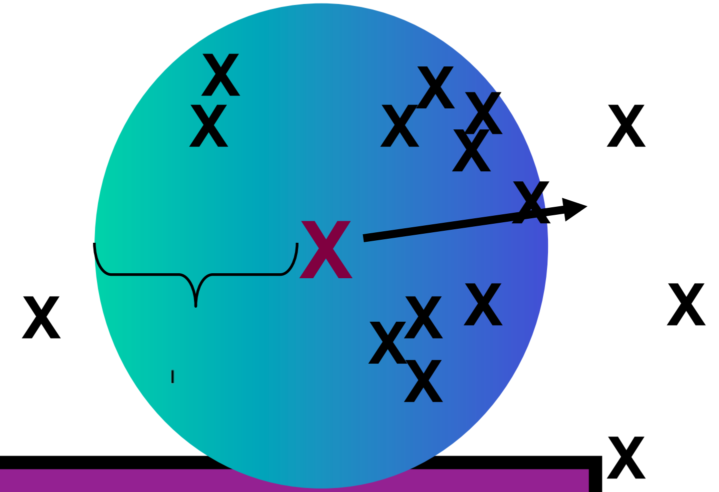
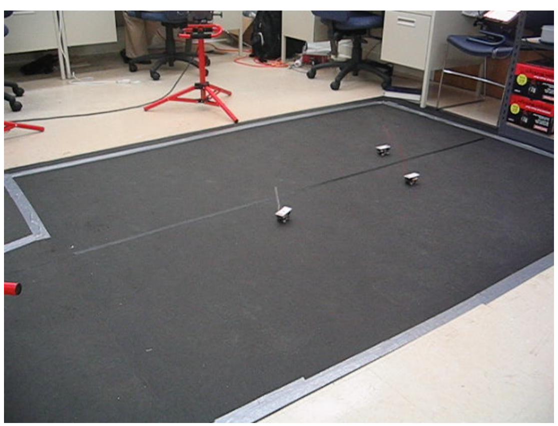
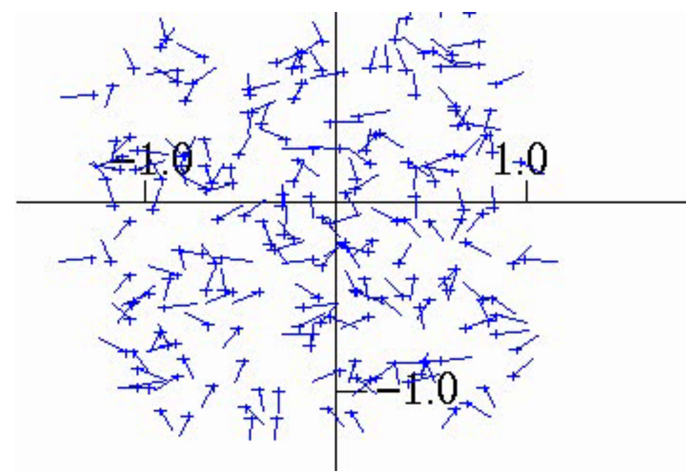
UCLA Nonlocal PDE workshop

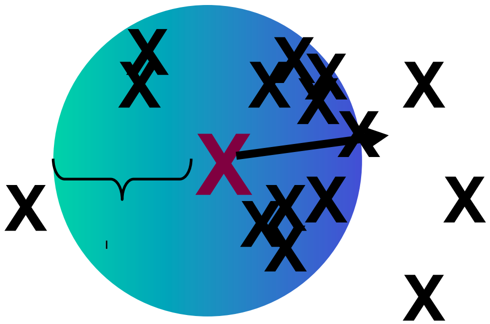


NONLOCAL AGGREGATION EQUATIONS and FLUID DYNAMICS

Andrea Bertozzi







The Problem

$$\rho_t + \nabla \cdot (\rho \nabla K * \rho) = 0$$

A density advected by a field that is the the gradient of K convolved with itself.

Active scalar problem in gradient flow format.

Analysis follows ideas from both *fluid mechanics* (active scalars – divergence free flow) and *optimal transport* (gradient flow).

Finite time singularities- general potentials

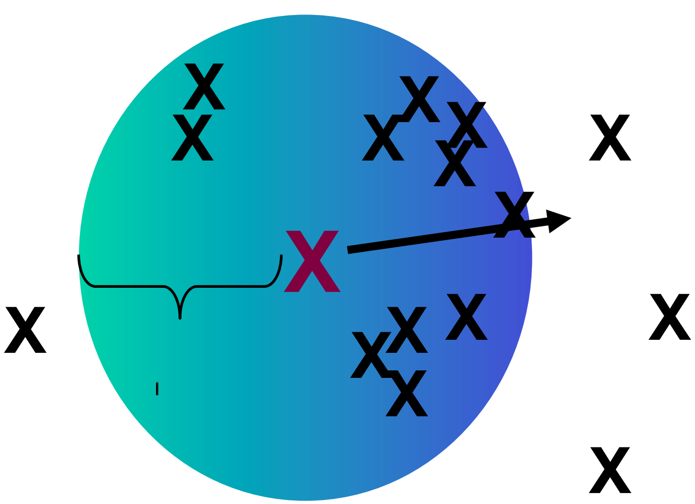
$$\rho_t + \nabla \cdot (\rho \nabla K * \rho) = 0$$

- <u>Previous Results</u>
- For smooth *K* the solution blows up in infinite time
- For `pointy' K (biological kernel such as $K=e^{-|x|}$) blows up in finite time for special radial data in any space dimension.

<u>New result</u>:
Osgood condition

$$\int_0^L \frac{1}{K'(r)} dr < \infty$$

is a necessary and sufficient condition for finite time blowup in any space dimension
(under mild monotonicity conditions).



Moreover-finite time blowup for pointy potential can not be described by `first kind' similarity solution in dimensions N=3,5,7,...

Osgood uniqueness criteria for ODEs

- dX/dt = F(X), system of first order ODEs
- Picard theorem – F is Lipschitz continuous
- Generalization of Picard is Osgood criteria(sharp) : w(x) is modulus of continuity of F, i.e.

  |F(X)-F(Y)|< w(|X-Y|)
- 1/w(z) is not integrable at the origin for unique solutions.
- Example: dx/dt = x |ln x| has unique solutions.
- Example: dx/dt = sqrt(x) does not have unique

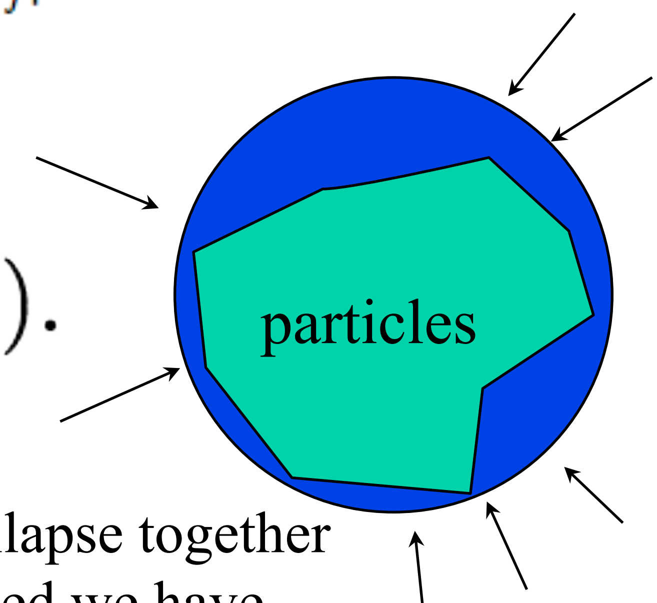
Collapse of particles in the ODE

Bertozzi, Carrillo, Laurent
Nonlinearity 2009

COMPARISON PRINCIPLE: Proof of finite time collapse for non-Osgood potentials – let $R(t)$ denote the particle farthest away from the center of mass (conserved), then

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\sum_{j\neq i} m_j \nabla K(x_i - x_j) = -\sum_{j\neq i} m_j \frac{x_i - x_j}{|x_i - x_j|} k'(|x_i - x_j|),$$

$$\dot{R} \leq -\frac{M}{2} K'(2R(t)).$$

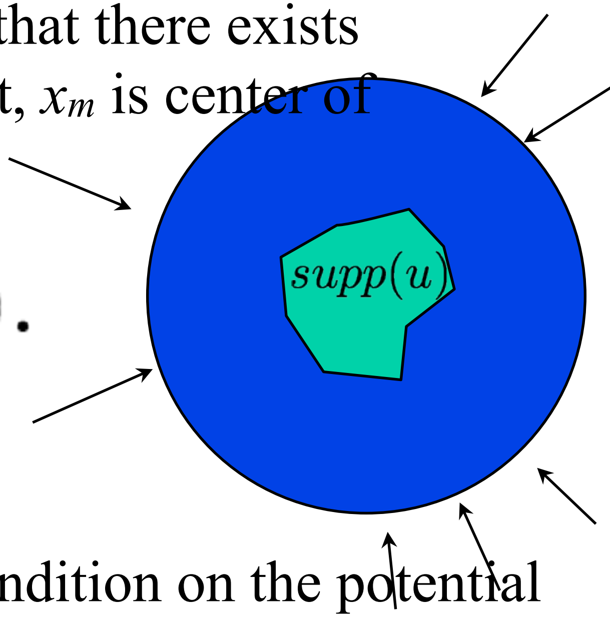


When the Osgood criteria is violated particles collapse together in finite time. When the Osgood criteria is satisfied we have global existence and uniqueness of a solution of the particle equations.

