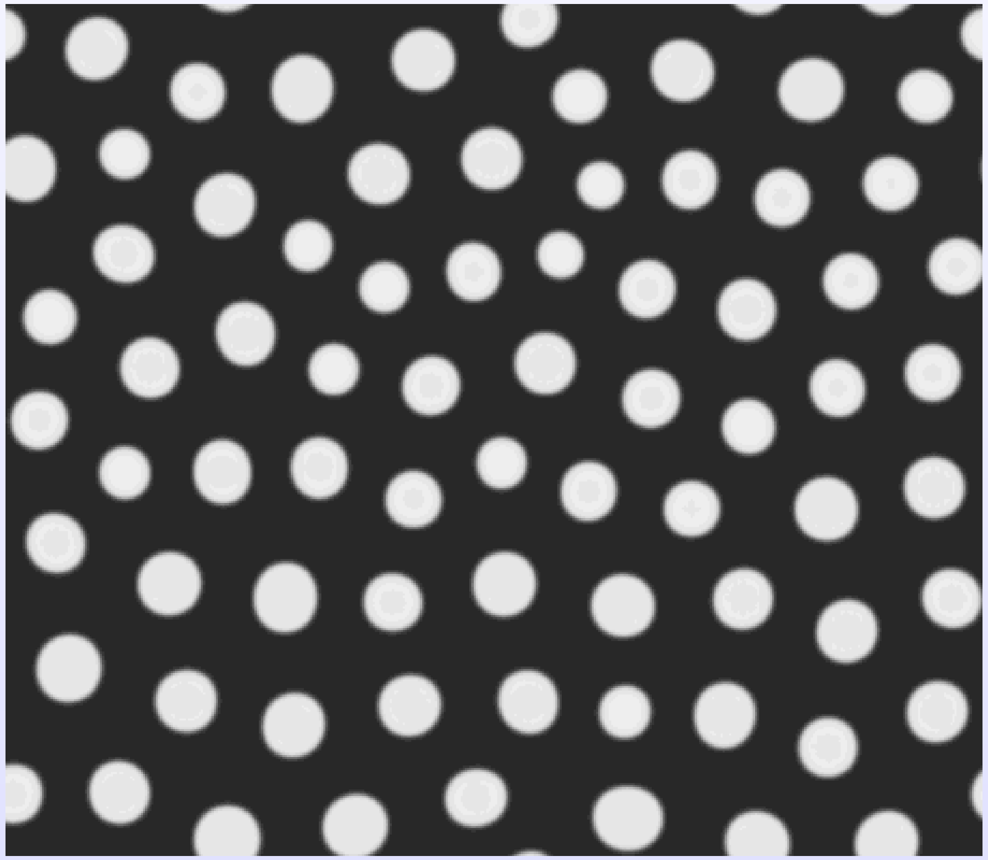
$\bar{u} \in (-1, 1), \quad \ell = O(1), \quad \varepsilon \ll 1 \quad \Rightarrow$ non-trivial minimizers

the rest of the talk is in two space dimensions

Non-trivial minimizers with high compositional asymmetry

- pattern with sharp interface
- identical disk-shaped droplets
- energy reduces to pair interactions (M'10):

$$V = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} G(x_i - x_j).$$



$\bar{u} = -0.5, \; \varepsilon = 0.025, \; W(u) = \frac{1}{4}(1-u^2)^2$
$\Omega = [0, 11.5) \times [0, 10)$

<u>note the similarity with Abrikosov vortices</u>

Is the minimizer a *hexagonal lattice?*

Energy of interacting droplets

$$G(x) = \frac{1}{2\pi} \sum_{\mathbf{n} \in \mathbb{Z}^2} K_0(\kappa|x - \mathbf{n}\ell|), \qquad G(x) = -\frac{1}{2\pi} \ln(\bar{\kappa}|x|) + O(|x|), \qquad |x| \ll 1$$

