Binary black hole initial data

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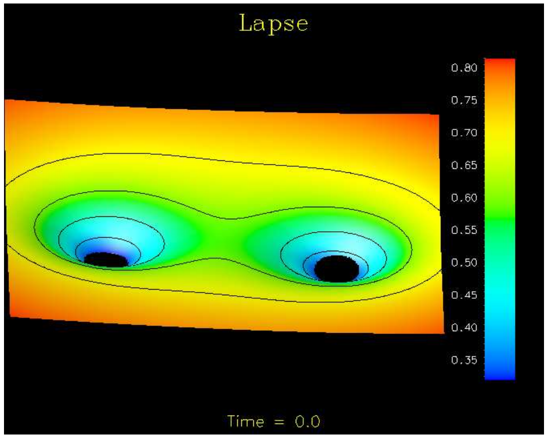


Relativistic Astrophysics Workshop, IPAM, UCLA, May 4, 2005

Black hole evolutions are important

See talks by

Joan Centrella
Frans Pretorius
Luis Lehner
Jerome Novak
Oscar Reula
Mark Scheel
Lee Lindblom
Matt Choptuik
Tvsi Piran
Saul Teukolsky

evolutions need initial data

Initial data for GR

- Initial data consists of **metric g_{ij}** and **extrinsic curvature K_{ij}** on one hypersurface Σ.

- It must **satisfy the constraints**

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j \left(K^{ij} - g^{ij}K\right) = 0$$

- **Challenges:**

1. How to solve the constraints
2. How to choose g_{ij} and K_{ij}

Outline of talk

1. *Construction of* **any** *initial data* – Conformal method
2. Some surprising properties – Non-uniqueness
3. *Construction of* **BBH** *initial data* – Quasi-equilibrium method
4. *Discussion & Thoughts* – How good is good enough?

Conformal method

Problem: Find solutions g_{ij}, K_{ij}
of the constraint equations

$$R + K^2 - K_{ij}K^{ij} = 0$$

$$\nabla_j\left(K^{ij} - g^{ij}K\right) = 0$$

Strategy: Split g_{ij} and K_{ij} into smaller pieces, some
freely specifiable, the rest **completely determined**.
Specifically, $g_{ij} = \psi^4\tilde{g}_{ij}$

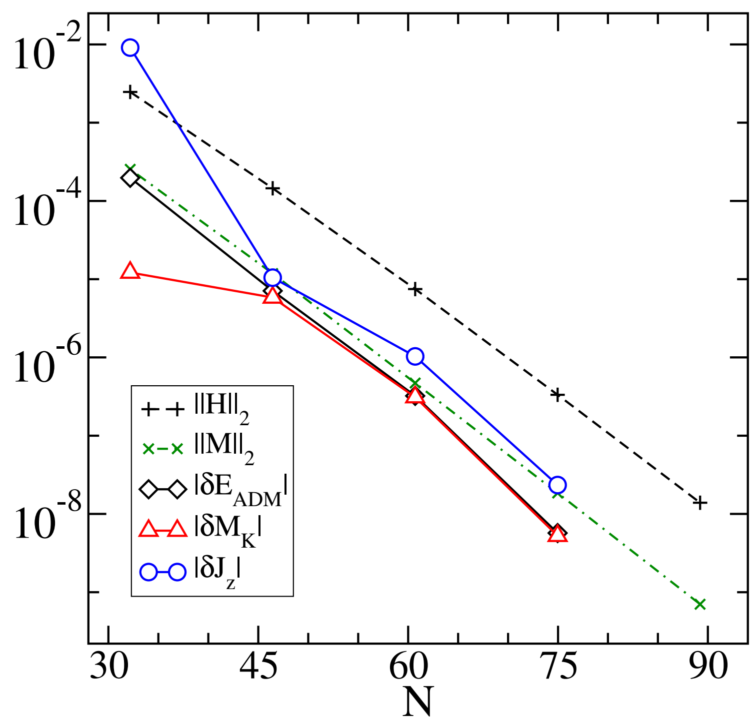
Choose free data $\Rightarrow$ **Solve** elliptic equations $\Rightarrow$ **Assemble** g_{ij}, K_{ij}

