On a higher order non-local equation arising in the modeling of hydraulic fractures

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Summary

- We consider the equation:

$$\partial_t u - \partial_x(u^3 \partial_x(-\partial_{xx})^{1/2}(u)) = 0 \qquad \text{for } x \in \Omega, \quad t > 0.$$

- Goal: Prove the existence of non-negative solutions (for non-negative initial data)

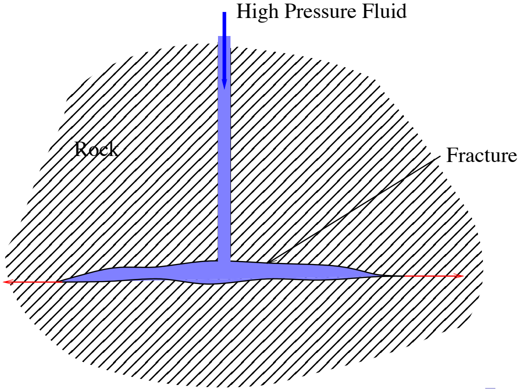
Outline

- Motivation (Hydraulic fracture) - Derivation of the equation
- Definition of the fractional laplacian on a bounded domain
- The porous media equation, the thin film equation and our equation
- Integral inequalities
- Existence results

Hydraulic Fracture

Hydraulic Fracturing: Propagation of fractures in a rock layer caused by the presence of a pressurized fluid

- Occur naturally (volcanic dikes causes by magma pressure)
- Fracking: Artificial injection of a highly-pressurized fluid to create new channels in the rock, to increase the extraction rate of oil and natural gas (shale gas)

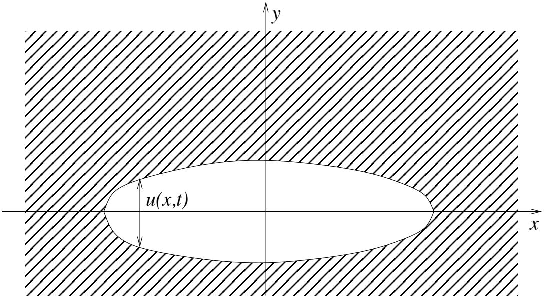


KGD model (Khristianovic, Geertsma and De Klerk)

Model developed by Khristianovic and Zheltov ('55) and Geertsma and De Klerk ('69).

- invariant with respect to z
- symmetric with respect to y

The fracture can then be entirely described by its opening $u(x,t)$ in the y direction:



Derivation: Lubrication approximation

- **Conservation of mass** for the fluid inside the fracture:

$$\partial_t(\rho u) + \partial_x(\rho u \overline{v}) = 0 \quad \text{in } \mathbb{R}$$

  where
  - ρ= density of the fluid (constant)
  - $\overline{v} = \frac{1}{u}\int_{-u/2}^{u/2} v_H(t,x,y)\,dy$ (averaged horizontal velocity of the fluid)
- **Lubrication approximation**: Navier-Stokes equations reduce to

$$\mu \frac{\partial^2 v_H}{\partial y^2}(t,x,y) = \partial_x p(x,t)$$

- Assuming a no-slip boundary condition $v = 0$ at $y = \pm u/2$, we deduce

$$\overline{v}(x,t) = -\frac{u^2}{12\mu}\partial_x p(x,t).$$

$$\boxed{\partial_t u - \partial_x\left(\frac{u^3}{12\mu}\partial_x p\right) = 0}$$

Pressure law (Ref.: A. Pierce)

Pressure $p(x,t)$ = Pressure exerted by the rock on the fluid.