<u>macroscopic limit:</u> $\varepsilon \to 0, \quad \ell \gtrsim 1$

assume droplets are disks, then to leading order

$$\begin{aligned} E_N(\{r_i\}, \{x_i\}) &= \sum_{i=1}^{N} \Big(2\pi\varepsilon r_i - 2\pi(1+\bar{u})\kappa^{-2} r_i^2 - \pi r_i^4 (\ln \bar{\kappa} r_i - \tfrac{1}{4}) \Big) \\ &+ 4\pi^2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} G(x_i - x_j) r_i^2 r_j^2. \end{aligned}$$

balancing terms:

$$\min E_N = O(\varepsilon^{4/3} |\ln \varepsilon|^{2/3})$$

$$r_i = O(\varepsilon^{1/3} |\ln \varepsilon|^{-1/3}) \qquad N = O(|\ln \varepsilon|) \qquad 1 + \bar{u} = O(\varepsilon^{2/3} |\ln \varepsilon|^{1/3})$$

the number of droplets diverges!

What is the limit behavior of the minimizers?

can be analyzed via the Euler-Lagrange equation, etc. (M'10)

Theorem. *Let* $W = \frac{9}{32}(1-u^2)^2$, *let* $\bar{u} = -1 + \varepsilon^{2/3}|\ln\varepsilon|^{1/3}\bar{\delta}$, *with some* $\bar{\delta} > 0$ *fixed, and let* $\kappa = \frac{2}{3}$. *Then*

(i) If $\bar{\delta} \leq \frac{1}{2}\sqrt[3]{9}\,\kappa^2$, *then* $\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min\mathcal{E} \to \frac{1}{2}\kappa^{-2}\bar{\delta}^2$,

(ii) If $\bar{\delta} > \frac{1}{2}\sqrt[3]{9}\,\kappa^2$, *then* $\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min\mathcal{E} \to \frac{\sqrt[3]{9}}{2}\left(\bar{\delta} - \frac{\sqrt[3]{9}}{4}\,\kappa^2\right)$,

as $\varepsilon \to 0$.

natural approach via Γ-convergence (an easier case is $\ell \sim \varepsilon^{1/3}$)

difficulty: (Ren, Wei'03)

$$\varepsilon \ll \varepsilon^{1/3}|\ln\varepsilon|^{-1/3} \ll |\ln\varepsilon|^{-1/2} \ll 1$$

multiple scales!

(see also Alberti, Choksi and Otto'08; Spadaro'09; Ren and Wei'07; Choksi and Peletier'10 and '11)

Setting for Γ-convergence

study via the sharp interface energy

$$E^\varepsilon[u] = \frac{\ell^2(1+\bar{u}^\varepsilon)^2}{2\kappa^2} + \sum_i \left\{ \varepsilon|\partial\Omega_i^+| - 2\kappa^{-2}(1+\bar{u}^\varepsilon)|\Omega_i^+| \right\} + 2\sum_{i,j} \int_{\Omega_i^+}\int_{\Omega_j^+} G(x-y)\,dx\,dy,$$

where Ω_i^+ are connected components of $\Omega^+ := \{u = +1\}$

introduce droplet area and perimeter (suitably rescaled):

$$A_i := \varepsilon^{-2/3}|\ln\varepsilon|^{2/3}|\Omega_i^+|, \qquad P_i := \varepsilon^{-1/3}|\ln\varepsilon|^{1/3}|\partial\Omega_i^+|.$$

droplet density:

$$d\mu(x) := \varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}\sum_i \chi_{\Omega_i^+}(x)dx = \frac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u)\,dx.$$