Finite time singularities- general potentials

COMPARISON PRINCIPLE: The proof for a finite number of particles extends naturally to the continuum limit. Proof of finite time blowup for non-Osgood potentials - assumes compact support of solution. One can prove that there exists an $R(t)$ such that $B_{R(t)}(x_m)$ contains the support, x_m is center of mass (conserved), and

$$\dot{R} \leq -\frac{M}{2} K'(2R(t)).$$



Thus the Osgood criteria provides a *sufficient* condition on the potential K for finite time blowup from bounded data. To prove the condition is also *necessary* we must do further potential theory estimates.

First an easier result - C^2 kernels

- A priori bound
- One can easily prove a Gronwall estimate
- L-infty of density controlled by L-infty of div v.
- But, div v = Laplacian of K convolved with the density, which we assume to be in L^1.
- If the kernel is C^2 we have an a priori bound.
- We need more refined potential theory estimates for general Osgood condition.

Proof of global existence:

Connection to 3D Euler

$$\begin{aligned} \rho_t + v \cdot \nabla \rho &= -(\nabla \cdot v)\rho \\ v &= \nabla K * \rho \end{aligned}$$

A priori bound for L^∞ norm follows similar approach as in BKM theorem (1984) for incompressible Euler.

$$\boxed{\begin{aligned} \omega_t + v \cdot \nabla \omega &= (\nabla v)\omega \\ v &= \vec{K}_3 * \omega \end{aligned}}$$

Vorticity Stream form of 3D Euler Equations
omega is vector vorticity and K_3 is Biot-Savart Kernel in 3D

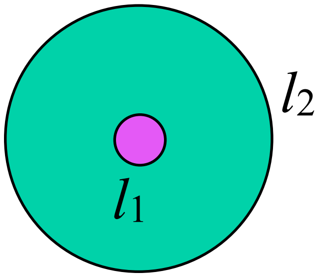
Finite time singularities- general potentials

BKM argument uses log-linear estimate for SIOs (see e.g. chapter 4 *Vorticity and Incompressible Flow*)

$$|\nabla v|_{L^\infty} \leq |\omega|_{L^\infty}[1 + ln(l_1/l_2)]$$

where

$$l_1 = (|\omega|_{L^\infty}/|\omega|_\gamma)^{1/\gamma} \quad l_2 = (|\omega|_{L^2}/|\omega|_{L^\infty})^{2/3}$$



Finite time singularities-general potentials

Similarly, the aggregation problem has the estimate

$$\partial_t |\rho|_{L^\infty} \leq |\rho|_{L^\infty} |\Delta K * \rho|_{L^\infty}$$

so the challenge is to show that $|\Delta K * \rho|_{L^\infty}$ is 'logarithmic' in ρ - in the sense of the Osgood criteria, in the case where ΔK is unbounded at the origin.

The key lengthscale in the problem is $\delta = (M/|\rho|_{L^\infty})^{1/N}$ where N is the space dimension and M is the mass,

$$M = \int \rho dx.$$

Finite time singularities-general potentials

One can show that $\delta = (M/|\rho|_{L^\infty})^{1/N}$ satisfies a differential inequality

$$\dot{\delta} \geq -C(N,M)K'(\delta)$$

which means that if K satisfies the Osgood condition, then δ is bounded away from zero for all time. This gives an a priori upper bound on $|\rho|_{L^\infty}$ since mass is conserved.

Shape of singularity- pointy potential

``Finite time blowup for `pointy' potential, K=|x|, can not be described by `first kind' similarity solution in dimensions N=3,5,7,…'' - proof Bertozzi, Carrillo, Laurent, Dai preprint – general N. What happens when the solution blows up? Let's compute it.

- Similarity solution of form $\rho(x,t) = \frac{1}{(T-t)^\alpha} w(\frac{x}{(T-t)^\beta})$
- The equation implies $\alpha = (n-1)\beta + 1$
- Conservation of mass would imply $\alpha = n\beta$ - NO
- Second kind similarity solution - no mass conservation
- Experimentally, the exponents vary smoothly with dimension of space, and there is no mass concentration in the blowup....

Simulations by Y. Huang

Second kind

Exact self-similar

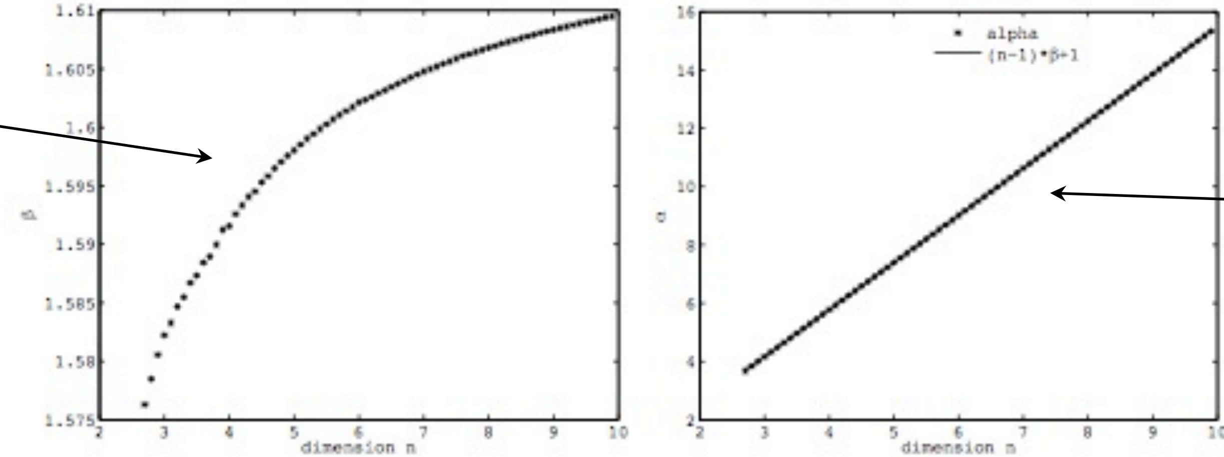


Figure 3: The exponents characterizing the blowup in different spatial dimensions: β(Left) and α(right). The comparison of the estimated α is in perfect agreement with the relation (11).

