NONLOCAL AGGREGATION EQUATIONS and FLUID DYNAMICS

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The Problem

\[ \rho_t + \nabla \cdot (\rho \nabla K \ast \rho) = 0 \]

A density advected by a field that is the gradient of \( K \) convolved with itself.

*Active scalar problem* in gradient flow format.

Analysis follows ideas from both *fluid mechanics* (active scalars – divergence free flow) and *optimal transport* (gradient flow).
Finite time singularities—general potentials

\[ \rho_t + \nabla \cdot (\rho \nabla K \ast \rho) = 0 \]

- Previous Results
  - For smooth \( K \) the solution blows up in infinite time
  - For `pointy` \( K \) (biological kernel such as \( K = e^{-|x|} \)) blows up in finite time for special radial data in any space dimension.

New result: Osgood condition
\[ \int_0^L \frac{1}{K'(r)} dr < \infty \]

is a necessary and sufficient condition for finite time blowup in any space dimension (under mild monotonicity conditions).

Moreover—finite time blowup for pointy potential can not be described by `first kind’ similarity solution in dimensions \( N = 3, 5, 7, \ldots \)
Osgood uniqueness criteria for ODEs

- \( \frac{dX}{dt} = F(X) \), system of first order ODEs
- Picard theorem – \( F \) is Lipschitz continuous
- Generalization of Picard is Osgood criteria (sharp): \( w(x) \) is modulus of continuity of \( F \), i.e.
  \[
  |F(X) - F(Y)| < w(|X - Y|)
  \]
- \( \frac{1}{w(z)} \) is not integrable at the origin for unique solutions.
- Example: \( \frac{dx}{dt} = x \ |\ln x| \) has unique solutions.
- Example: \( \frac{dx}{dt} = \sqrt{x} \) does not have unique solutions.
COMPARISON PRINCIPLE: Proof of finite time collapse for non-Osgood potentials – let $R(t)$ denote the particle farthest away from the center of mass (conserved), then

$$\frac{dx_i}{dt} = - \sum_{j \neq i} m_j \nabla K(x_i - x_j) = - \sum_{j \neq i} m_j \frac{x_i - x_j}{|x_i - x_j|} k'(|x_i - x_j|),$$

$$\dot{R} \leq - \frac{M}{2} K'(2R(t)).$$

When the Osgood criteria is violated particles collapse together in finite time. When the Osgood criteria is satisfied we have global existence and uniqueness of a solution of the particle equations.
COMPARISON PRINCIPLE: The proof for a finite number of particles extends naturally to the continuum limit. Proof of finite time blowup for non-Osgood potentials - assumes compact support of solution. One can prove that there exists an $R(t)$ such that $B_{R(t)}(x_m)$ contains the support, $x_m$ is center of mass (conserved), and

$$\dot{R} \leq -\frac{M}{2} K'(2R(t)).$$

Thus the Osgood criteria provides a sufficient condition on the potential $K$ for finite time blowup from bounded data. To prove the condition is also necessary we must do further potential theory estimates.
First an easier result - $C^2$ kernels

- A priori bound
- One can easily prove a Gronwall estimate
- $L$-infty of density controlled by $L$-infty of $\text{div } v$.
- But, $\text{div } v = \text{Laplacian of } K$ convoluted with the density, which we assume to be in $L^1$.
- If the kernel is $C^2$ we have an a priori bound.
- We need more refined potential theory estimates for general Osgood condition.
Proof of global existence:

Connection to 3D Euler

\[ \rho_t + v \cdot \nabla \rho = -(\nabla \cdot v)\rho \]

\[ v = \nabla K * \rho \]

A priori bound for \( L^\infty \) norm follows similar approach as in BKM theorem (1984) for incompressible Euler.