Setting for Γ-convergence

The rescaled energy:

$$\bar{u}^\varepsilon := -1 + \varepsilon^{2/3}|\ln\varepsilon|^{1/3}\bar{\delta}.$$

$$E^\varepsilon[u] = \varepsilon^{4/3}|\ln\varepsilon|^{2/3}\left(\frac{\bar{\delta}^2\ell^2}{2\kappa^2} + \bar{E}^\varepsilon[u]\right), \quad \bar{E}^\varepsilon[u] := \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2}A_i^\varepsilon\right) + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y).$$

sequences of bounded energy $\bar{E}^\varepsilon$ have:

$$\frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon = \int_{\mathbb{T}_\ell^2} d\mu^\varepsilon$$

$$\limsup_{\varepsilon\to 0}\frac{1}{|\ln\varepsilon|}\sum_i P_i^\varepsilon < +\infty, \qquad \limsup_{\varepsilon\to 0}\frac{1}{|\ln\varepsilon|}\sum_i A_i^\varepsilon < +\infty,$$

since:

$$\bar{E}^\varepsilon[u] \geq -\frac{2\delta}{\kappa^2}\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu^\varepsilon(x)d\mu^\varepsilon(y)$$

$$\geq -\frac{2\bar{\delta}}{\kappa^2}\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon + \frac{2}{\kappa^2\ell^2}\left(\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon\right)^2,$$

compactness w.r.t. convergence of measures

Sharp interface energy

a suitable notion of convergence is, therefore, in terms of weak convergence of measures

Main result:

Theorem. **(Γ-convergence of E^ε)** *Fix $\bar\delta > 0$, $\kappa > 0$ and $\ell > 0$, and let E^ε and $\bar u_\varepsilon$ be as before. Then, as $\varepsilon \to 0$ we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}E^\varepsilon \xrightarrow{\Gamma} E^0[\mu] := \frac{\bar\delta^2\ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar\delta}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)\,d\mu(x)\,d\mu(y),$$

where $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$.

Corollary. *For given $\bar\delta > 0$, $\kappa > 0$ and $\ell > 0$, let $(u^\varepsilon) \in BV(\{-1,+1\})$ be minimizers of E^ε. Then, as $\varepsilon \to 0$ we have*

$$\mu^\varepsilon \rightharpoonup \begin{cases} 0 \\ \frac{1}{2}(\bar\delta - \bar\delta_c) \end{cases} \quad \text{in } (C(\mathbb{T}_\ell^2))^*, \qquad \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min E^\varepsilon \to \begin{cases} \frac{\bar\delta^2}{2\kappa^2} \\ \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c), \end{cases}$$

when $\bar\delta \le \bar\delta_c$ or $\bar\delta > \bar\delta_c$, respectively, with $\bar\delta_c := \frac{1}{2}3^{2/3}\kappa^2$. (M'10)

Sharp interface energy

characterization of almost minimizers:

Theorem. *Let* $(u^\varepsilon) \in \mathcal{A}$ *be a sequence of almost minimizers of* E^ε *with prescribed limit density* μ. *For every* $\gamma \in (0,1)$ *define the set* $I^\varepsilon_\gamma := \{i \in \mathbb{N} : 3^{2/3}\pi\gamma \le A^\varepsilon_i \le 3^{2/3}\pi\gamma^{-1}\}$. *Then*

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_i \left(P^\varepsilon_i - \sqrt{4\pi A^\varepsilon_i} \right) = 0,$$

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_{i \in I^\varepsilon_\gamma} \left(A^\varepsilon_i - 3^{2/3}\pi \right)^2 = 0,$$

$$\lim_{\varepsilon \to 0} \frac{1}{|\ln \varepsilon|} \sum_{i \notin I^\varepsilon_\gamma} A^\varepsilon_i = 0.$$

$\Rightarrow$ most droplets are nearly circular of radius $r = 3^{1/3}\varepsilon^{1/3}|\ln \varepsilon|^{-1/3}$.