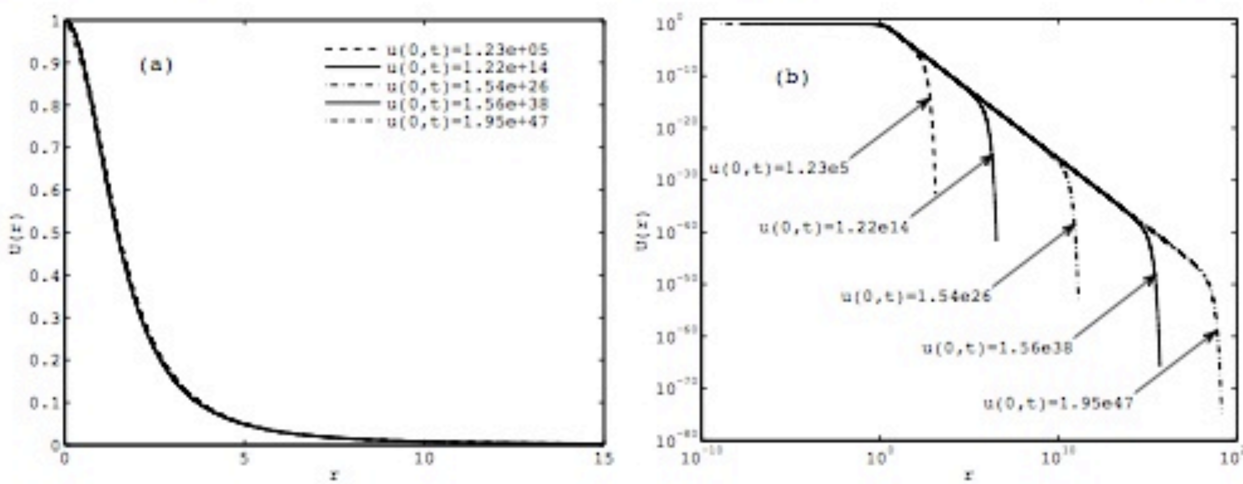


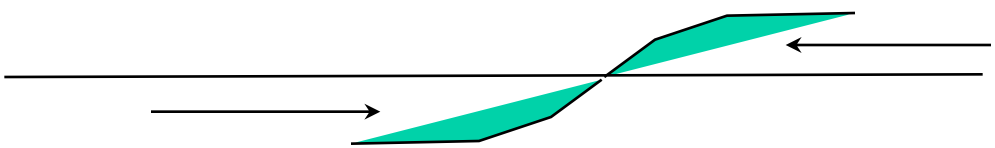
Figure 4: The convergence of the normalized profiles in dimension three. (a) Near the origin, all the profiles are indistinguishable. (b) Far away from the origin, the blowup dynamics adjusts the algebraic decay of the tail.

Huang and Bertozzi
SIAP 2010

Shape of singularity- pointy potential

- CONNECTION TO BURGERS SHOCKS
- In one dimension, K(x) = |x|, even initial data, the problem can be transformed exactly to *Burgers equation* for the integral of u.

$$\psi = \int_0^x u(x')dx', \quad \phi = C - 2\psi, \quad \phi_t + \phi\phi_x = 0.$$

- *Burgers equation* for odd initial data has an exact similarity solution for the blowup - it is an initial shock formation, with a 1/3 power singularity at x=0.
- 

$\phi(x,t)$

- There is no jump discontinuity at the initial shock time, which correponds to a zero-mass blowup for the aggregation problem. However immediately after the initial shock formation a jump discontinuity opens up - corresponds to mass concentration in the aggregation problem instantaneously after the initial blowup.
- This Burgers solution is (a) self-similar, (b) of `second kind', and (c) generic for odd initial data. There is a one parameter family of such solutions (also true in higher D).
- For the original u equation, this corresponds to beta = 3/2.

ALB, Laurent, Rosado, CPAM 2011

Local existence of L^p solutions

$$\nabla K \in W^{1,q} \quad \Longrightarrow \quad \text{Local existence of } L^p\text{-solution.}$$

Why ?

$$\nabla K \in W^{1,q} \quad \& \quad \rho \in L^p \quad \Longrightarrow \quad v = \nabla K * \rho \quad \in C^1$$

The velocity field is smooth enough to build characteristics.

Example: $K(x) = |x|$

- $\nabla K \in W^{1,q}$ for all $q < d$
- Local existence of solution in L^p for all $p > \frac{d}{d-1}$.

More on L^p

ALB, Laurent, Rosado, CPAM 2011

- Local existence using method of characteristics and some analysis (ALB, Laurent)
- Uniqueness using optimal transport theory (Rosado)
- Global existence vs local existence – the Osgood criteria comes back again. Why?
  - Mass concentration eventually happens in finite time for non-Osgood kernel
  - A priori L^p bound for Osgood kernel – see next slide

Review L^p well-posedness

for general potential

- Local existence of solutions in L^p provided that

$$\nabla K \in W^{1,q}(R^N)$$

- where q is the Holder conjugate of p (characteristics).
  - Existence proof constructs solutions using characteristics, in a similar fashion to weak L^∞ solutions (B. and Brandman *Comm. Math. Sci.* - to appear).
- Global existence of the same solutions in L^p provided that K satisfies the *Osgood condition* (derivation of a priori bound for L^p norm - similar to refined potential theory estimates in BCL 2009).
- When *Osgood condition* is violated, solutions blow up in finite time - implies blowup in L^p for all $p>p_c$.

Review L^p well-posedness for general potential

- *Ill-posedness* of the problem in L^p for p less than the Holder-critical p_c associated with the potential K.
- *Ill-posedness* results because one can construct examples in which mass concentrates instantaneously (for all $t>0$).
- For $p> p_c$, uniqueness in L^p can be proved for initial data also having bounded second moment, the proof uses ideas from *optimal transport*.
- The problem is *globally well-posed* with measure-valued data (preprint of Carrillo, DiFrancesco, Figalli, Laurent, and Slepcev - using optimal transport ideas).
- Even so, for *non-Osgood* potentials K, there is loss of information as time increases.
- Analogous to information loss in the case of compressive shocks for scalar conservation laws.

Followup on Lp

- Recent paper by Hongjie Dong proving that the pc is sharp for all powerlaw kernels
- Recent preprint by ALB, Garnett, Laurent studying monotonicity of radially symmetric solutions with mass concentration –delta
  - Existence of solutions for all powers down to Newtonial potential – requires Lagrangian form of the equation.
  - Newtonial potential is easy because radial symmetry reduces the PDE to Burgers equation in 1D and you can prove everything.
  - Uniqueness is open for powerlaw kernels between |x| and Newtonian case.

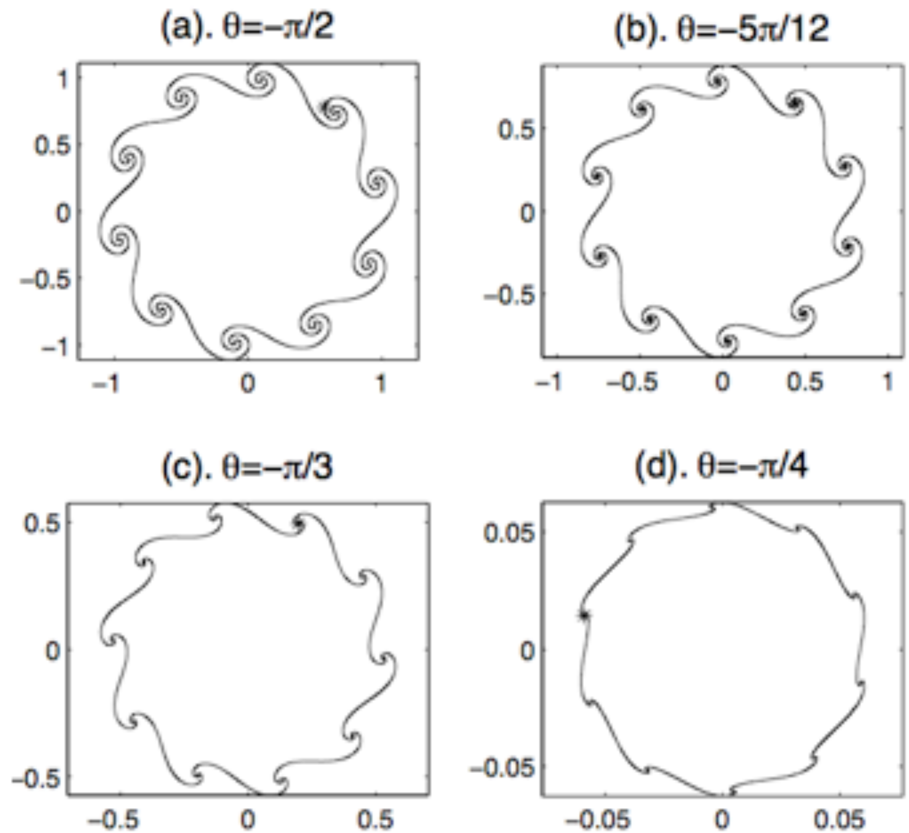
Generalize Birkhoff-Rott Equation

H. Sun, D Uminsky, and ALB, preprint 2011.

To appear in SIAP

$$\partial_t \rho + \nabla \cdot (u\rho) = 0, (t, x) \in (0, \infty) \times \mathbb{R}^2$$
$$u = M\nabla\Delta^{-1}\rho, \qquad \rho|_{t=0} = \rho_0$$

