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Gravitational Wave Extraction based on the Cauchy-Characteristic Method

(*based on gr-qc/0501008*)

Abstract

- We implement a code to find the gravitational News at future null infinity.
- We use data from a Cauchy code as a boundary data for a characteristic code.
- The Cauchy-characteristic extraction (CCE) technique allows the extraction of gravitational waves from numerical simulation.
- We test this method on flat and static black hole spacetimes in various gauges.
- We show the convergence of the algorithm and illustrate its succes in cases where other wave extraction methods fail.

Introduction

- Accurate 3D numerical simulations of isolated sources like binary black holes are needed to provide gravitational waveform templates for gravitational wave detectors.
- The commonly codes used in numerical simulations are based on a "3+1" spacetime slicing
- Two problems arise:
  1. artificial boundary conditions must be placed on the computational domain
  2. information, such as the gravitational news, cannot be extracted at null infinity.
- To avoid these problems, two possible approaches have been suggested:

Introduction

- To evolve the entire space, for example:
  a) In the hyperboloidal slicing of the spacetime information propagates to ***infinity in finite time (S. Husa '01, P. Hubner '***01, J. Frauendiener '04)
  b) In the conformal compactification, infinity is placed at finite computational location.
  c) In the characteristic approaches based on a Bondi-Sachs line element, the radiation region is naturally evolved out to infinity (N. Bishop '97, R. Gomez '97, F. Siebel '02, S. Husa '01, B. Szilagyi '02). Characteristic numerical codes are used to study: tail decay, critical phenomenta, singularity structure and fluid collapse.

  Characteristic methods develop caustics!
  This can be avoided by evolving the strong field region with a standard Cauchy slicing (CCM).

Introduction

II. The standard way of avoiding those problems is the use of the perturbation theory, for example:
  a) the quadrupole formula or first order gauge invariant formalism (Zerilli '70, Moncrief '74)
  b) perturbative boundary conditions can be provided to the truncated Cauchy slice, in the Cauchy-perturbative matching approach

**The full nonlinear Cauchy-characteristic matching has many appealing properties:**

- ✔ The characteristic description of the exterior allows for a natural extraction of gravitational information such as the news.
- ✔ The use of a match to a standard Cauchy code in the interior is a natural way of avoiding the problems of caustics and artificial boundaries.

Introduction

- In this work we take a step towards the full CCM approach by using a Cauchy code to provide boundary for the characteristic code

  i. The radiation-region is described by the Pitt Null Code (L. Lehner '99, N. Bishop '96, R. Gomez '01)
  ii. The link between the Cauchy and the characteristic is done by a non-linear 3D CCE algorithm (first version J. Winicour '98, B. Szilagyi '00)
  iii. At the exterior of the Bondi code the News Extraction Module calculates the gravitational news (Y. Zlochower '02, '03)
  iv. Here the CCE algorithm have been improved and the combined CCE-NE codes imported into Cactus (G. Allen ' 01, T. Goodale '03, CactusWeb)
  v. We also test the robustness of the CCE approach and compare the results with the Zerilli method.

Geometry

- The geometry is described by two separate foliations, neither one covering the entire spacetime.
- The interaction between the Cauchy and the characteristic foliations are in terms of the ADM metric:

$$ds^2 = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j$$

- This data it is given initial and at the boundary in either the BSSN or generalized harmonic formulation.
- An artificial boundary is placed by a world-tube that intersects t-const Cauchy slices in Cartesian spheres.
- The world-tube provides then the inner boundary for the Bondi-Sachs characteristic foliation:

$$ds^2 = -(e^{2\beta} V/r - r^2 h_{AB} U^A U^B) du^2 - 2 e^{2\beta} du dr \\ -2 r^2 h_{AB} U^B du dy^A + r^2 h_{AB} dy^A dy^B$$