Vorticity Stream form of 3D Euler Equations

\[ \omega_t + v \cdot \nabla \omega = (\nabla v)\omega \]

\[ v = \vec{K}_3 * \omega \]

omega is vector vorticity and \( K_3 \) is Biot-Savart Kernel in 3D
Finite time singularities—
general potentials

BKM argument uses log-linear estimate for SIOs
(see e.g. chapter 4 Vorticity and Incompressible Flow)

\[ |\nabla v|_{L^\infty} \leq |\omega|_{L^\infty} [1 + \ln(l_1/l_2)] \]

where

\[ l_1 = (|\omega|_{L^\infty}/|\omega|_{\gamma})^{1/\gamma} \quad l_2 = (|\omega|_{L^2}/|\omega|_{L^\infty})^{2/3} \]
Similarly, the aggregation problem has the estimate

$$\partial_t |\rho|_{L^\infty} \leq |\rho|_{L^\infty} |\Delta K * \rho|_{L^\infty}$$

so the challenge is to show that $|\Delta K * \rho|_{L^\infty}$ is ‘logarithmic’ in $\rho$ - in the sense of the Osgood criteria, in the case where $\Delta K$ is unbounded at the origin.

The key lengthscale in the problem is $\delta = (M/|\rho|_{L^\infty})^{1/N}$ where $N$ is the space dimension and $M$ is the mass,

$$M = \int \rho dx.$$
One can show that \( \delta = (M/|\rho|_{L^\infty})^{1/N} \) satisfies a differential inequality

\[
\dot{\delta} \geq -C(N,M)K'(\delta)
\]

which means that if \( K \) satisfies the Osgood condition, then \( \delta \) is bounded away from zero for all time. This gives an a priori upper bound on \( |\rho|_{L^\infty} \) since mass is conserved.
``Finite time blowup for ‘pointy’ potential, $K=|x|$, cannot be described by ‘first kind’ similarity solution in dimensions $N=3,5,7,...$” - proof Bertozzi, Carrillo, Laurent, Dai preprint – general $N$. What happens when the solution blows up? Let’s compute it.

- Similarity solution of form $\rho(x, t) = \frac{1}{(T-t)^\alpha} w\left(\frac{x}{(T-t)^\beta}\right)$

- The equation implies $\alpha = (n - 1)\beta + 1$

- Conservation of mass would imply $\alpha = n\beta$ - no

- Second kind similarity solution - no mass conservation

- Experimentally, the exponents vary smoothly with dimension of space, and there is no mass concentration in the blowup....
Figure 3: The exponents characterizing the blowup in different spatial dimensions: $\beta$ (Left) and $\alpha$ (right). The comparison of the estimated $\alpha$ is in perfect agreement with the relation (11).

Figure 4: The convergence of the normalized profiles in dimension three. (a) Near the origin, all the profiles are indistinguishable. (b) Far away from the origin, the blowup dynamics adjusts the algebraic decay of the tail.
• CONNECTION TO BURGERS SHOCKS

• In one dimension, $K(x) = |x|$, even initial data, the problem can be transformed exactly to Burgers equation for the integral of $u$.

\[ \psi = \int_0^x u(x') \, dx', \quad \phi = C - 2\psi, \quad \phi_t + \phi \phi_x = 0. \]

• Burgers equation for odd initial data has an exact similarity solution for the blowup - it is an initial shock formation, with a $1/3$ power singularity at $x=0$.

• There is no jump discontinuity at the initial shock time, which corresponds to a zero-mass blowup for the aggregation problem. However immediately after the initial shock formation a jump discontinuity opens up - corresponds to mass concentration in the aggregation problem instantaneously after the initial blowup.

• This Burgers solution is (a) self-similar, (b) of `second kind`, and (c) generic for odd initial data. There is a one parameter family of such solutions (also true in higher D).

• For the original $u$ equation, this corresponds to $\beta = 3/2$. 
Local existence of $L^p$ solutions

\[ \nabla K \in W^{1,q} \implies \text{Local existence of } L^p\text{-solution.} \]

Why?

\[ \nabla K \in W^{1,q} \quad \& \quad \rho \in L^p \implies v = \nabla K \ast \rho \in C^1 \]

The velocity field is smooth enough to build characteristics.