Linear elasticity:

- The **stress tensor** components are denoted by $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx}$ and $\sigma_{yz} = \sigma_{zy}$.
- The **displacement** components are denoted by u_x, u_y and u_z.
- The **strain tensor** is related to the displacement components as follows:

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \; e_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \; e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

Equations for plane-strain

We assume that the solid is in a **state of plane-strain**:

$$u_z = 0, \quad u_x \text{ and } u_y \text{ independent of the } z \text{ coordinate}$$

- **Equilibrium conditions**:

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 \\ \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{xy}}{\partial x} = 0 \end{cases}$$

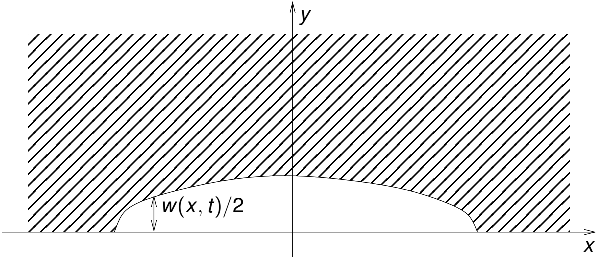
- **Stress-strain relations**

$$\begin{cases} e_{xx} = \frac{1}{2G}[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})] \\ e_{yy} = \frac{1}{2G}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})] \\ e_{xy} = \frac{1}{2G}\sigma_{xy} \end{cases}$$

where ν is Poisson's ratio and G is the shear modulus.

We get 5 equations with 5 unknowns: σ_{xx}, σ_{yy}, σ_{xy}, u_x, u_y.

Fracture in an infinite solid



$$\sigma_{xy}(x,0) = 0, \qquad u_y(x,0) = w(x)/2$$

- **Airy stress function**: There exists a bi-harmonic potential $U(x, y)$ such that
$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y}.$$
Equilibrium conditions are satisfied.
- Use **Fourier transform** with respect to x to solve the stress-strain relations.

Fracture in an infinite solid, cont.

Using the boundary conditions, we find

$$\widehat{U}(k,y) = A(k)(1+y|k|)e^{-|k|y} \quad \text{for all } k \in \mathbb{R},\ y>0.$$

with

$$A(k) = \frac{G}{2(1-\nu)}\frac{1}{|k|}\widehat{w}(k).$$

The **pressure** exerted by the rock in the y direction along $y=0$ is given by

$$\widehat{p}(k) := -\widehat{\sigma_{yy}}(k,0) = k^2\widehat{U}(k,0) = \frac{G}{2(1-\nu)}|k|\widehat{w}(k) \quad \text{for all } k \in \mathbb{R}.$$

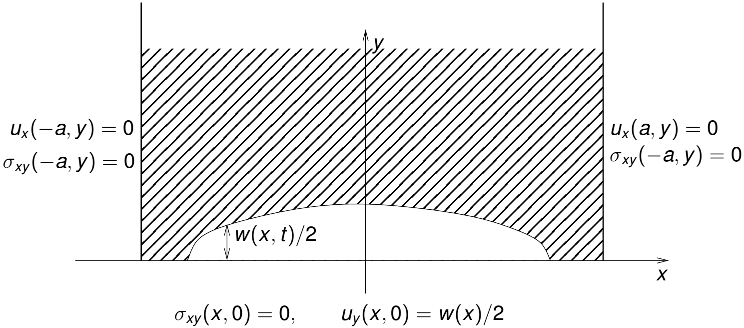
$$\boxed{p(x) = \frac{G}{2(1-\nu)}(-\Delta)^{1/2}w(x) \qquad \text{for } x \in \mathbb{R}.}$$

$$\boxed{\partial_t u - \partial_x(u^3\partial_x(-\partial_{xx})^{1/2}(u)) = 0}$$

Fracture in a constrained solid

The solid is constrained by rigid lubricated walls on both sides:

- the horizontal displacement is zero
- the wall does not transmit a shear stress to the solid



Remark: On the lateral boundary, we have $e_{xy} = 0$ and so

$$\frac{\partial u_y}{\partial x} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}.$$

Fracture in a contraint solid, cont.