in the limit the charge separates into droplets *equally*

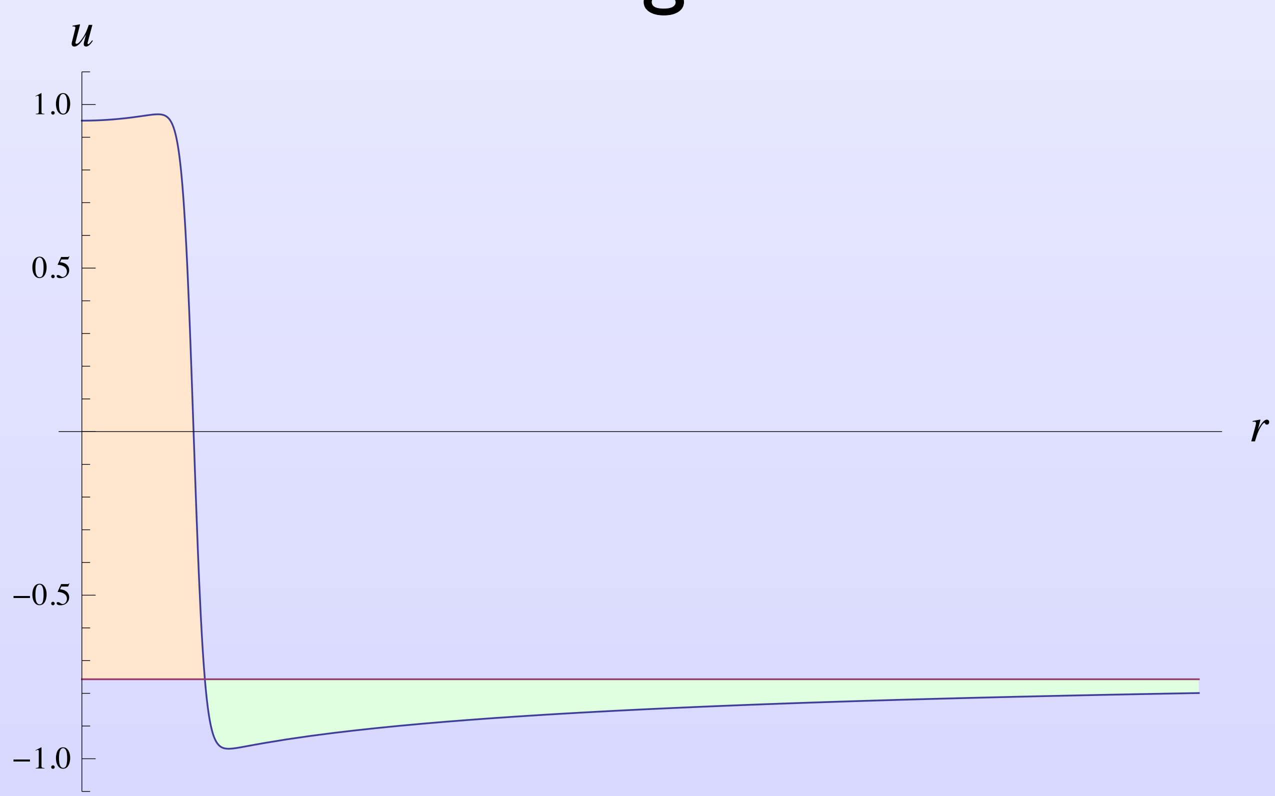
Diffuse interface energy

sharp interface results *cannot* be applied directly:

$$\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon$$ is not fixed on the sharp interface level, *but*

$$\int_{\mathbb{T}_\ell^2} d\mu^\varepsilon = \frac{1}{2}\bar{\delta}\ell^2$$ on the diffuse interface level

intimately related to screening:



need to filter out the screening charges

Diffuse interface energy

introduce:

$$u_0^\varepsilon(x) := \begin{cases} +1, & u^\varepsilon(x) > 0, \\ -1, & u^\varepsilon(x) \le 0, \end{cases} \qquad d\mu_0^\varepsilon := \tfrac{1}{2}\varepsilon^{-2/3}|\ln\varepsilon|^{-1/3}(1+u_0^\varepsilon(x))dx.$$

Main result:

Theorem. (Γ-convergence of $\mathcal{E}^\varepsilon$) *Fix $\bar\delta > 0$ and $\ell > 0$, and let $W(u) = \frac{9}{32}(1-u^2)^2$. Then, as $\varepsilon \to 0$ we have that*

$$\varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\mathcal{E}^\varepsilon \overset{\Gamma}{\to} E^0[\mu] := \frac{\bar\delta^2\ell^2}{2\kappa^2} + \left(3^{2/3} - \frac{2\bar\delta}{\kappa^2}\right)\int_{\mathbb{T}_\ell^2} d\mu + 2\int_{\mathbb{T}_\ell^2}\int_{\mathbb{T}_\ell^2} G(x-y)d\mu(x)d\mu(y),$$

where $\mu \in \mathcal{M}(\mathbb{T}_\ell^2) \cap H^{-1}(\mathbb{T}_\ell^2)$ and $\kappa = \frac{2}{3}$.

Corollary (for almost minimizers):

$$\mu_0^\varepsilon \rightharpoonup \begin{cases} 0 \\ \frac{1}{2}(\bar\delta - \bar\delta_c) \end{cases} \quad in\ (C(\mathbb{T}_\ell^2))^*, \quad \varepsilon^{-4/3}|\ln\varepsilon|^{-2/3}\ell^{-2}\min\mathcal{E}^\varepsilon \to \begin{cases} \frac{\bar\delta^2}{2\kappa^2}, \\ \frac{\bar\delta_c}{2\kappa^2}(2\bar\delta - \bar\delta_c), \end{cases}$$

when $\bar\delta \le \bar\delta_c$ or $\bar\delta > \bar\delta_c$, respectively, with $\bar\delta_c := \frac{1}{2}3^{2/3}\kappa^2$ and $\kappa = \frac{2}{3}$.

Key points of proofs

rescaled interfacial energy:

$$\begin{aligned}\bar{E} &= |\ln\varepsilon|^{-1}\big(|\partial\bar{\Omega}^+| - 2\bar{\delta}\kappa^{-2}|\bar{\Omega}^+|\big)\\ &+ 2|\ln\varepsilon|^{-2}\int_{\bar{\Omega}^+}\int_{\bar{\Omega}^+} G\big(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}(\bar{x}-\bar{y})\big)d\bar{x}\,d\bar{y}.\end{aligned}$$

a priori estimates:

$$\begin{aligned}|\bar{\Omega}^+| &\leq C|\ln\varepsilon|\\ |\partial\bar{\Omega}^+| &\leq C|\ln\varepsilon|\end{aligned}$$

$$\mathrm{diam}(\bar{\Omega}_i^+) \leq C|\ln\varepsilon|$$

allows to expand the kernel

insensitive to shape!

$$\frac{1}{|\ln\varepsilon|}G(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}(\bar{x}-\bar{y})) = \frac{1}{6\pi} - \frac{\ln|\ln\varepsilon|}{6\pi|\ln\varepsilon|} - \frac{1}{2\pi|\ln\varepsilon|}\ln(\bar{\kappa}|\bar{x}-\bar{y}|) + o(\varepsilon^{1/3})$$

Key points of proofs (cont.)

lower bound = isoperimetric inequality + expansion of the kernel

$$\bar{E}^\varepsilon[u^\varepsilon] \geq I^\varepsilon_{\mathrm{def}} + \frac{1}{|\ln\varepsilon|}\sum_i \left(\sqrt{4\pi A_i^\varepsilon} - \left(\frac{2\bar{\delta}}{\kappa^2}+\delta\right)A_i^\varepsilon + \frac{1}{3\pi}|\tilde{A}_i^\varepsilon|^2\right) + 2\iint G_\rho(x-y)\,d\mu^\varepsilon(x)\,d\mu^\varepsilon(y).$$