Example: $K(x) = |x|$
More on $L^p$

*ALB, Laurent, Rosado, CPAM 2011*

- Local existence using method of characteristics and some analysis (ALB, Laurent)
- Uniqueness using optimal transport theory (Rosado)
- Global existence vs local existence – the Osgood criteria comes back again. Why?
  - Mass concentration eventually happens in finite time for non-Osgood kernel
  - A priori $L^p$ bound for Osgood kernel – see next slide
• Local existence of solutions in $L^p$ provided that
  \[ \nabla K \in W^{1,q}(\mathbb{R}^N) \]
• where $q$ is the Holder conjugate of $p$ (characteristics).
• Global existence of the same solutions in $L^p$ provided that $K$ satisfies the *Osgood condition* (derivation of a priori bound for $L^p$ norm - similar to refined potential theory estimates in BCL 2009).
• When *Osgood condition* is violated, solutions blow up in finite time - implies blowup in $L^p$ for all $p>p_c$. 
Ill-posedness of the problem in $L^p$ for $p$ less than the Holder-critical $p_c$ associated with the potential $K$.

Ill-posedness results because one can construct examples in which mass concentrates instantaneously (for all $t>0$).

For $p > p_c$, uniqueness in $L^p$ can be proved for initial data also having bounded second moment, the proof uses ideas from optimal transport.

The problem is globally well-posed with measure-valued data (preprint of Carrillo, DiFrancesco, Figalli, Laurent, and Slepcev - using optimal transport ideas).

Even so, for non-Osgood potentials $K$, there is loss of information as time increases.

Analogous to information loss in the case of compressive shocks for scalar conservation laws.
Followup on Lp

• Recent paper by Hongjie Dong proving that the pc is sharp for all powerlaw kernels

• Recent preprint by ALB, Garnett, Laurent studying monotonicity of radially symmetric solutions with mass concentration –delta
  – Existence of solutions for all powers down to Newtonian potential – requires Lagrangian form of the equation.
  – Newtonian potential is easy because radial symmetry reduces the PDE to Burgers equation in 1D and you can prove everything.
  – Uniqueness is open for powerlaw kernels between $|x|$ and Newtonian case.
Generalize Birkhoff-Rott Equation

H. Sun, D Uminsky, and ALB, preprint 2011.

\[ \partial_t \rho + \nabla \cdot (u \rho) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^2 \]

\[ u = M \nabla \triangle^{-1} \rho, \quad \rho \big|_{t=0} = \rho_0 \]

\[ M(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \]

Superfluids Example
Generalize Birkhoff-Rott Equation

H. Sun, D Uminsky, and ALB, preprint 2011.

\[ u = \lambda_1 \nabla K \ast \rho + \lambda_2 \nabla^\perp K \ast \rho \]

\[ K = \frac{1}{d^2} e^{-r^2/d^2} \]

Swarming Example

Fig. 4.9. The solution at time \( t=25 \) for \( \lambda_2 = 1 \) and varying values of \( \lambda_1 \).
Mixed Potentials – the World Cup Example

joint work with T. Kolokolnikov, H. Sun, D. Uminsky

*Phys. Rev. E* 2011

\[ K'(r) = \tanh((1-r)a) + b \]

Patterns as Complex as The surface of a A soccer ball.
Predicting pattern formation in particle interactions (3D linear theory)

to appear in M3AS, von Brecht, Uminsky, Kolokolnikov, ALB
Linear stability of spherical shells

- Funk-Hecke theorem for spherical harmonics
- Analytical computation predicts shapes of patterns

- Pattern

- Linearly Unstable mode
Fully nonlinear theory for multidimensional sheet solutions

- Joint work with James von Brecht
- Existence/uniqueness of solutions
- Local well-posedness depends on the kernel – sometimes not locally well-posed even for ‘reasonable’ kernels (fission to non-sheet behavior)
- Collapse in finite time – Osgood condition comes back again
- Also can expand to infinity in finite time
Aggregation Patches
Joint work with Flavien Leger and Thomas Laurent, to appear in M3AS

• These are like vortex patches….only different
• $V = \text{grad } N^* u$ where $N$ is Newtonian potential
• Flow is orthogonal to the case of the vortex patch
• Solution will either contract or expand depending on the sign of the kernel
Key aspects of the problem