We use Fourier sine series for u_x and Fourier cosine series for u_y:

Theorem

Assume $\Omega = (0, \pi)$. If

$$w(x) = \sum_{k\in\mathbb{N}} \widehat{w}(k) \cos(kx),$$

then the pressure $p(x)$ is given by

$$\boxed{p(x) = \frac{G}{2(1-\nu)} \sum_{k\in\mathbb{N}} k\widehat{w}(k) \cos(kx).}$$

We denote

$$p(x) = I(w)$$

where I is a **nonlocal elliptic operator of order** 1.

$$\boxed{\partial_t u - \partial_x(u^3 \partial_x I(u)) = 0}$$

Other definitions of the operator I

- **Spectral definition:** We denote by (λ_k, φ_k) eigenvalues/eigenfunctions of the Laplace operator with Neumann boundary conditions on Ω.

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \longmapsto \sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k$$

 (The same operator, with Dirichlet boundary condition, is studied by Cabré-Tan '10)

- **Dirichlet-to-Neumann map:** Let v be the following harmonic extension of w in $\Omega \times (0, \infty)$:

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \times (0,\infty), \\ v(x,0) = w(x) & \text{on } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{1}$$

Then

$$I(w) = -\partial_y v(\cdot, 0).$$

Other definition of the operator I

- **Singular integral definition:**

$$I(w)(x) = \int_\Omega [w(x) - w(y)]\, \nu(x,y) dy$$

where

$$\nu(x,y) = \frac{\pi}{2}\left(\frac{1}{1-\cos(\pi(x-y))} + \frac{1}{1-\cos(\pi(x+y))}\right)$$

(if $\Omega = (0,1)$).

The Hydraulic Fracture Equation

Conclusion:

- The opening $u(x,t)$ of the fracture solves a degenerate nonlocal parabolic equation of order 3:

$$\partial_t u - \partial_x(u^3 \partial_x I(u)) = 0 \qquad \text{for } x \in \Omega, \quad t > 0$$

where I is a nonlocal elliptic operator of order 1 (half laplacian with Neumann boundary conditions).

- Boundary conditions:

$$\partial_x u = 0, \quad u^3 \partial_x I(u) = 0 \qquad \text{for } x \in \partial\Omega, \quad t > 0$$

(the second condition ensures zero fluid loss)

- Initial condition:

$$u(x,0) = u_0(x) \quad \text{for } x \in \Omega.$$

References

- Derivation of the model
  - Khristianovic and Zheltov ('55)
  - Geertsma and de Klerk ('69)
- Numerical computations and formal asymptotic results
  - Spence and Sharp ('85)
  - Adachi and Detournay ('94)
  - Mitchel, Kuske and Peirce ('06,'07)
  - Adachi and Peirce ('08)
  - . . .

The porous media equation, the thin film equation and our equation

A (slightly) more general model

The lubrication approximation gave us

$$\partial_t u - \partial_x(u^3 \partial_x p) = 0$$

If we replace the no-slip condition for the fluid in contact with the rock by a Navier slip condition, we get a coefficient of the form

$$u^3 + \Lambda u \sim \Lambda u \qquad \text{for small } u$$

or

$$u^3 + \Lambda u^2 \sim \Lambda u^2 \qquad \text{for small } u$$

instead of u^3.

So it makes sense to consider the general equation

$$\boxed{\partial_t u - \partial_x(u^n \partial_x I(u)) = 0}$$

for $n \geq 1$.

Porous media and thin film equations

This equation belongs to the general class of equations of the form:

$$\partial_t u - \partial_x (u^n \partial_x I^\alpha(u)) = 0$$

where

$$I^\alpha(u) = (-\partial_{xx})^\alpha(u)$$

Then

- $\alpha < 0$, $n = 1$: Non-local porous media equation
- $\alpha = 0$, $n > 0$: Porous media equation
- $\alpha = 1/2$, $n \in [1, 3]$: Hydraulic fracture equation
- $\alpha = 1$, $n > 0$: Thin film equation

For $\alpha > 0$, there is no maximum principle: the existence of non-negative solutions is non-trivial, and uniqueness results are very difficult to obtain.

For $\alpha \in (0, 1)$, the equation is non-local.