where

$$\tilde{A}_i^\varepsilon := \begin{cases} A_i^\varepsilon, & \text{if } A_i^\varepsilon < 3^{2/3}\pi\gamma^{-1} \\ (3^{2/3}\pi\gamma^{-1})^{1/2}|A_i^\varepsilon|^{1/2} & \text{if } A_i^\varepsilon \geq 3^{2/3}\pi\gamma^{-1} \end{cases}$$

$$I^\varepsilon_{\mathrm{def}} := \frac{1}{|\ln\varepsilon|}\sum_i\left(P_i^\varepsilon - \sqrt{4\pi A_i^\varepsilon}\right)$$

$$f(x) := \frac{2\sqrt{\pi}}{\sqrt{x}} + \frac{1}{3\pi}x$$

optimization over droplet areas:

$$\begin{aligned}\sqrt{4\pi A_i^\varepsilon} + \frac{1}{3\pi}|\tilde{A}_i^\varepsilon|^2 - \left(\frac{2\bar{\delta}}{\kappa^2}+\delta\right)A_i^\varepsilon &= A_i^\varepsilon\left(\frac{2\sqrt{\pi}}{\sqrt{A_i^\varepsilon}} + \frac{1}{3\pi}A_i^\varepsilon - \frac{2\bar{\delta}}{\kappa^2} - \delta\right) \\ &= A_i^\varepsilon\left(f(A_i^\varepsilon) - \frac{2\bar{\delta}}{\kappa^2} - \delta\right) \\ &\geq \left(3^{2/3} - \frac{2\bar{\delta}}{\kappa^2} - \delta\right)A_i^\varepsilon + \frac{1}{2}A_i^\varepsilon f''\left(3^{2/3}\pi\gamma^{-1}\right)(A_i^\varepsilon - 3^{2/3}\pi)^2,\end{aligned}$$

pass to the limit $\varepsilon \to 0$, then $\rho \to 0$, then $\delta \to 0$.

Key points of proofs (cont.)

upper bound: use construction for the magnetic GL vortices

(Sandier and Serfaty'00)

approximate: $d\mu(x) = g(x)dx, \qquad c \le g \le C.$

place $N(\varepsilon) = \frac{1}{3^{2/3}} \frac{|\ln\varepsilon|}{\pi} \mu(\mathbb{T}_\ell^2) + o(|\ln\varepsilon|)$ droplets of optimal radius

$r = 3^{1/3} \varepsilon^{1/3} |\ln\varepsilon|^{-1/3}$ satisfying $d(\varepsilon) := \min |a_i - a_j| \ge \frac{C}{\sqrt{N(\varepsilon)}}$

into disjoint squares $\{K_i\}$ of side length $|\ln\varepsilon|^{-1/2} \ll \delta \ll 1.$

$$N_{K_i}(\varepsilon) = \left\lfloor \frac{1}{3^{2/3}} \frac{|\ln\varepsilon|}{\pi} \mu(K_i) \right\rfloor \qquad \text{dist } (a_i, \partial K_i) \ge \frac{C}{\sqrt{N(\varepsilon)}}, \qquad N(\varepsilon) := \sum_i N_{K_i}.$$

Open problems

back to:

$$\mathcal{E}[u] = \int_{\mathbb{T}_\ell^d} \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx + \frac{1}{2} \int_{\mathbb{T}_\ell^d} \int_{\mathbb{R}^d} \frac{(u(x) - \bar{u})(u(y) - \bar{u})}{|x - y|^\alpha} dx\, dy$$

main difficulty for $\alpha > 0$ is to *minimize*:

$$E[u] = \int_{\mathbb{R}^d} |\nabla u|\, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x)u(y)}{|x - y|^\alpha}\, dx\, dy, \qquad u \in BV(\mathbb{R}^d, \{0, 1\}) : \int_{\mathbb{R}^d} u\, dx = m.$$

isoperimetric problem with a competing non-local term

solutions *exist* and are balls for $m \ll 1$

solutions *fail to exist* for $m \gg 1$

(Knupfer and M'11)