- General equations
  \[\rho_t + v \cdot \nabla \rho = -(\text{div}v)\rho, \quad v = -\nabla K * \rho.\]

- Newtonian case
  \[\rho_t + v \cdot \nabla \rho = \rho^2.\]

- Density is specified along particle paths \(\rho(t) = (1/\rho_0 - t)^{-1}\)

- It means there are exact solutions that are patches – like the vortex patch only they blow up in finite time, and the measure of the support shrinks to zero.

  \[\rho(x, t) = \rho(t)\chi_{\Omega_t}, \quad \rho(t) = (1/\rho_0 - t)^{-1}\]

- These solutions exist in any dimension.
Expanding case – aggregation patch

• In the expanding case the patch grows at a known rate
• In the long time limit the expanding patch converges in L1 to an exact similarity solution
• The similarity solution is an expanding ball:

\[
\Omega(t) = B_{R(t)}(0), \quad R(t)^d = R_0^d \frac{1}{\rho(t)}.
\]

• Proof of convergence to the similarity solution has a power-law rate – proved to be sharp in 2D
Aggregation Patch “Kirchoff Ellipse”

- Exact analytic solution – aggregation analogue of the Kirchoff ellipse – collapse onto a line segment – weighted measure
Aggregation Patches – attractive case 2D – collapse onto skeletons

(a) Rounded triangle: boundary at $t = 0$

(b) Rounded triangle: boundary at $t = 0.92$

(c) Rounded triangle: boundary at $t = 0.99995$

(d) A pentagon - 1000 points

(e) A random shape - 5000 points

(f) An annulus - 1000 points
Movie Aggregation Patch 3D Cube
Attractive Case

initial state
Movie Aggregation Patch – Teapot – Attractive Case

initial state
3D knot collapse

initial state
2D repulsive patch – rescaled variables
Repulsive case – 2D particles rescaled variables
Papers- Inviscid Aggregation Equations – Analysis

• ALB, J. B. Garnett, T. Laurent, - existence of solutions with measure and singular potentials, 1D radial, SIMA to appear 2012.


• ALB, Jose A. Carrillo, and Thomas Laurent, \textit{Nonlinearity}, 2009. - Osgood criteria for finite time blowup, similarity solutions in odd dimension

• ALB, Thomas Laurent, Jesus Rosado, \textit{CPAM} 2011.
  – Full \(L^p\) theory

  – \(L \) infinity weak solutions of the aggregation problem

  – Finite time blowup in all space dimensions for pointy kernels
Numerics and Asymptotics

• Kolokolnikov, Sun, Uminsky, Bertozzi, ‘world cup’ Phys. Rev. E 2011
• von Brecht, Uminsky, Kolokolnikov, ALB, M3AS, vol. 22, Supp. 1, 1140002, 2012,
• Yanghong Huang, ALB, SIAP 2010.
  – Simulation of finite time blowup
• Huang, Witelski, and ALB. Asymptotic theory to explain selection of anomalous exponents for blowup solution – in dimensions 3 and 5 for K=|x|. Need more general theory.
• Huang and ALB, General scaling of blowup solutions for powerlaw kernels in general dimensions, DCDS – Tom Beale issue, 2012.
• Sun, Uminsky, ALB – extension of 2D vortex sheets to problems with aggregating kernels –SIAP, 2012.
Papers and preprints on viscous aggregation equations


Bertozzi and Slepcev, *CPAA*, 2010

Rodriguez and Bertozzi – *M3AS* 2010 crime models.

Von Brecht, Uminsky, ALB– *M3AS* to appear

Bedrossian, Rodriguez, and Bertozzi, *Nonlinearity* 2011

Students and Postdocs and Collaborators

• Masters: Flavien Leger
• PhD students: Jeremy Brandman, Yanghong Huang, Hui Sun, Nancy Rodriguez, Jacob Bedrossian, Jesus Rosado, James von Brecht
• Postdocs: Chad Topaz, Thomas Laurent, Dejan Slepcev, David Uminsky
• Collaborators: Jose Antonio Carrillo, John Garnett, Theo Kolokolnikov, Tom Witelski