Lipschitz regularity of solutions of nonlinear elliptic integro-differential equations

joint work with Barles, Chasseigne, Ciomaga

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Nonlocal PDEs, Variational Problems and their Applications

Equations with composed/mixed ellipticity

A "composed" elliptic integro-differential equation

$$(1)\quad \begin{cases} \Lambda_1(x)(-\Delta)u + (1-\Lambda_1(x))(-\Delta)^{\frac{\beta}{2}}u + f(x) = 0 \quad \text{in } \mathbb{R}^d \\ (-\Delta)^{\frac{\beta}{2}} = \text{fractional Laplacian} \\ 0 \le \Lambda_1(x) \le 1, \text{Hölder continuous} \end{cases}$$

A "mixed" elliptic integro-differential equation

$$(2)\quad \begin{cases} (-\Delta_{x_1})u + (-\Delta_{x_2})^{\frac{\beta}{2}}u = f(x_1, x_2) \quad \text{in } \mathbb{R}^d \\ (-\Delta_{x_2})^{\frac{\beta}{2}} = \text{partial fractional Laplacian} \end{cases}$$

Question: are solutions Hölder/Lipschitz continuous in $\mathbb{R}^d$?

▸ Main results ▸ Composed and Mixed

Equations with composed/mixed ellipticity

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Question: are solutions Hölder/Lipschitz continuous in $\mathbb{R}^d$?

Generalization to non-linear versions of these standing examples.

▸ Main results ▸ Composed and Mixed

Outline of the talk

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1. Motivations
2. Main results
3. The Ishii-Lions method

Fractional Laplacian

Fractional Laplacian ($\beta \in (0,2)$)

- Fourier multiplier: $(-\Delta)^{\frac{\beta}{2}} u = \mathcal{F}^{-1}(|\xi|^{\beta} \mathcal{F} u)$
- Singular integral: $(-\Delta)^{\frac{\beta}{2}} u = -c \int [u(x+z) - u(x)] \frac{dz}{|z|^{d+\beta}}$
- "Dirichlet-to-Neumann" formula

Properties

- Regularizing effect
- Positive max principle: $u(x) = \max u \Rightarrow (-\Delta)^{\frac{\beta}{2}} u(x) \geq 0.$

“Nice” singular integral operators

Lévy measure μ

$$\int \min(1, |z|^2)\mu(dz) < +\infty$$

Lévy operators

$$L[u](x) = -\int [u(x+z) - u(x)]\mu_x(dz)$$

with $\forall x, \mu_x =$ Lévy measure

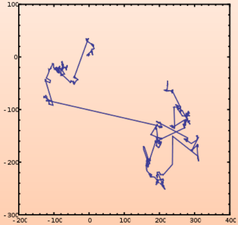
Lévy-Itô operators

$$L_{\text{LI}}[u](x) = -\int [u(x + j(x,z)) - u(x)]\mu(dz)$$

with $\mu =$ Lévy measure

Lévy processes

Stochastic processes with stationary and independent increments



$X_t = \text{drift} + \text{diffusion} + \text{jumps}$

Infinitesimal generators

$$Lu = \underbrace{b \cdot Du}_{\text{drift}} + \underbrace{AD^2u}_{\text{diffusion}} + \underbrace{L[u]}_{\text{jumps}}$$

Stochastic control

Control of a SDE

$$dX_t = \underbrace{b(X(t), \gamma(t))dt}_{\text{drift}} + \underbrace{\sigma(X(t), \gamma(t))dW_t}_{\text{diffusion}} + \int \underbrace{j(X(t), \gamma(t), z)\tilde{N}(dt, dz)}_{\text{jumps}}$$

Cost functional

$$J(x, \gamma(\cdot)) = \mathbb{E}\left[\int_0^\infty e^{-\lambda t} f(X(t), \gamma(t))\, dt\right].$$