Geometry

- The free variables in the Bondi-Sachs metric are:

$$V, \beta, U^A, h_{AB}$$

- The intrinsic metric of the r=const surfaces describes the world-tube geometry in a 2+1 foliation:

$$\gamma_{ij} dy^i dy^j = -e^{2\beta} V/r \, du^2 + r^2 h_{AB} (dy^A - U^A du)(dy^B - U^B du)$$

- Here u labels the outgoing null hypersurfaces, y^A the null rays emanating from the world-tube, and r the radial surface area distance.
- The 2+1 decomposition identifies $r^2 h_{AB}$ as the metric of the u=const 2-surfaces, $-U^A$ as the shift vector, and $e^{2\beta} V/r$ the square of the lapse.

Geometry

- The CCE algorithm takes the data given in the ADM form in a neighborhood of the world-tube and transform it into the boundary data for the Bondi metric.
- Because the world-tube is not a surface of constant r, the data is first determined in terms of an affine parameter along the radial direction and the Bondi data it is filled by a Taylor expansion.
- The Bondi code uses the hypersurface equations to evolve the data. The outgoing characteristic cones are constructed along the r coordinate, null oriented and normal to the world-tube and the News are extracted at I^+
- As we do not take the complex step of using the characteristic code to provide boundary data for the Cauchy code, we might expect a contamination of the wavesignal.
- The algorithm is applied to four “null” tests : flat spacetime, random perturbations of flat spacetime, static black hole in centered and in oscillating ingoing EF frame.

The CCE Algorithm

- The CCE algorithm starts by interpolating the Cauchy metric, lapse, shift and spatial derivatives onto the world-tube.
- The time derivatives are computed via backwards finite differencing along the worldtube:

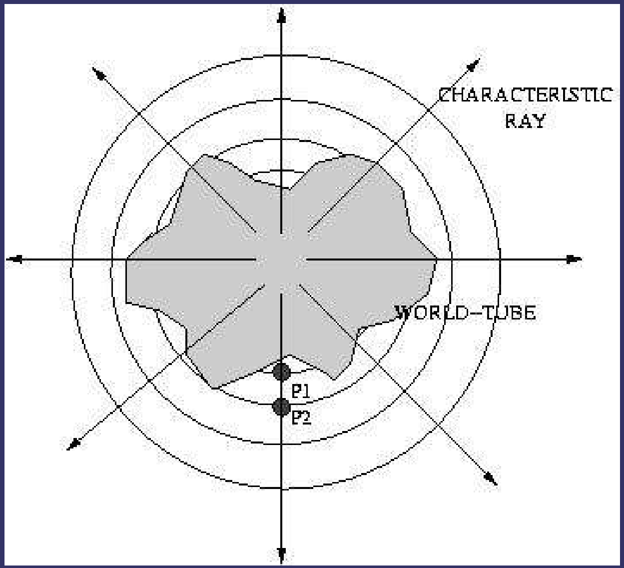
$$\partial_t F_{[n]} = \frac{1}{2\Delta t}\left(3F_{[n]} - 4F_{[n-1]} + F_{[n-2]}\right) + O((\Delta t)^2)$$

- We fill the Bondi gridpoints around the world-tube with the affine metric as a Taylor expansion:

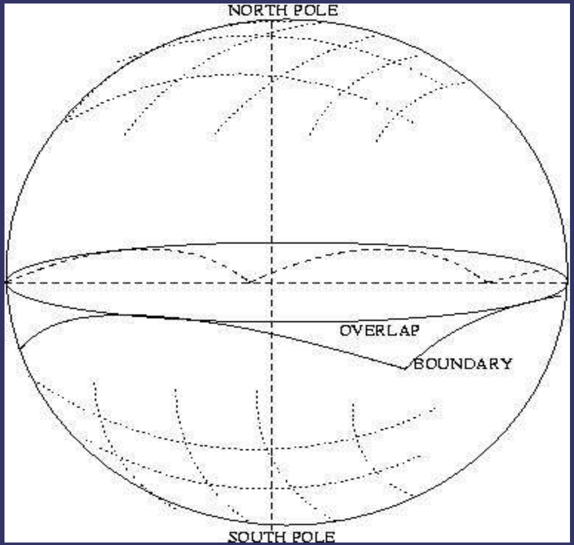
$$\tilde{\eta}^{\tilde{\alpha}\tilde{\beta}} = (\tilde{\eta}^{\tilde{\alpha}\tilde{\beta}})_\Gamma + \lambda(\tilde{\eta}^{\tilde{\alpha}\tilde{\beta}}_{,\lambda})_\Gamma + O(\lambda^2)$$