Numerical method: Spectral elliptic solver

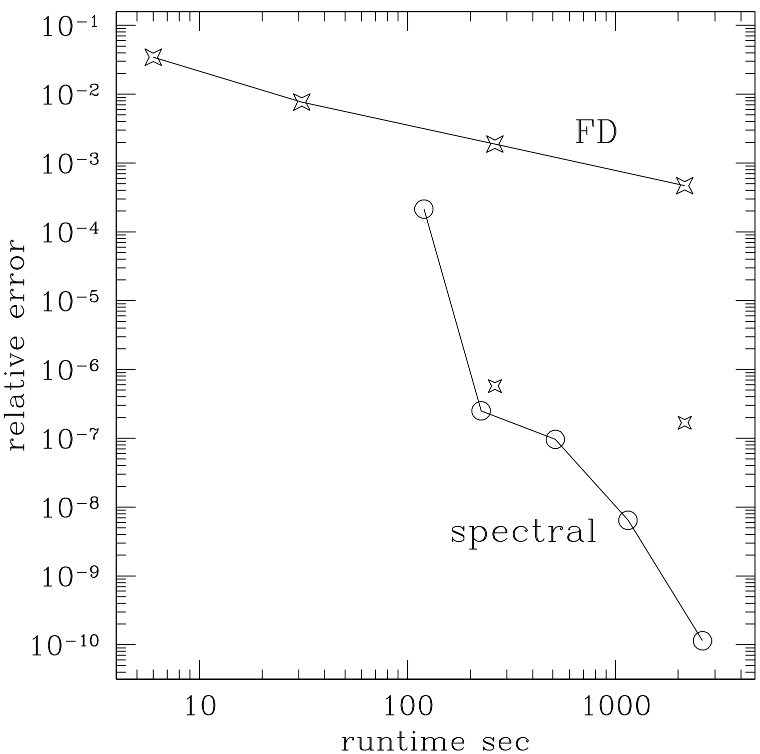
(HP, Kidder, Scheel, Teukolsky 2003)

Expand solution in basis-functions & solve for expansion-coefficients
Smooth solutions $\Rightarrow$ exponential convergence

- *Superior accuracy:* Numerical errors $\ll$ physical effects
- *Superior efficiency:* Large parameter studies
- *Domain decomposition:* Multiple length-scales



Binary black hole ID (Cook & HP, 2004)



Bowen-York ID (HP, Kidder *et al*, 2003)

Standard and extended conformal thin sandwich

Standard CTS

Free data $\tilde{g}_{ij}, \partial_t\tilde{g}_{ij}, K, \tilde{N}$

Four elliptic equations

$$\tilde{\nabla}^2\psi - \frac{1}{8}\tilde{R}\psi - \frac{1}{12}K^2\psi^4 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}\mathbb{L}\beta^{ij}\right) - \frac{2}{3}\psi^6\tilde{\nabla}^i K - \tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}\tilde{u}^{ij}\right) = 0$$

+ Quite a few existence and uniqueness results (especially for $K = 0$)

Extended CTS

Free data $\tilde{g}_{ij}, \partial_t\tilde{g}_{ij}, K, \partial_t K$

Five elliptic equations

$$\tilde{\nabla}^2\psi - \frac{1}{8}\tilde{R}\psi - \frac{1}{12}K^2\psi^4 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij} = 0$$

$$\tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}\mathbb{L}\beta^{ij}\right) - \frac{2}{3}\psi^6\tilde{\nabla}^i K - \tilde{\nabla}_j\left(\frac{1}{2\tilde{N}}\tilde{u}^{ij}\right) = 0$$

$$\tilde{\nabla}^2(\tilde{N}\psi^7) - \tilde{N}\psi^7\left(\frac{1}{8}\tilde{R} + \frac{5}{12}K^2\psi^4 + \frac{7}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right) = -\psi^5(\partial_t - \beta^k\partial_k)K$$

– Mathematical *terra incognita*

Some results on the standard system

Mathematics:

1. asymptotically flat
2. no inner boundaries
3. maximal slice $K = 0$

$\longrightarrow$ Yamabe constant $\mathcal{Y}[\tilde{g}_{ij}]$:

$\mathcal{Y}[\tilde{g}_{ij}] > 0 \Leftrightarrow$ existence & uniqueness

(Cantor 1977, Murray & Cantor 1981, Maxwell 2005)