Value function

$$u(x) = \inf_{\gamma(\cdot)} J(x, \gamma(\cdot)).$$

Bellman equation

General form

$$\sup_{\gamma} \left\{ L_{\text{LI}}^{\gamma}[u] - \text{Tr}(A_{\gamma} D^2 u) - b_{\gamma} \cdot Du - f_{\gamma} \right\} + \lambda u = 0$$

with $A_{\gamma} = \frac{1}{2}\sigma_{\gamma}\sigma_{\gamma}^{T}$

Example

$$(-\Delta)^{\frac{\beta}{2}} u + b(x)|Du|^{k+\tau} + |Du|^{r} + \lambda u = f(x)$$

Elliptic nonlinear PIDE

Partial Integro-Differential Equations (PIDE)

Wide range of applications

- Finance
- Dislocations (cf. Monneau's talk)
- Hydraulic fractures (cf. Mellet's talk)
- Combustion
- Fluid dynamics (cf. Kiselev's talk)
- Life sciences (cf. Bertozzzi's, Carillo's, Gonzalez's and Topaz's talks)
- Image (cf. Guidotti's and Osher's talks)
- Statistical Physics (cf. Lebowitz's talk)
- ...

Regularity for PIDE

Linear equations (probability)

- Bass, Kassmann, Levin, Song, Vondracek...

Viscosity solutions

- Sayah, Jakobsen, Karlsen, CI, Barles, Chasseigne, Ciomaga, CI...

Fully non-linear elliptic equations

- Caffarelli, Silvestre ...

Quasi-geostrophic equation

- Caffarelli, Vasseur, Kiselev, Nazarov, Volberg, Silvestre, Dabkowski, Lemarié-Rieusset ...

Fractional Burgers

- Biler, Funaki, Karch, Woyczynski, Alibaud, Droniou, Vovelle, CI, Kiselev, Nazarov, Chan, Czubak, Silvestre, Du, Dong, Li ...

"Under divergence form"

- Komatsu, Kassmann, Barlow, Bass, Chen, Caffarelli, Chan, Vasseur...

Outline of the talk

Main result

Recall equations ▸ Equations

Theorem

Assume Λ_1 is Hölder continuous. Solutions of Equations (1) and (2) are Hölder continuous if $\beta \leq 1$ and Lipschitz continuous if $\beta > 1$.

Remarks

- Non-linear equations (Bellman *etc*)
- Explicit Hölder exponent ($\forall \alpha < \beta \leq 1$)

- First order terms: "Ellipticity-Growth conditions"
- General Lévy measures
- General Lévy-Itô operators

Extension (I) : Lévy *Vs.* Lévy-Itô

x-dependent Lévy measures	x-dependent jumps
$-\int(u(x+z)-u(x))\mu_x(dz)$	$-\int(u(x+j(x,z))-u(x))\mu(dz)$
Comparison principle: ok?	Comparison principle: ok!

Hölder continuity of coefficients

$$\int_{B(0,\delta)} |z|^2|\mu_x-\mu_y|(dz) \leq C\delta^{2-\beta}|x-y|^\gamma$$

$$\int_{\mathbb{R}^d\setminus B(0,\delta)} |z||\mu_x-\mu_y|(dz) \leq C\delta^{1-\beta}|x-y|^\gamma \qquad (\beta\neq 1)$$

Extension (II): Composed *Vs.* Mixed

Recall equations ▸ Equations

- Eq. (1): composed — Barles-Chasseigne-CI (JEMS, 2011)
- Eq. (2): mixed — Barles-Chasseigne-Ciomaga-CI (JDE)

General form of the equation

$$F_0(u, Du, D^2u, L_0[u]) + \sum_{i=1,2} F_i(x_i, D_{x_i}u, D^2_{x_i}u, L[x_i, u]) = 0$$

- F_0 is proper and Lipschitz w.r.t the last variable
- F_1 satisfies a Growth-Ellipticity condition
- F_2 satisfies a uniqueness-type condition