- Next the Bondi-Sachs null coordinate system metric is computed:

$$(\eta^{\mu\nu})_\Gamma = \left(\frac{\partial y^\mu}{\partial \tilde{y}^{\tilde{\alpha}}}\frac{\partial y^\nu}{\partial \tilde{y}^{\tilde{\beta}}}\tilde{\eta}^{\tilde{\alpha}\tilde{\beta}}\right)_\Gamma$$



The CCE Algorithm



- In order to avoid the angular coordinates singularities at the poles, we use a complex vector:

$$\xi = \tilde{y}^2 + I\tilde{y}^3$$

- The sphere is covered stereographical by 2 patches:

$$\xi_{north} = \sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}\, e^{I\phi},\ \xi_{south} = \frac{1}{\xi_{north}}$$

- The complex vector on the sphere is constructed as:

$$q^A = \frac{1+\xi\bar{\xi}}{2}(\delta_2^A + I\delta_3^A),\ \bar{q}_A q^A = 2$$

- The tensor fields are represented by spin-weighted functions using a computational version of the *eth* formalism (R. Gomez '97, N. Bishop '97)

The CCM Algorithm

- The unit sphere metric satify the relations:

$$q_{AB} = \frac{1}{2}(q_A \bar{q}_B + \bar{q}_A q_B),\ \det(h_{AB}) = \det(q_{AB})$$

- The conformal metric h_{AB} is encoded onto the functions:

$$J = \frac{1}{2} q^A q^B h_{AB},\ K = \frac{1}{2} q^A \bar{q}^B h_{AB},\ K^2 = 1 + J\bar{J}$$

- The surface area coordinate r is defined as:

$$r = \left(\frac{\det(\tilde{\eta}_{\tilde{A}\tilde{B}})}{\det(q_{AB})} \right)^{\frac{1}{4}}$$

- The expansion of the light rays propagating outwards is:

$$\beta = -\frac{1}{2} \log(r_{,\lambda})$$

- The other components of the Bondi-Sachs metric are:

$$\beta = -\frac{1}{2} \log(-\eta^{rr}),\ U \equiv U^A q_A = \frac{\eta^{rA}}{\eta^{ru}},\ W \equiv \frac{V - r}{r^2}$$

The CCM Algorithm

- We introduce auxiliary variables to eliminate the need of explicit second angular derivatives in the Bondi code and to avoid interpolating second derivatives:

$$\nu = \bar{\eth} J = \frac{1}{2} h_{AB,C} q^A q^B \bar{q}^C, \; B = \eth \beta = \beta_A q_A$$

$$k = \eth K = \frac{1}{2} h_{AB,C} q^A \bar{q}^B q^C + 2 \xi K$$

- The radial derivative of the Bondi-Sachs metric is required for the boundary data:

$$J, \beta, U, \partial_r U, W, \nu, k, B$$

- The *eth* correction between the affine and Bondi frame for the auxiliary angular variables on the world-tube is:

$$\eth F = \tilde{\eth} F - \frac{F_{,\lambda}}{r_{\lambda}} \tilde{\eth} r$$

The Bondi Code

- The inner working of the Bondi code had been described elsewhere (Bishop96, Gomez97, Lehner98)
- The variables of the code are:

$$J, \beta, U, \partial_r U, W, \nu, k, B$$

- Only J contains time derivatives and is updated with via an evolution stencil that involves two time-levels.
- The rest of the variables are integrated radially from the world-tube out to plus null infinity.
- The integration constraints are set by CCE.
- The U equation contains $U_{,rr}$ and requires two constants and the need to provide U and $U_{,r}$ at the worldtube.
- All the other radial integration equations are first differential order in r.