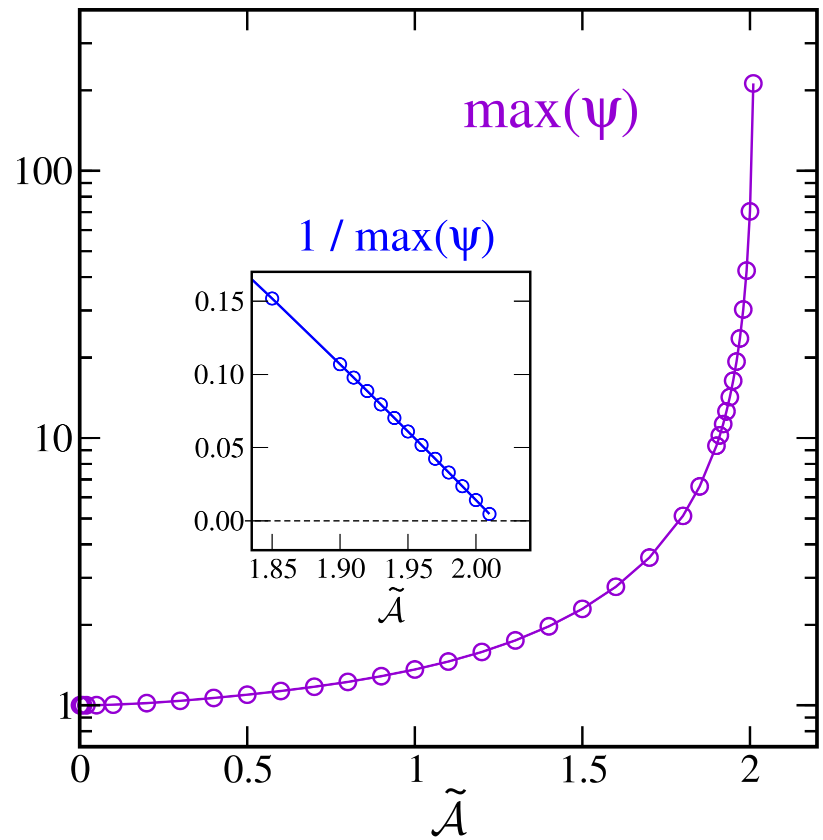
Free data based on "Teukolsky wave" ingoing, $M=0$, odd parity, centered at $r=20$

(HP, Kidder, Scheel, Shoemaker 2005)