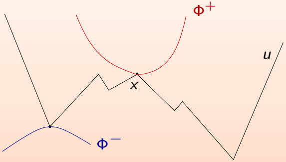
$\Rightarrow$ partial regularity w.r.t. x_1 variables

Viscosity solutions for PIDE

- Soner (1986)
- Sayah, Lenhart
- Arisawa, Pham
- Ishii, Koike
- Alvarez, Tourin, Karlsen, Jakobsen
- Barles, CI
- Barles, Chasseigne, Ciomaga
- ...

Definition of viscosity solutions

Test functions



$$D\Phi^+(x) = Du(x)$$
$$D^2\Phi^+(x) \geq D^2u(x)$$
$$\Phi^+(x+z) - \Phi^+(x) \geq u(x+z) - u(x)$$

Viscosity solutions

- (Subsolution) Φ^+ touches u from above at $x \Rightarrow F[\Phi^+](x) \leq 0$
- (Supersolution) Φ^- touches u from below $\Rightarrow F[\Phi^-](x) \geq 0$

Examples

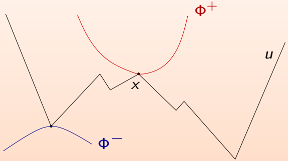
$$(-\Delta)^{\frac{\alpha}{2}}\Phi^+(x) - \mathrm{Tr}(A\ \ (x)D^2\Phi^+(x))$$

$$- b\ \ (x)\cdot D\Phi^+(x) - f\ \ (x) \quad + \lambda u(x) \leq 0$$

(Linear equation)

Definition of viscosity solutions

Test functions



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Examples

$$\inf_{\gamma_1} \sup_{\gamma_2} \Big\{ (-\Delta)^{\frac{\alpha}{2}} \Phi^+(x) - \mathrm{Tr}(A_\gamma(x) D^2\Phi^+(x)) - b_\gamma(x) \cdot D\Phi^+(x) - f_\gamma(x) \Big\} + \lambda u(x) \leq 0$$

(Bellman equation)

Definition of viscosity solutions

Non-local equations $\Phi^{\pm}$ should be *globally* above or below u

$$-\int [\Phi^+(x+z) - \Phi^+(x)]\mu(dz) \leq -\int [u(x+z) - u(x)]\mu(dz)$$

Equivalent definition split the integral

$$-\int_{B(0,r)} [\Phi^+(x+z) - \Phi^+(x)]\mu(dz) - \int_{\mathbb{R}^d \setminus B(0,r)} [u(x+z) - u(x)]\mu(dz) \leq -\int [u(x+z) - u(x)]\mu(dz)$$

References Sayah'91, CI'05, Barles-CI'08

Outline of the talk

The Ishii-Lions method: local case (I)

Linear equation

$$-\Delta u - b \cdot Du - f + \lambda u = 0$$

What we want

$$u(x) - u(y) \le L_1 |x - y|^\alpha$$

Localization

$$u(x) - u(y) \le L_1 |x - y|^\alpha + L_2 |x - x_0|^2$$

Proof by contradiction

Assume $\left| \begin{array}{l} M = \sup_{x,y} u(x) - u(y) - \Phi(x-y) - \Gamma(x) > 0 \\ \text{for all } \alpha \in (0,1),\ L_1 > 0,\ L_2 > 0 \end{array} \right.$

In particular $L_1 |x - y|^\alpha \le \|u\|_\infty + \|v\|_\infty$

The Ishii-Lions method: local case (II)

Linear equation $$-\Delta u \underbrace{-b \cdot Du - f + \lambda u}_{\text{lot}} = 0$$

Assume the solution u is smooth

- optimality condition: $\left| \begin{array}{l} Du(x) = D\Phi(x-y) + D\Gamma(x) \\ Du(y) = D\Phi(x-y) \end{array} \right.$
- Second order optimality condition:

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} Z & -Z \\ -Z & Z \end{pmatrix}$$

with $X = D^2u(x) - D^2\Gamma(x)$, $Y = D^2u(y)$, $Z = D^2\Phi(x-y)$.