The News Algorithm

- The calculation of Bondi news is based on the algorithm developed by the Pitt team (N. Bishop '97) with modifications (Y. Zlochower '02, '03).
- The Bondi-Sachs metric close to infinity, in compactified coordinates $(u, x^A, l=1/r)$ is:

$$l^2 ds^2 = O(l^2) du^2 + 2 e^{2\beta} du\, dl - 2 h_{AB} U^B du\, dx^A + h_{AB} dx^A dx^B$$

- The metric variables have asymptotic expansions:

$$\beta = H + O(l^2),\ U^A = L^A + O(l),\ h_{AB} = H_{AB} + l c_{AB} + O(l^2)$$

- We define a complex vector so that: $F^{(A} \bar{F}^{B)} = H^{AB}$:

$$F^a = (0, F^A, 0),\ F^A = q^a \sqrt{(K_0+1)/2} - J_0 \bar{q}^A \sqrt{1/2 (K_0+1)}$$

- The metric functions J_0, K_0 and U_0 are:

$$J_0 = q^A q^B H_{AB}/2,\ K_0 = q^A \bar{q}^B H_{AB}/2,\ U_0 = q_A L^A$$

The News Algorithm

- We construct the News in a conformal Bondi frame that corresponds asymptotically to an inertial frame.
- We fix a null-tetrad on I^+ in inertial coordinates:
  $\tilde{n}=(1,0,0,0)$, $\tilde{l}=(0,0,0,1)$, $\tilde{m}=(1/\tilde{P}/2, I\tilde{P}/2, 0)$
- The Bondi News up to a phase factor $e^{-2I\Delta}$ is given by:

$$N=\frac{1}{2}\omega^{-2}e^{-2H}F^{A}F^{B}\left((\partial_u+L)c_{AB}-\frac{1}{2}c_{AB}D_C L^C+2\omega D_A[\omega^{-2}D_B(\omega e^{2H})]\right)$$

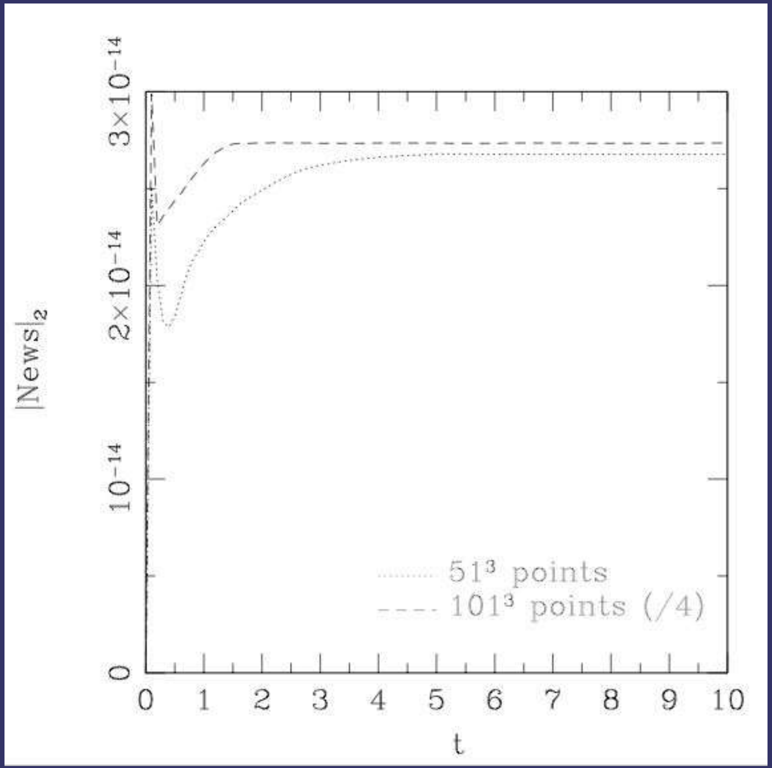
- The Bondy News is calculated in three steps:
  - first is evaluated, ignoring the phase factor, as a function of the evolution coordinates;
  - then is interpolated onto a fixed inertial angular grid and multiplied by the phase factor.
  - finally, in post-processing, the News is interpolated in time onto fixed inertial time slices.