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{A}\, h_{ij}$$

$$\tilde{u}_{ij} = \tilde{A}\, \partial_t h_{ij}$$

$$K = 0, \quad \tilde{N} = 1$$

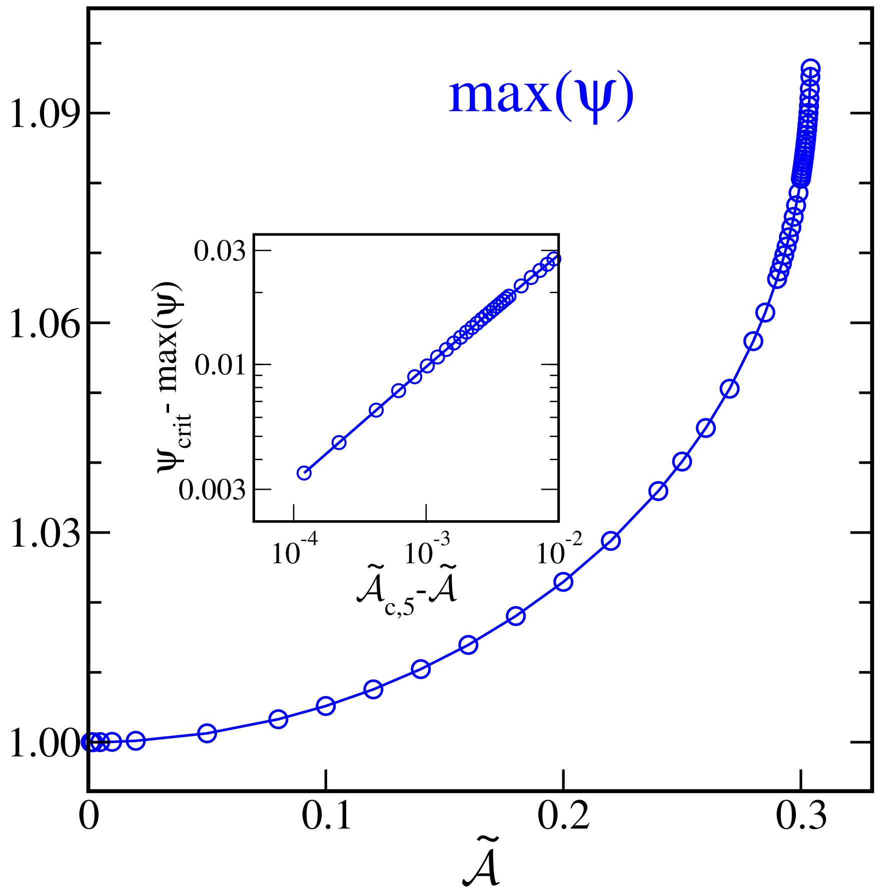


Extended system (HP & York, gr-qc/0504142)