Both the porous media and thin film equation have nice properties:

- Existence of a non-negative solution (for non-negative initial data)
- Optimal regularity result
- Finite speed of propagation of the support (compact support remains compact)
- Existence of source-type solutions
- Initial waiting time phenomenon

References for the thin film equation:

- Bernis-Friedman ('90)
- Bernis ('86,'96,'96,'96)
- Beretta-Bertsch-Dal Passo ('95)
- Bertozzi-Pugh ('94,'96)
- Bertozzi ('98)
- Grün ('95,'01,'01,'02,'03)
- ...

A free boundary problem

The derivation of the equation for $u(x,t)$ was only valid inside the fracture, so we should really write a **free boundary problem**:

$$\partial_t u - \partial_x(u^n \partial_x I(u)) = 0 \qquad \text{for } x \in \{u > 0\}, \quad t > 0$$

On $\partial\{u > 0\}$, we have

$$u = 0, \quad u^n \partial_x I(u) = 0$$

(zero width and no fluid loss at the tip of the fracture).

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(zero width and no fluid loss at the tip of the fracture).

We need an additional free boundary condition:

Spence and Sharp ('85) show that at the tip of the fracture, we should have

$$\boxed{u(x,t) \sim K(x_0 - x)_+^{1/2} \qquad \text{as } x \to x_0 \in \partial\{u > 0\}}$$

for some coefficient $K \geq 0$ related to the **rock toughness**.

Similar situation arises with the thin film equation. In that case, the free boundary condition is $u(x,t) \sim \alpha(x_0 - x)_+$ (**contact angle condition**).

Zero toughness model

Instead of considering this free boundary problem, we assume that the equation is satisfied throughout Ω.

- This is the standard approach for the porous media equation.
- This approach is classically used for the thin film equation.
- In both cases, the support propagates with finite speed (so a free boundary is implicitly defined).
- For the thin film equation, this leads to solutions satisfying a zero contact angle condition (**complete wetting**, or **precursor film** regime).

For our problem, we expect support of the solutions to propagate with finite speed and we should have

$$u(x,t) = o\left((x_0 - x)_+^{1/2}\right) \qquad \text{at the tip of the crack}$$

This is the zero toughness model ($K = 0$) or **pre-fractured** rock.

The exponent $n = 3$

In 1971, Huh and Scriven noted that for $n = 3$, the motion of the contact line leads to infinite dissipation of energy for the **thin film equation**:

The velocity of the fluid is given by

$$v = -u^2 \partial_x p$$

and the dissipation of energy is given by

$$D(u) = \int u^3 (\partial_{xxx} u)^2 \, dx = \int \frac{v^2}{u}$$

If $u \sim x_+$ and $v \neq 0$ at the free boundary this is infinite.

For the **Hydraulic fractures**, we have $u \sim x_+^{1/2}$, so the dissipation is finite even when $v \neq 0$.
In fact, a similar argument shows that $n = 4$ is the critical exponent for our equation.

Existence results

Our goal: Existence theorem for the zero toughness regime

Remark:

- No maximum principle
- Integral inequalities similar to the thin film equation
- Non-locality of the operator I complicates things considerably.

Integral inequalities (Lyapunov functionals)

Three important inequalities

- Conservation of mass

$$\int_\Omega u(t,x)\,dx = \int_\Omega u_0(x)\,dx$$

- Energy inequality

$$\frac{d}{dt}\int_\Omega u\, I(u)\,dx + \int_\Omega u^n(\partial_x I(u))^2\,dx = 0$$

- Entropy inequality

$$\frac{d}{dt}\int_\Omega G(u)\,dx + \int_\Omega u_x\, I(u)_x\,dx = 0$$

where G is a non-negative convex function satisfying $G''(s) = s^{-n}$.