Tests

- We apply the algorithm described above to four tests
- In the first set (Minkowsky and small perturbations of Minkowsky) we show the stability of the code, and the errors due to transformation between different coordinate systems and variables
- In the second set (Schwarzchild black hole in ingoing EF coordinates) we indicate the accuracy of the code in non-trivial space-times.
- In all cases the extracted gravitational wave signal should vanish identically.
- For the News, we plot the L_2 norm over the all grid points at I^+, without weighting by area element.
- For the Zerilli, we use a norm over all angles and we expect the signal to vanish for all angles, so this norm should be identically zero.
- To test convergence, all grids are scaled by factor 2.

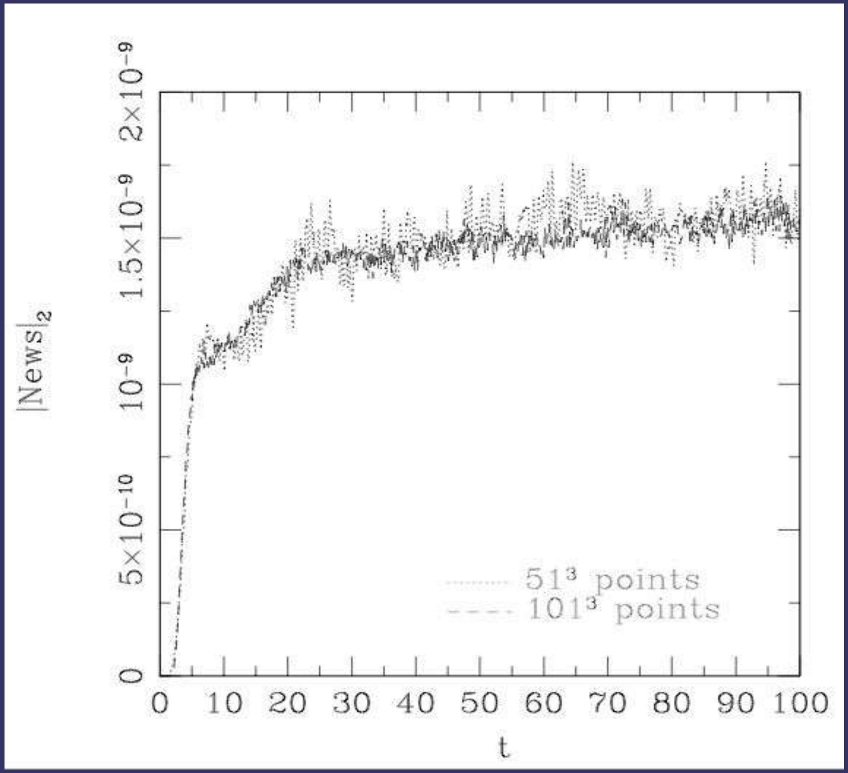
Tests: Flat Spacetime