$$\tilde{g}_{ij} = \delta_{ij} + \tilde{\mathcal{A}}\, h_{ij}$$

$$\tilde{u}_{ij} = \tilde{\mathcal{A}}\, \partial_t h_{ij}$$

$$K = 0, \quad \partial_t K = 0$$

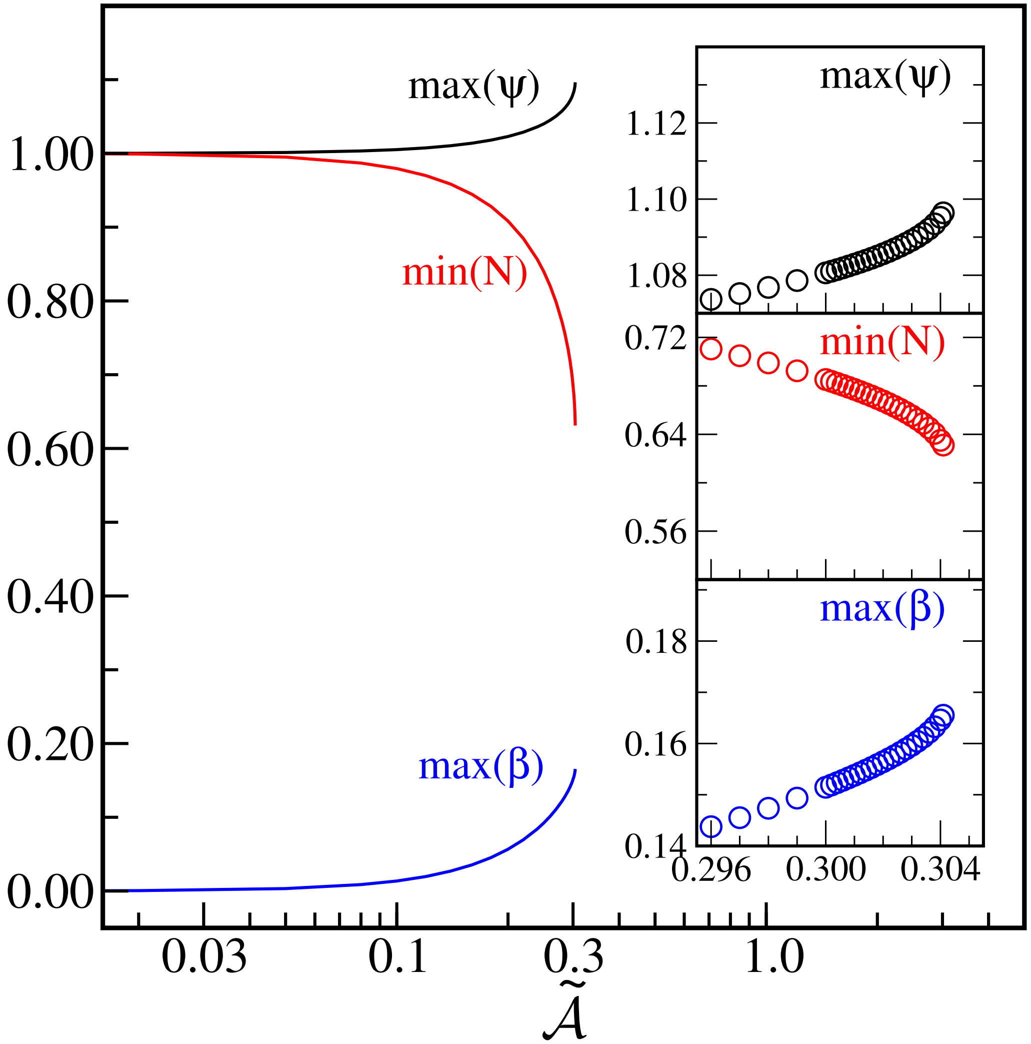


ψ **finite** as $\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}_c$

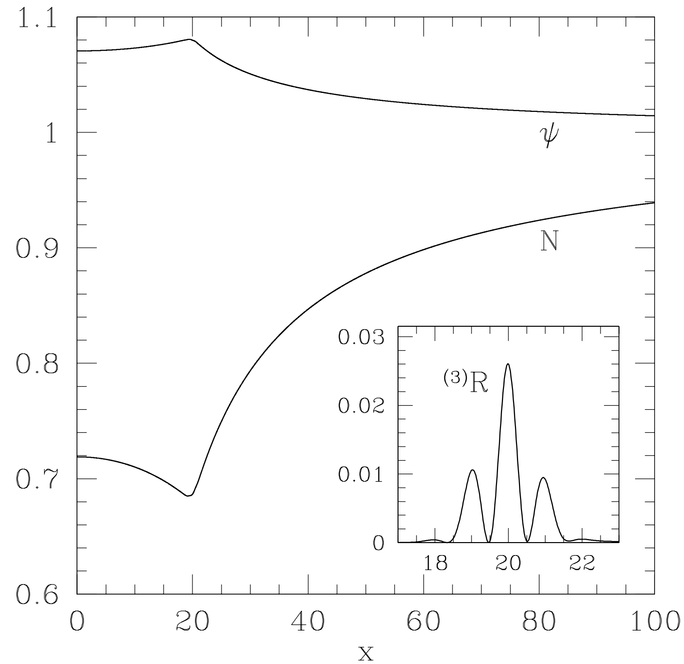
Parabolic behavior

$\psi \approx \psi_c - \text{const.}(\delta\tilde{\mathcal{A}})^{1/2}$

A more comprehensive look



Equatorial plane, $\tilde{\mathcal{A}} = 0.3$



A second branch

- So far $\quad \mathbf{u}_-(\tilde{\mathcal{A}}) = \mathbf{u}_c - \mathbf{v}_c\sqrt{\delta\tilde{\mathcal{A}}}\,, \qquad \mathbf{u} = \Big(\psi, \beta^i, N\Big)$

- *Two* branches??

$$\mathbf{u}_\pm(\tilde{\mathcal{A}}) = \mathbf{u}_c \pm \mathbf{v}_c\sqrt{\delta\tilde{\mathcal{A}}}$$