$$M(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$



Superfluids Example

Generalize Birkhoff-Rott Equation

H. Sun, D Uminsky, and ALB, preprint 2011.

$$u = \lambda_1 \nabla K * \rho + \lambda_2 \nabla^{\perp} K * \rho$$

$$K = \frac{1}{d^2} e^{-r^2/d^2}$$

Swarming Example

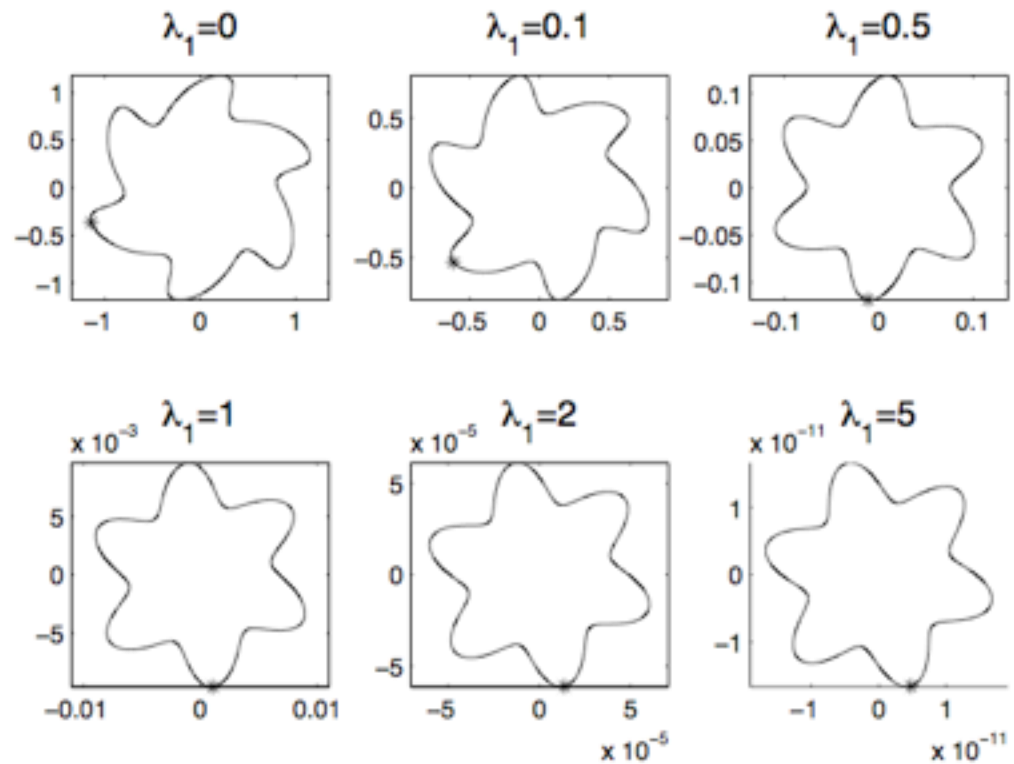
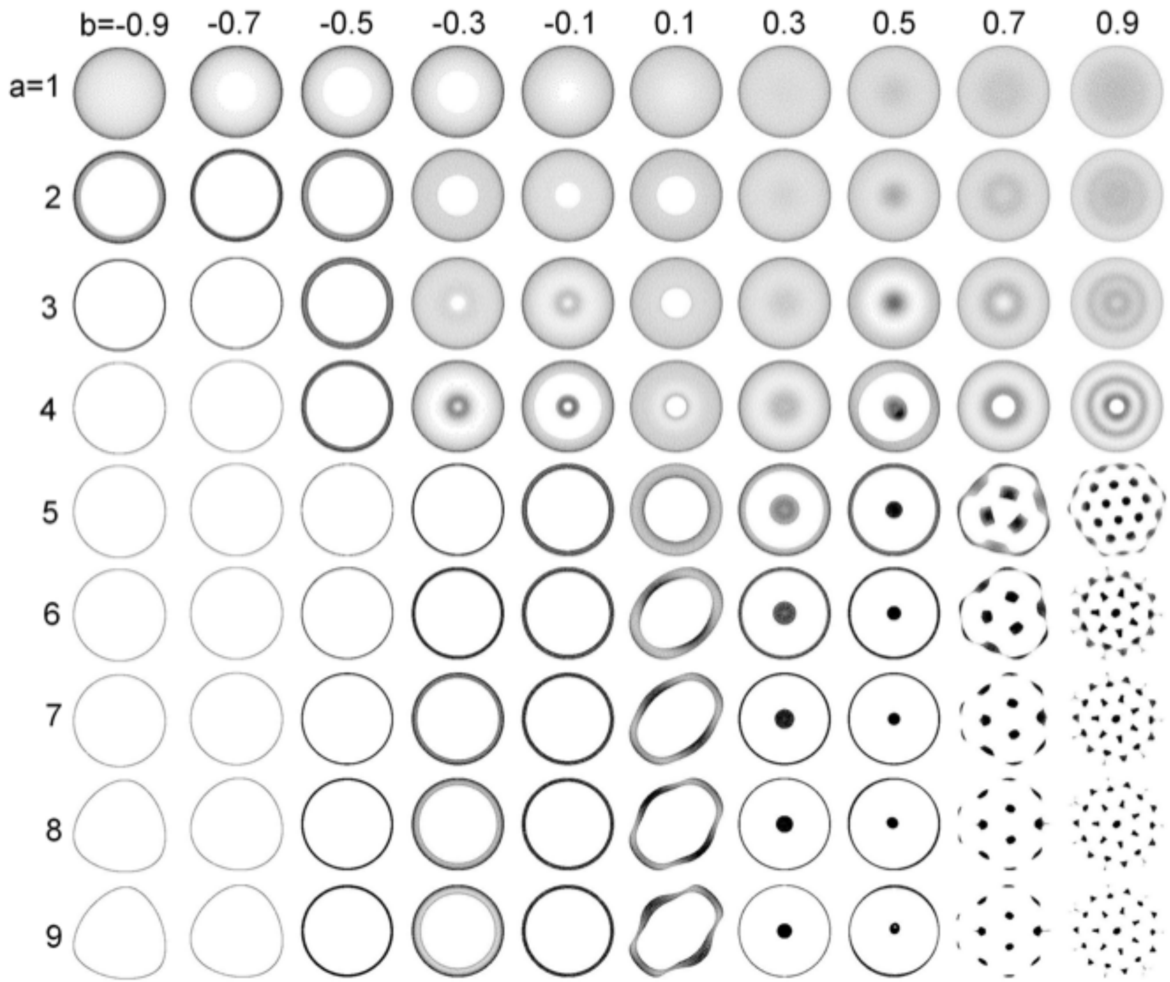


FIG. 4.9. *The solution at time t=25 for $\lambda_2 = 1$ and varying values of λ_1.*