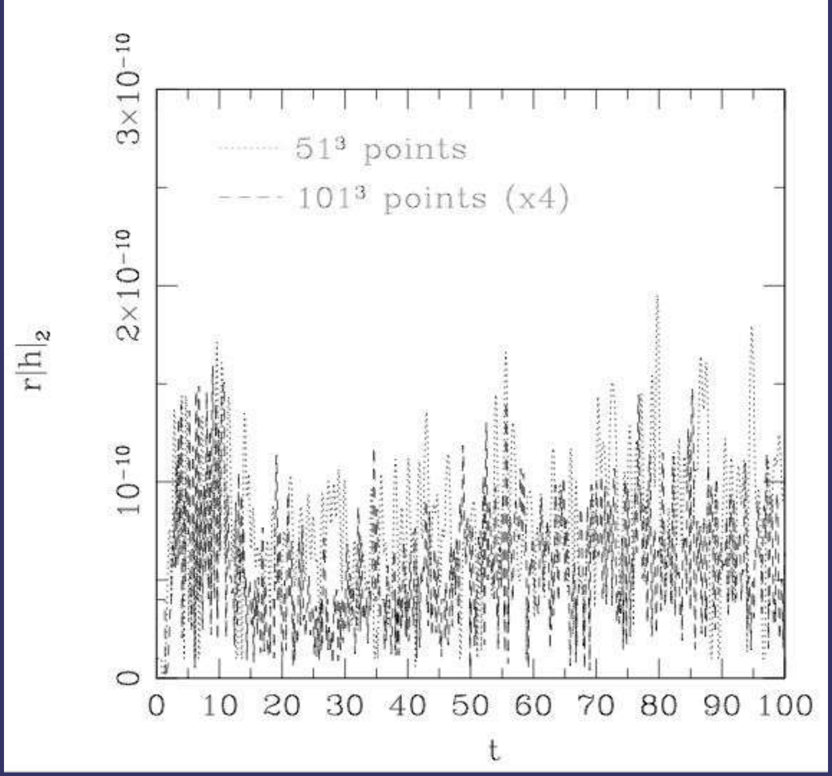
- Minkowski space in standard coordinates is evolved in the harmonic Abigel code (B. Szilagyi '04).
- The domain is [-10,10], the coarsest grid 50^3 (35^2x31) with the worldtube at r=7.
- The test indicates the round off error in transforming from Cauchy to Bondi variables on the world-tube
- News sensitivity to noise in Cauchy data is expected, and is not a problem.
- Zerilli method gives identically zero results.



Tests: Flat Spacetime

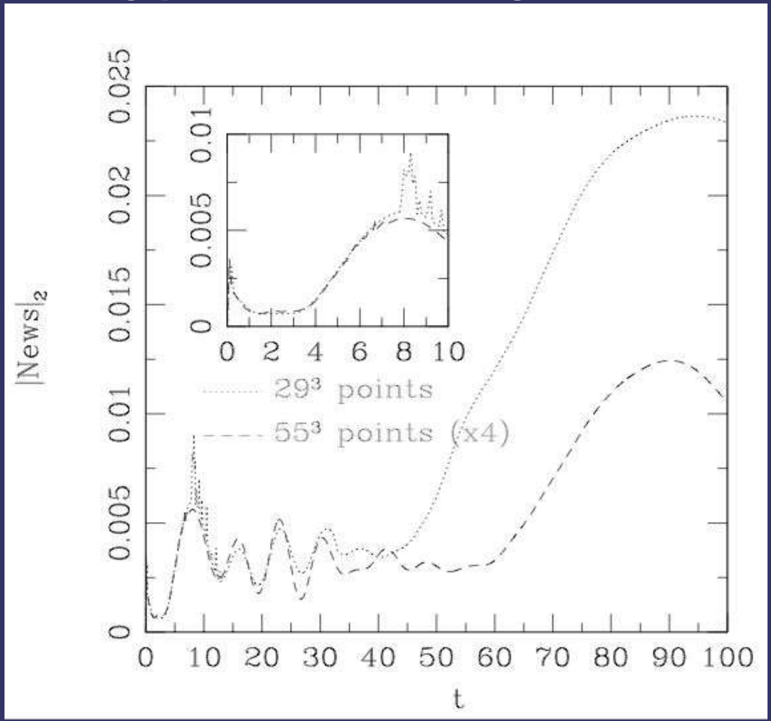
- Random perturbations around Monkowsky is a robust stability test for both evolution and extraction codes.
- The growth in the noise is independent of resolution showing the code stability against small perturbations.
- In contrast with the News, the Zerilli code gives a small-er wavesignal but does converge with resolution.