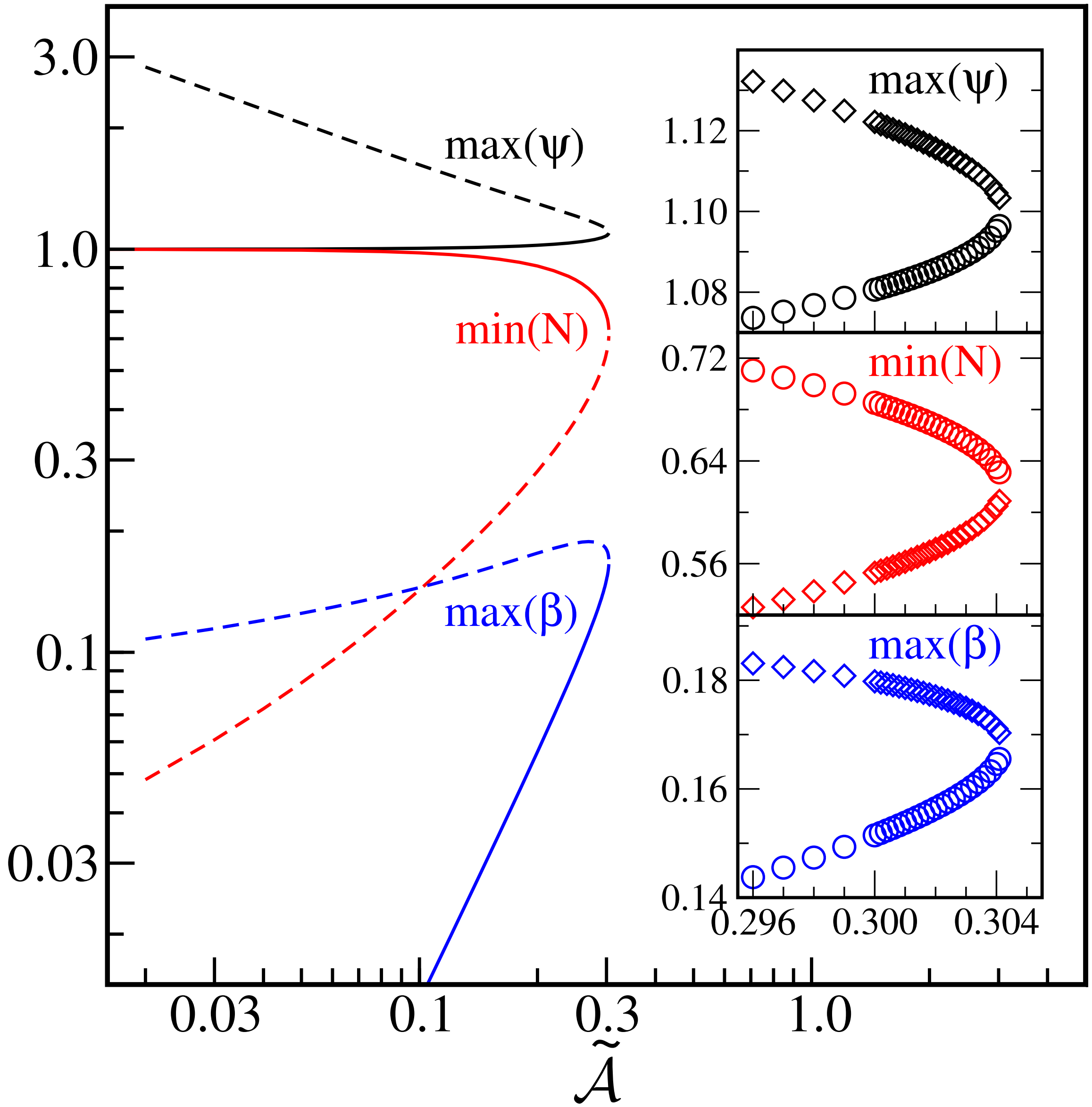
- Problem: With "simple" initial guess, elliptic solver converges always to $\mathbf{u}_-$; need good guess to converge to $\mathbf{u}_+$.

$$\frac{\mathrm{d}\mathbf{u}_-(\tilde{\mathcal{A}})}{\mathrm{d}\tilde{\mathcal{A}}} = \frac{1}{2\sqrt{\delta\tilde{\mathcal{A}}}}\mathbf{v}_c$$

$$\Rightarrow \mathbf{u}_+(\tilde{\mathcal{A}}) \approx \mathbf{u}_-(\tilde{\mathcal{A}}) + 4\,\delta\tilde{\mathcal{A}}\,\frac{\mathrm{d}\mathbf{u}_-(\tilde{\mathcal{A}})}{\mathrm{d}\tilde{\mathcal{A}}}$$