Mixed Potentials – the World Cup Example

joint work with T. Kolokolnikov, H. Sun, D. Uminsky
Phys. Rev. E 2011

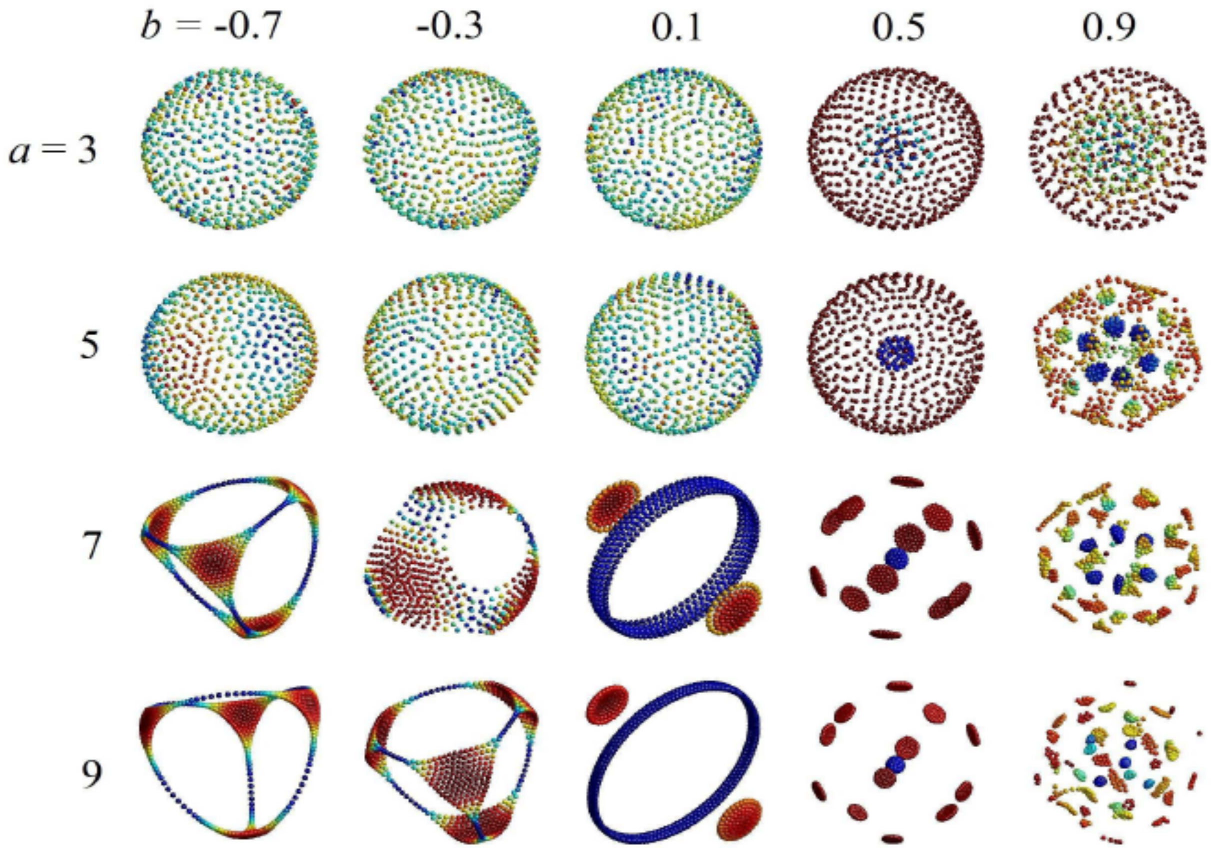


- K'(r) = Tanh((1-r)a) +b

Patterns as Complex as The surface of a A soccer ball.

Predicting pattern formation in particle interactions (3D linear theory)

to appear in M3AS, von Brecht, Uminsky, Kolokolnikov, ALB



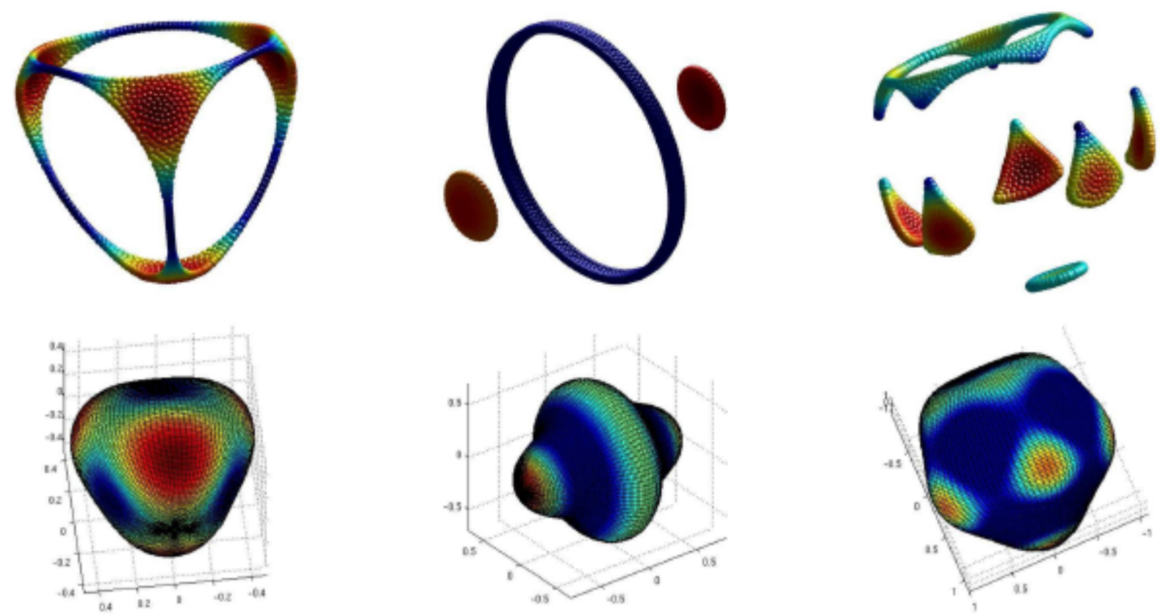
Linear stability of spherical shells

- Funk-Hecke theorem for spherical harmonics
- Analytical computation predicts shapes of patterns

- Pattern

- Linearly

Unstable mode



(a) (b) (c)

Fully nonlinear theory for multidimensional sheet solutions

- Joint work with *James von Brecht*
- Existence/uniqueness of solutions
- Local well-posedness depends on the kernel – sometimes not locally well-posed even for `reasonable’ kernels (fission to non-sheet behavior)
- Collapse in finite time – Osgood condition comes back again
- Also can expand to infinity in finite time

Aggregation Patches

Joint work with *Flavien Leger and Thomas Laurent*, to appear in *M3AS*

- These are like vortex patches….only different
- V = grad N*u where N is Newtonian potential
- Flow is orthogonal to the case of the vortex patch
- Solution will either contract or expand depending on the sign of the kernel

Key aspects of the problem

- General equations

$$\rho_t + v \cdot \nabla \rho = -(div v)\rho, \quad v = -\nabla K * \rho.$$

- Newtonian case

$$\rho_t + v \cdot \nabla \rho = \rho^2.$$

- Density is specified along particle paths $\rho(t) = (1/\rho_0 - t)^{-1}$
- It means there are exact solutions that are patches – like the vortex patch only they blow up in finite time, and the measure of the support shrinks to zero.

$$\rho(x,t) = \rho(t)\chi_{\Omega_t}, \quad \rho(t) = (1/\rho_0 - t)^{-1}$$

- These solutions exist in *any dimension*.

Expanding case – aggregation patch

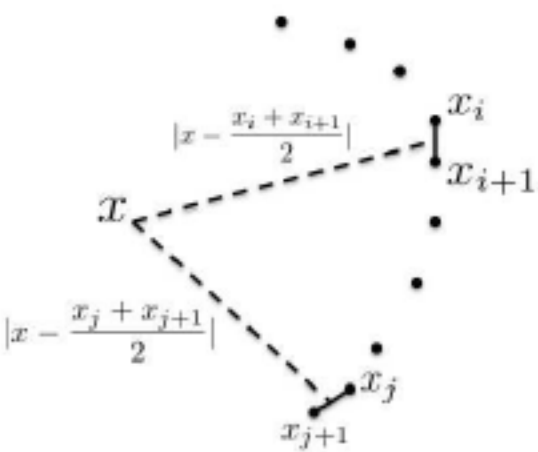
- In the expanding case the patch grows at a known rate
- In the long time limit the expanding patch converges in L1 to an exact similarity solution
- The similarity solution is an expanding ball:

$$\Omega(t) = B_{R(t)}(0), \quad R(t)^d = R_0^d \frac{1}{\rho(t)}.$$

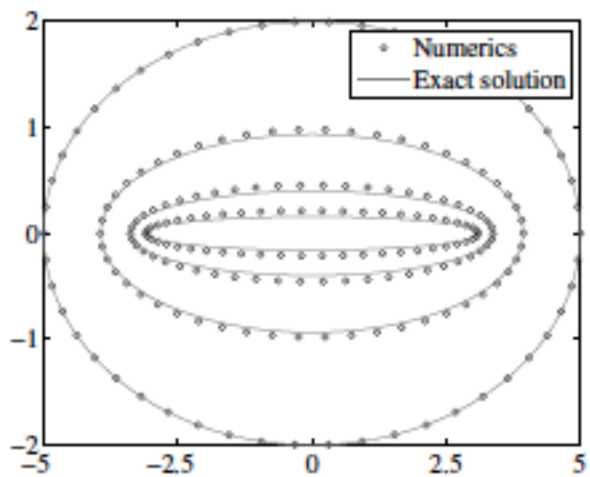
- Proof of convergence to the similarity solution has a power-law rate – proved to be sharp in 2D