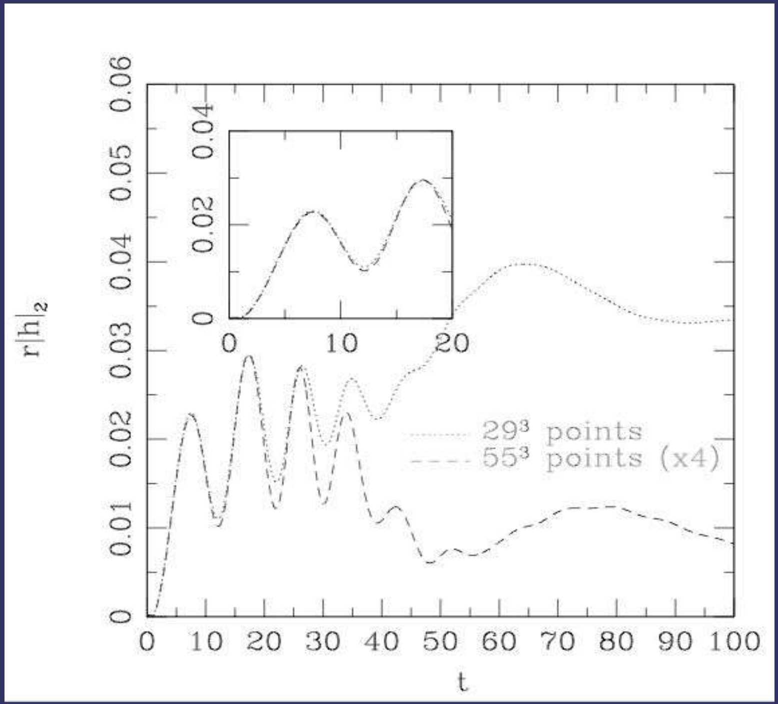




Tests: BH Spacetime

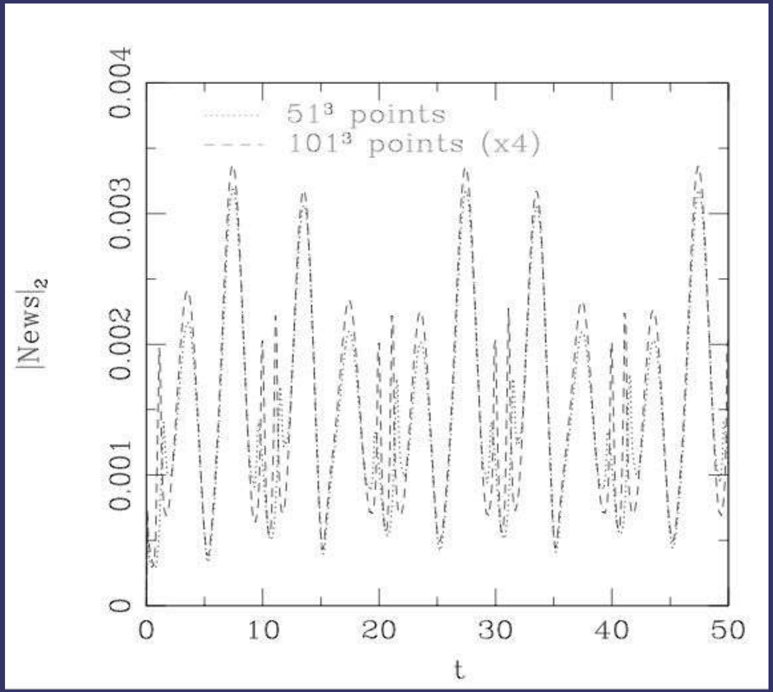
- BH in centered frame test uses the metric of Schwazchild black hole in ingoing Eddington-Finkenstein coordinates.
- The spacetime is evolved using the BSSN code.
- The convergent errors are caused by finite differencing errors in the strong fienld regime.
- The non-convergent errors are caused by the Sommer-feld type boundary conditions on the Cauchy grid.