Use the equation twice and combine them

$$\underbrace{O(L_2)}_{\text{lot}} \leq \text{Tr}(X) - \text{Tr}(Y)$$

The Ishii-Lions method: local case (III)

Recall

$\Phi(z) = L_1|z|^\alpha$ and $|x-y|^\alpha \leq \frac{1}{L_1}$

$Z = L_1 D^2|\cdot|^\alpha = L_1|\cdot|^{\alpha-2}(I - (2-\alpha)\widehat{x-y} \otimes \widehat{x-y})$

Use the matrix inequality

$\Rightarrow \mathrm{Tr}(X-Y) \leq -\frac{L_1(1-\alpha)}{|x-y|^{2-\alpha}}$

$\Rightarrow O(L_2) \leq -\frac{L_1(1-\alpha)}{|x-y|^{2-\alpha}}$

If u is not smooth, use viscosity solution techniques

Jensen-Ishii's lemma needed

The Ishii-Lions method: local case (IV)

Main idea in the previous proof

Use the concavity of $|\cdot|^{\alpha}$ to create a "large" negative term

For non-local case

Use the concavity "around" a given direction

Lipschitz regularity

Use $\Phi = L_1|\cdot| - \sigma|\cdot|^{1+\alpha}$.

The Ishii-Lions method: Non-local case (combined - I)

Optimality condition ($\Gamma \equiv 0$)

$$u(x+z) - u(y+z') - \Phi(x-y+z'-z) \leq u(x) - u(y) - \Phi(x-y)$$

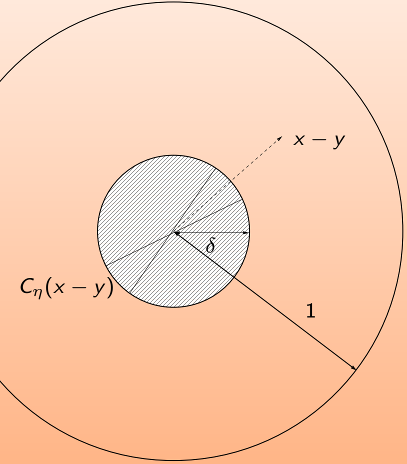
This implies

$$u(x+z) - u(x) \leq \Phi(x-y-z) - \Phi(x-y)$$
$$u(y) - u(y+z) \leq \Phi(x-y+z') - \Phi(x-y)$$

From smooth to viscosity solutions

Jensen-Ishii's lemma should be adapted
Joint work with G. Barles (Annales IHP, 2008)

The Ishii-Lions method: Non-local case (combined - II)



$$\int (\Phi(x-y+z) - \Phi(x-y)) \frac{dz}{|z|^{d+\alpha}}$$

$$\|$$

$$\int_{\mathbb{R}^d \setminus B} (\dots) \frac{dz}{|z|^{d+\alpha}}$$ [bounded]

$$+$$

$$\int_{B \setminus C} (\dots) \frac{dz}{|z|^{d+\alpha}}$$ [controlled!]

$$+$$

$$\int_{C} (\dots) \frac{dz}{|z|^{d+\alpha}}$$ [good!!]

The Ishii-Lions method: Non-local case (mixed)

Prove

$$u(x_1, x_2) - u(y_1, y_2) \leq L_1 |x_1 - y_1|^\alpha + \frac{|x_2 - y_2|^2}{2\varepsilon}.$$

Function Φ

$$\Phi(x_1, x_2) = L_1 |x_1|^\alpha + \frac{|x_2|^2}{2\varepsilon}.$$

Assumptions

- F_1 Growth-Ellipticity condition
- F_2 uniqueness-type condition

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