Aggregation Patch "Kirchoff Ellipse"

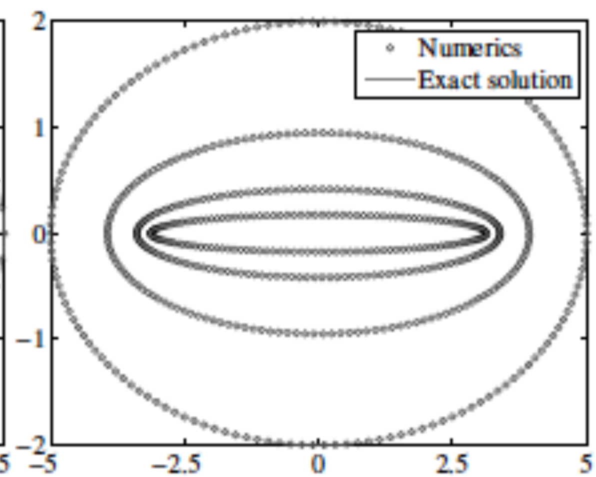
- Exact analytic solution – aggregation analogue of the Kirchoff ellipse – collapse onto a line segment – weighted measure



(a) Quadrature schematic

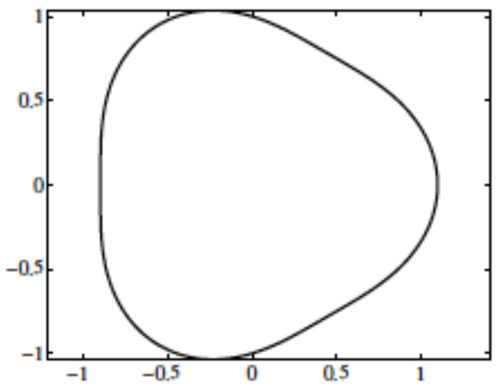


(b) Ellipse 50 points

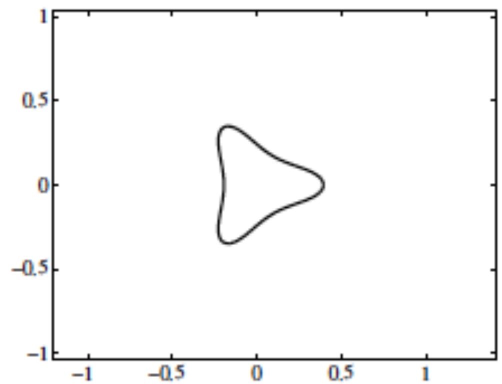


(c) Ellipse 150 points

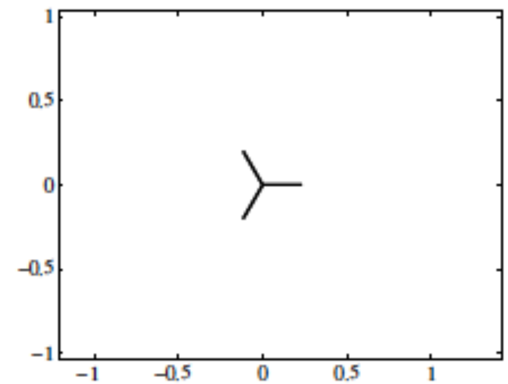
Aggregation Patches – attractive case 2D – collapse onto skeletons



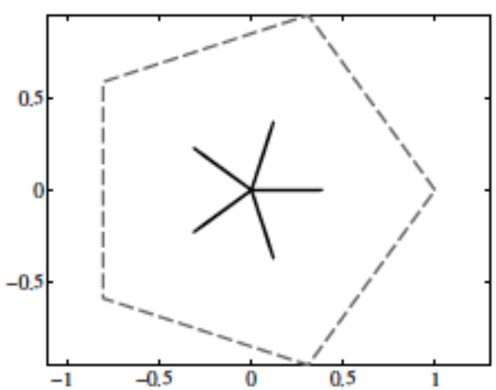
(a) Rounded triangle: boundary at $t = 0$



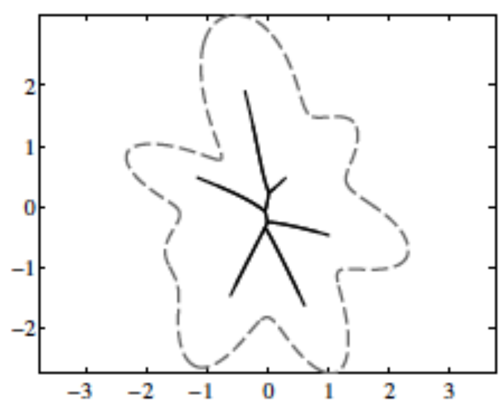
(b) Rounded triangle: boundary at $t = 0.92$



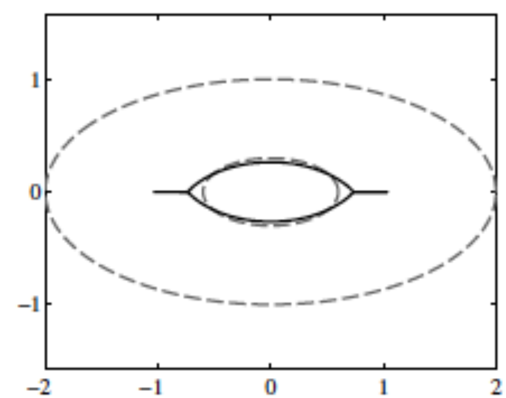
(c) Rounded triangle: boundary at $t = 0.99995$