Functional spaces

Recall that $I : \sum_{k=0}^{\infty} c_k \varphi_k \longmapsto \sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k$

We define

$$H^s_N(\Omega) = \left\{ u = \sum_{k=0}^{\infty} c_k \varphi_k \ ; \ \sum (1 + \lambda_k^s) c_k^2 < \infty \right\}$$

with the norm

$$\|u\|^2_{H^s_N} = \sum (1 + \lambda_k^s) c_k^2$$

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with the norm

$$\|u\|^2_{H^s_N} \quad = \quad \sum (1+\lambda_k^s) c_k^2$$

- If $0 \leq s < 3/2$, then $H^s_N(\Omega) = H^s(\Omega)$
- If $3/2 < s < 7/2$, then $H^s_N(\Omega) = \{u \in H^s(\Omega) \, ; \; u_\nu = 0 \text{ on } \partial\Omega\}$.
- if $s = 3/2$, then

$$H^{3/2}_N(\Omega) = \left\{ u \in H^{3/2}(\Omega) \, ; \; \int_\Omega \frac{u_x^2}{d(x)} \, dx < \infty \right\}$$

Integral inequalities (Lyapunov functionals)

With these norms, we can rewrite the integral inequalities as follows:

- Conservation of mass

$$\int_\Omega u(t,x)\,dx = \int_\Omega u_0(x)\,dx$$

- Energy inequality

$$\frac{d}{dt}\|u\|^2_{\dot{H}^{\frac{1}{2}}} + \int_\Omega u^n(\partial_x I(u))^2\,dx = 0$$

- Entropy inequality

$$\frac{d}{dt}\int_\Omega G(u)\,dx + \|u\|^2_{\dot{H}^{\frac{3}{2}}_N} = 0$$

Existence theorem

Theorem (Imbert-M. (2011))

Consider $u_0 \in H^{\frac{1}{2}}(\Omega)$ *such that* $\int_\Omega G(u_0)dx < +\infty$. *There exists a non-negative weak solution u.*

Moreover, for a.e. $t \in (0,T)$,

- *(Mass)* $\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx;$
- *(Energy)* $\|u(t)\|^2_{H^{\frac{1}{2}}(\Omega)} + 2\int_0^t \int_\Omega g^2(s,x)\,dx\,ds \leq \|u_0\|^2_{H^{\frac{1}{2}}(\Omega)}$
  where $g = \partial_x(u^{n/2} I(u)) - \frac{n}{2} u^{\frac{n}{2}-1}\partial_x u\; I(u) \in L^2$
- *(Entropy)* $\int_\Omega G(u(t))\,dx + \int_0^t \|u(s)\|^2_{\dot{H}_N^{\frac{3}{2}}(\Omega)}\,ds \leq \int_\Omega G(u_0)\,dx$

Idea of the proof

- **General strategy:** Regularize the diffusion coefficient $u^n \to f_\varepsilon(u)$, and prove the existence of a solution u^ε. Then pass to the limit $\varepsilon \to 0$.
- Energy inequality + mass conservation give a bound on u^ε in

$$H^{1/2}(\Omega) \subset L^p(\Omega) \quad \text{for all } p < \infty$$

and shows that the flux $h^\varepsilon = f_\varepsilon(u^\varepsilon) I(u^\varepsilon)_x$ is bounded in $L^2(0, T, L^{2-}(\Omega))$.

- Since $G'' = u^{-n}$, we have $G(s) = +\infty$ for $s < 0$ whenever $n \geq 1$, so the entropy inequality implies $\lim u^\varepsilon \geq 0$.
- However, for $n \geq 2$, $G(0) = +\infty$, so the entropy inequality requires positive initial data, which is a major limitation of the result.

Idea of the proof

We have a solution of

$$\partial_t u^\varepsilon - \partial_x h^\varepsilon = 0.$$

We have to show that

$$h^\varepsilon = f_\varepsilon(u^\varepsilon) I(u^\varepsilon)_x \longrightarrow u^n I(u)_x.$$

Idea:

- $h^\varepsilon \to 0$ wherever $\lim u^\varepsilon = 0$
- $I(u^\varepsilon)_x$ bounded in L^2 wherever $\lim u^\varepsilon \geq \delta > 0$.

Difficulty: to identify $\lim I(u^\varepsilon)_x$ in $\mathcal{D}'$, we need u continuous, which we do not have.