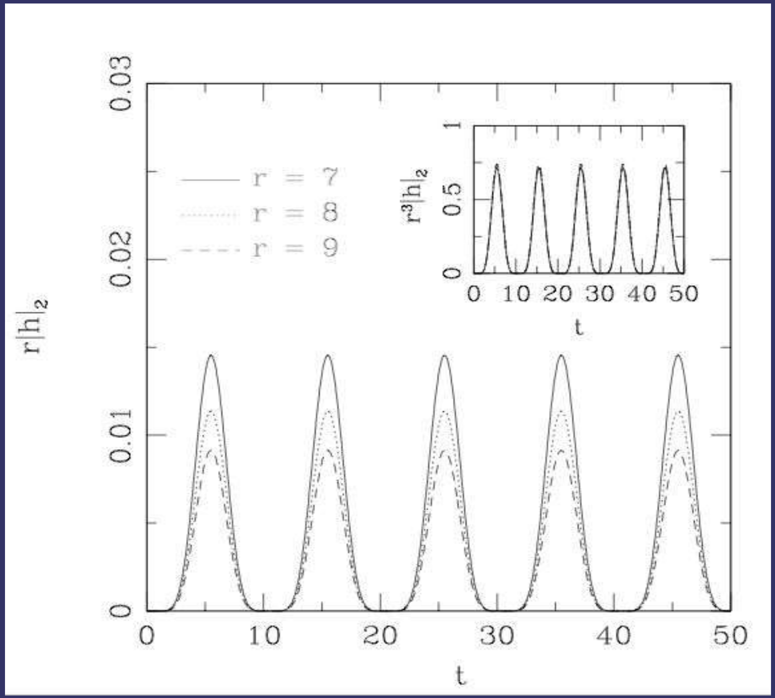




BH Spacetime

- BH in an oscillating frame test uses a moving coordinate frame given by: $t=t'$, $x^i=x^{i'}+B^i b(t)$, $b=\sin(\Omega(t-t_0))^3$.
- The metric is given analitically to avoid boundary effects
- The News converges to zero as it should. However, Zerilli extraction does not give a signal that converge to zero with the grid resolution refinement.





Conclusions

- We have demonstrated that it is possible to numerically extract the gravitational news at null infinity with existing codes in full 3D numerical relativity.
- The interface at the world-tube between Cauchy and characteristic codes works with BSSN and harmonic formulations (M. Alcubierre '02, B. Szilagyi '02)
- The extracted news is stable against small perturbations on the Cauchy grid.
- In non-trivial black hole spacetimes the code performs as expected.
- The results of this paper show that as a stand alone wave extraction method it produces the correct results in situations where other methods, such as Zerrilli extraction, may fail to converge to zero due to the pure gauge motion of the center of the mass.

Conclusions

- However, one limitation on the Cauchy-characteristic extraction is that the method is extremely sensitive to errors on the Cauchy slice, particularly those introduced by inconsistent Cauchy boundary conditions.
- Another limitation, inherent in characteristic formulations, is that the code will break down if the coordinate system forms caustics, that is, if the world-tube is not time-like, then caustics form on the tube itself.
- Finally, these tests show the limitations of the current implementation of the code. Any angular dependence in the solution leads to effects that are poorly resolves by the characteristic code, especially where the stereographic patches overlap.
- We are inversigating the use of multiple patch implementation (Thornburgh '04)

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- We are grateful for the help of the Cactus team while importing the extraction code into the Cactus infrastructure

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