(d) A pentagon - 1000 points

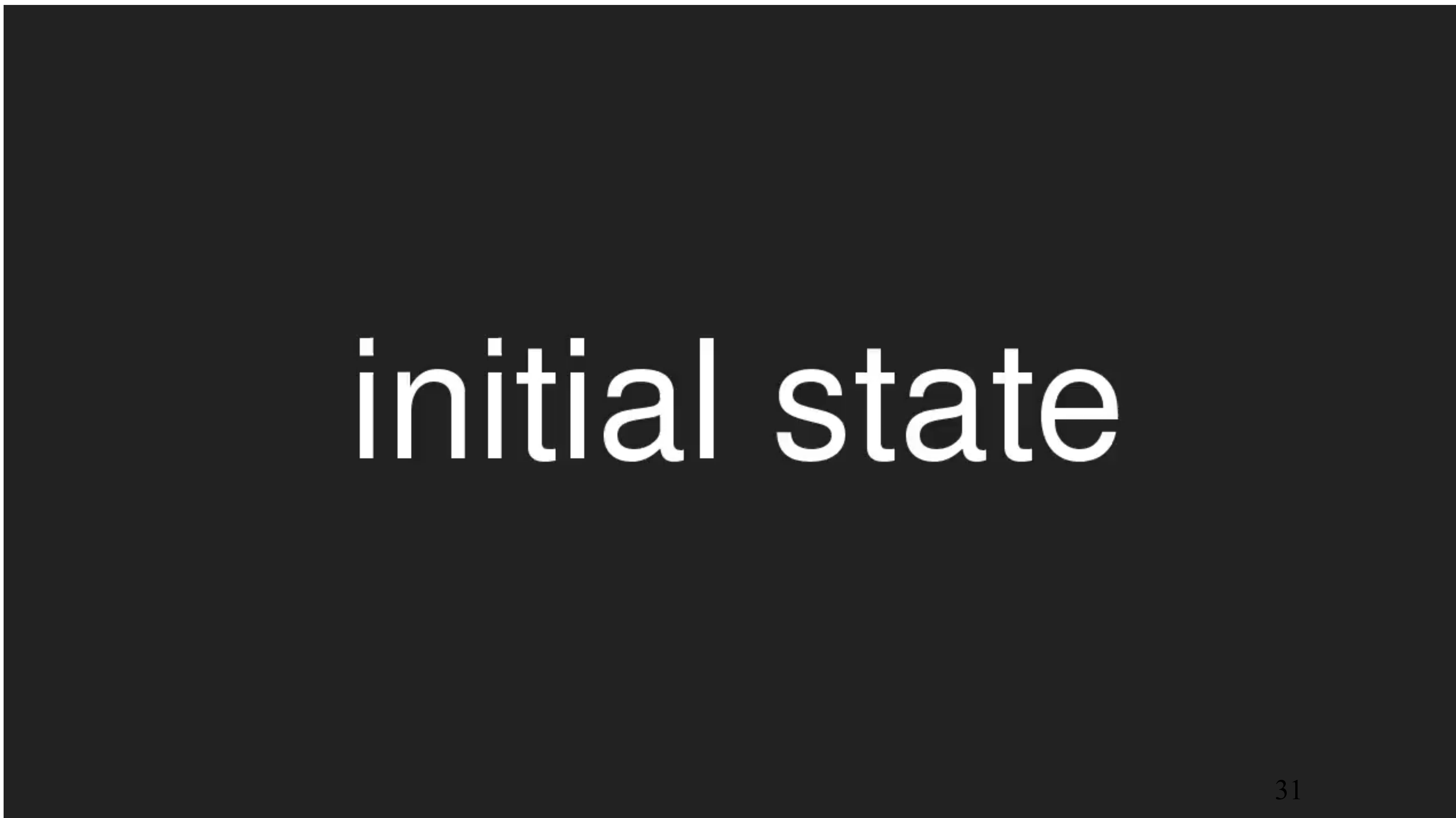


(e) A random shape - 5000 points

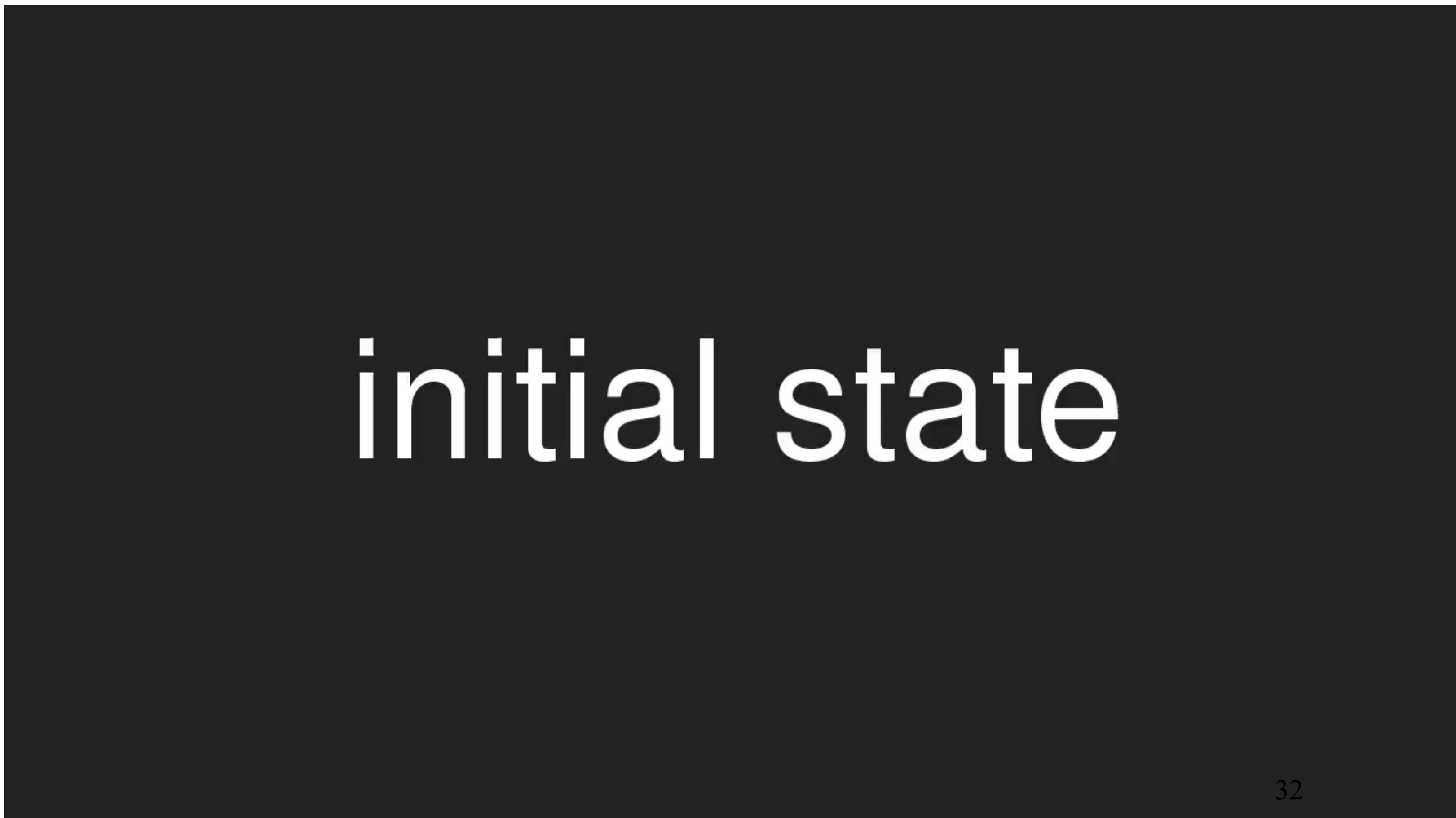


(f) An annulus - 1000 points

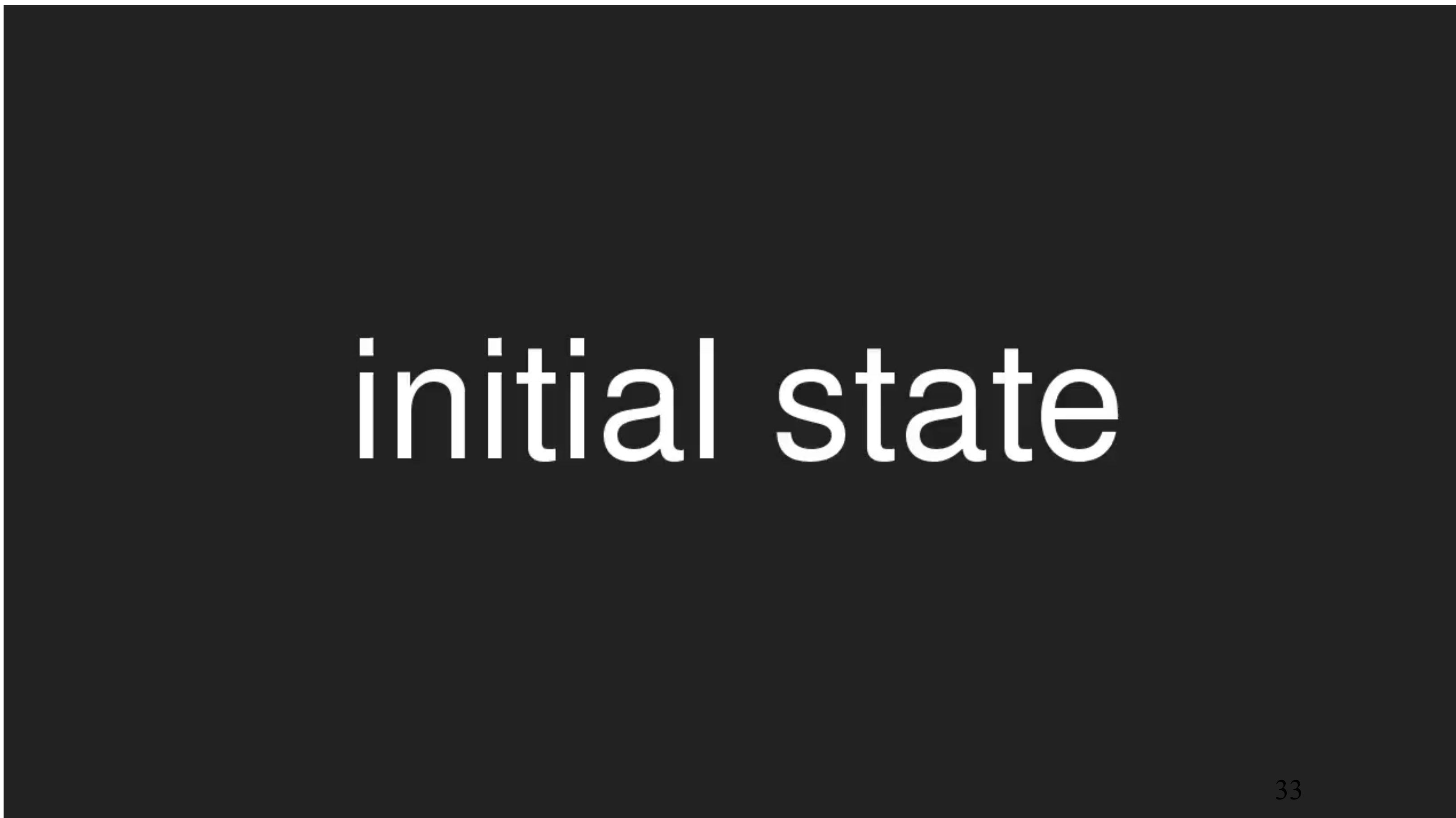
Movie Aggregation Patch 3D Cube Attractive Case