Remark: Our equation is critical in the sense that if $\alpha > 1/2$ (e.g. for the thin film equation), the energy inequality gives a bound in

$$H^\alpha(\Omega) \subset L^\infty(\Omega) \cap C^{0,\beta}(\Omega).$$

Instead, we use the entropy dissipation ($H_N^{3/2}$ bound)

Weak formuation

With two integrations by parts, we obtained the following weak formulation:

$$\iint_Q u\,\partial_t\varphi\,dx\,dt - \iint_Q nu^{n-1}\partial_x u\,I(u)\,\partial_x\varphi\,dx\,dt - \iint_Q u^n\,I(u)\,\partial_{xx}\varphi\,dx\,dt$$
$$= -\int_\Omega u_0\varphi(0,\cdot)\,dx$$

for all $\varphi \in \mathcal{D}(\overline{\Omega}\times[0,T))$ satisfying $\partial_x\varphi|_{\partial\Omega} = 0$.

This formulation makes sense for solutions in

$$L^\infty(0,T,H^{\frac{1}{2}}(\Omega)) \cap L^2(0,T,H_N^{\frac{3}{2}}(\Omega)).$$

Conclusion:

- We get the existence of non-negative weak solutions for all $n \geq 1$, but for $n \geq 2$, we cannot have compactly supported initial data.
- When $n \in [1, 2)$ (Lubrication approximation with certain Navier-Slip conditions), the result includes compactly supported initial data.

  We note that the condition $u \in L^2(0, T, H_N^{\frac{3}{2}}(\Omega))$ yield

$$u = o\left((x - x_0)_+^{1/2}\right) \quad \text{on } \partial\{u > 0\}$$

  (zero toughness condition)

Regularity and positivity when $n > 3$

We can improve the result when $n > 3$:

Theorem

If $n > 3$, then the solution u previously constructed satisfies

- $u(t,\cdot) > 0$ *in* Ω *for a.e.* t;
- $u(t,\cdot)$ *is* $C^{0,\alpha}$ *for all* $\alpha \in (0,1)$ *for a.e.* t;
- u *solves* $\partial_t u + \partial_x J = 0$ *with* $J = u^n \partial_x I(u) \in L^1(\Omega)$.

Further regularity (for $n \in [2,3]$)?

In order to get stronger existence results when $n \in [2,3]$, we need further regularity results.

- The energy dissipation gives

$$\int u^n [I(u)_x]^2 \, dx \, dt < \infty.$$

In the case of the thin film equation, we have some beautiful integral inequalities (due to Bernis '96):

$$\int ((u^{\frac{n+2}{2}})_{xxx})^2 \, dx \leq C \int u^n [u_{xxx}]^2 \, dx \quad \text{for } n \in (\tfrac{1}{2}, 3).$$

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$$\int ((u^{\frac{n+2}{2}})_{xxx})^2 \, dx \le C \int u^n [u_{xxx}]^2 \, dx \quad \text{for } n \in (\tfrac{1}{2}, 3).$$
- For the thin film equation, the so-called α-entropy yields further regularity (Bertozzi-Pugh 96): For $\alpha \in (\max(-1, \frac{1}{2} - n), 2 - n)$, $\alpha \neq 0$, the solutions of the thin film equation satisfy:
$$\int u^{\alpha+1}(\cdot, t) \, dx + C \int_0^t \int \left(|\partial_x u^{\frac{\alpha+n+1}{4}}|^4 + |\partial_{xx} u^{\frac{\alpha+n+1}{2}}|^2 \right) dx \, dt \le \int u_0^{\alpha+1} \, dx.$$

Conclusions

- **Proved:**
  - Existence of non-negative solution for general initial data when $n \in [1, 2)$.
  - Existence of non-negative solution for positive initial data when $n \geq 2$.
- **To be proved:**
  - Existence of solution for compactly supported initial data when $n \geq 2$
  - L^∞ bound and continuity of the solution
  - Finite speed of propagation of the support
  - Optimal regularity of the solution
  - Existence of solution for the free boundary problem with non-zero toughness