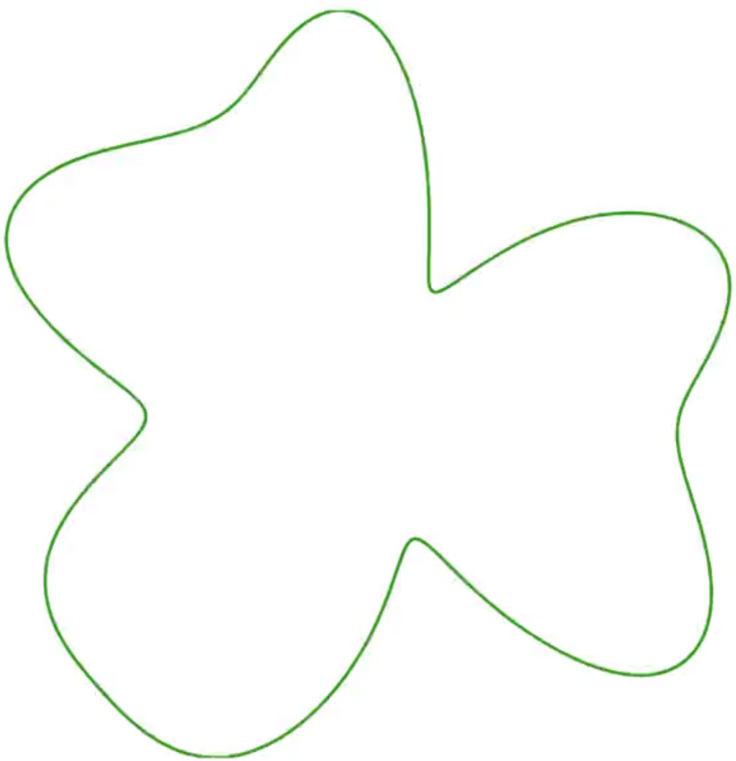
Movie Aggregation Patch – Teapot – Attractive Case



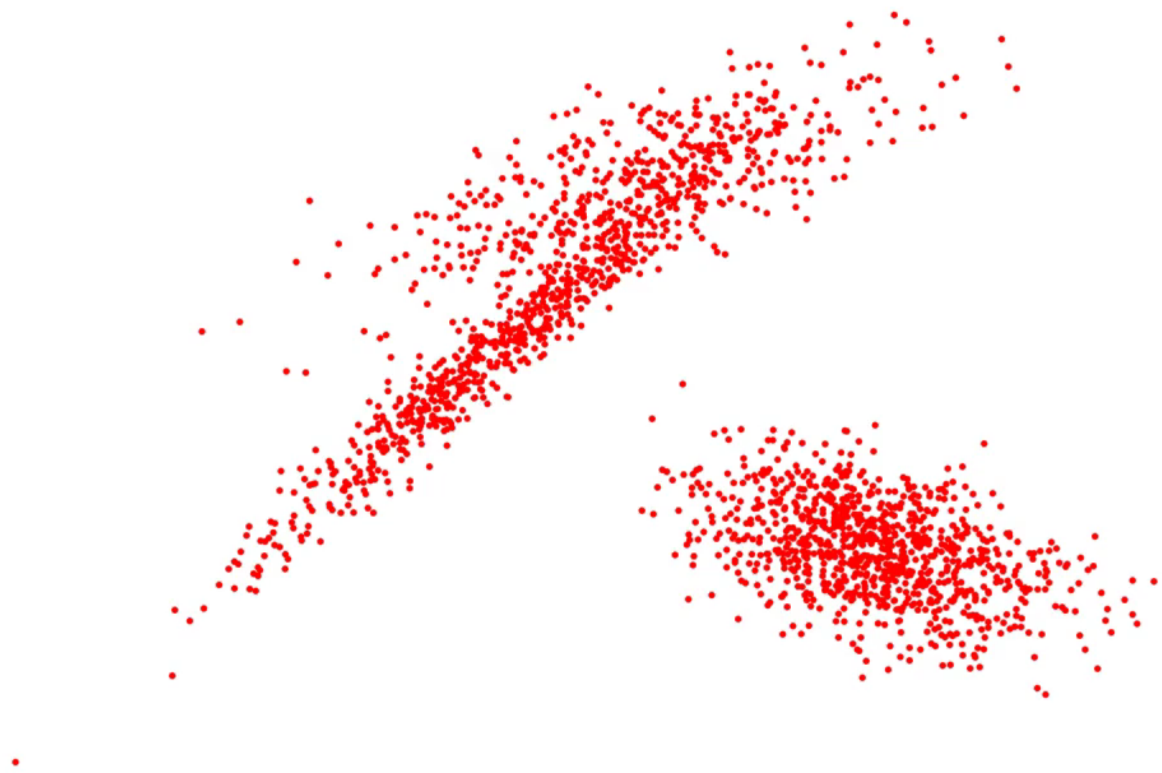
3D knot collapse



2D repulsive patch – rescaled variables



Repulsive case – 2D particles rescaled variables



Papers- Inviscid Aggregation Equations –Analysis

- ALB, J. B. Garnett, T. Laurent, - existence of solutions with measure and singular potentials, 1D radial, SIMA to appear 2012.
- von Brecht and ALB – general analysis of "sheet solutions" diffeomorphic to spherical shells – submitted 2012.
- ALB, F. Leger, T. Laurent, aggregation patches and the Newtonial Potential, *M3AS* 2012.
- ALB, Jose A. Carrillo, and Thomas Laurent, *Nonlinearity*, 2009. - Osgood criteria for finite time blowup, similarity solutions in odd dimension
- ALB, Thomas Laurent, Jesus Rosado, *CPAM* 2011.
  - Full L^p theory
- ALB and Jeremy Brandman, *Comm. Math. Sci*, 2010.
  - L infinity weak solutions of the aggregation problem
- ALB and T. Laurent, *Comm. Math. Phys.*, 274, p. 717-735, 2007.
  - Finite time blowup in all space dimensions for pointy kernels

Numerics and Asymptotics

- Kolokolnikov, Sun, Uminsky, Bertozzi, `world cup' Phys. *Rev. E 2011*
- von Brecht, Uminsky, Kolokolnikov*, ALB,* M3AS, vol. 22, Supp. 1, 1140002, 2012,
- Von Brecht and Uminsky, to appear in J. Non. Sci. 2012.
- Yanghong Huang, ALB, *SIAP* 2010.
  - Simulation of finite time blowup
- Huang, Witelski, and ALB. Asymptotic theory to explain selection of anomalous exponents for blowup solution – in dimensions 3 and 5 for K=|x|. Need more general theory.
- Huang and ALB, General scaling of blowup solutions for powerlaw kernels in general dimensions, *DCDS* – Tom Beale issue, 2012.
- Sun, Uminsky, ALB – extension of 2D vortex sheets to problems with aggregating kernels –*SIAP, 2012*.

Papers and preprints on viscous aggregation equations

Topaz, Bertozzi, and Lewis, *Bull. Math. Bio*. 2006.

Bertozzi and Slepcev, *CPAA*, 2010

Rodriguez and Bertozzi – *M3AS* 2010 crime models.

Von Brecht, Uminsky, ALB– M3AS to appear

Bedrossian, Rodriguez, and Bertozzi, *Nonlinearity* 2011

Bedrossian – *CMS* 2011, *AML* 2011.

Students and Postdocs and Collaborators

- Masters: Flavien Leger
- PhD students: Jeremy Brandman, Yanghong Huang, Hui Sun, Nancy Rodriguez, Jacob Bedrossian, Jesus Rosado, James von Brecht
- Postdocs: Chad Topaz, Thomas Laurent, Dejan Slepcev, David Uminsky
- Collaborators: Jose Antonio Carrillo, John Garnett, Theo Kolokolnikov, Tom Witelski