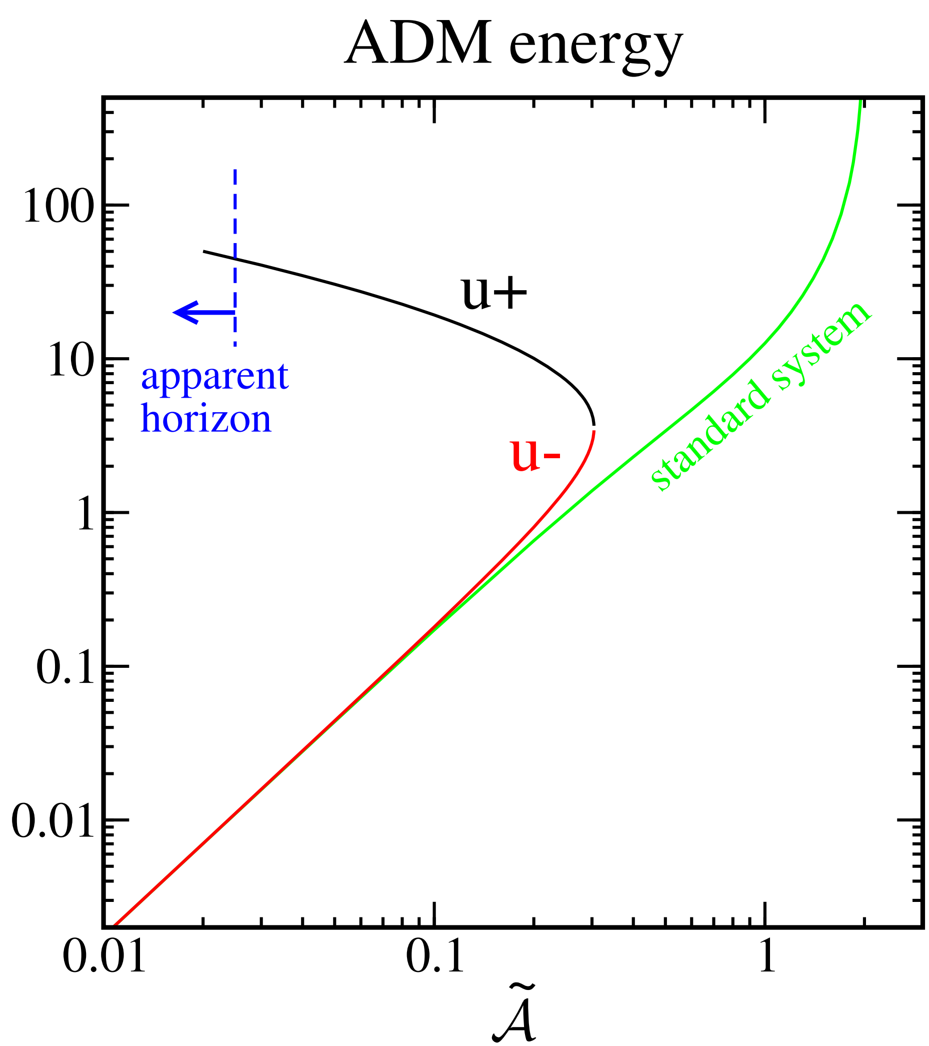
- Take two numeric solutions $\mathbf{u}_-$ of five coupled 3-D nonlinear elliptic equations, and **finite-difference** them !!

Constructing the upper branch $\mathbf{u}_+$

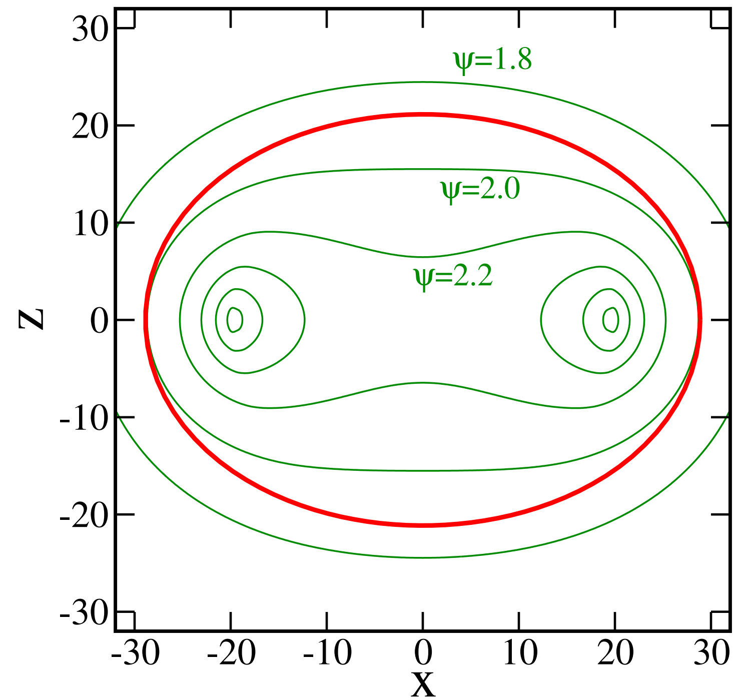


- Parabolic close to $\tilde{\mathcal{A}}_c$
- $\mathbf{u}_+$ and $\mathbf{u}_-$ meet at $\tilde{\mathcal{A}}_c$
- $\mathbf{u}_+$ deviates strongly from Minkowski at small $\tilde{\mathcal{A}}$
- **Two solutions for arbitrarily small $\tilde{\mathcal{A}}$!!**

Energy & Apparent horizon

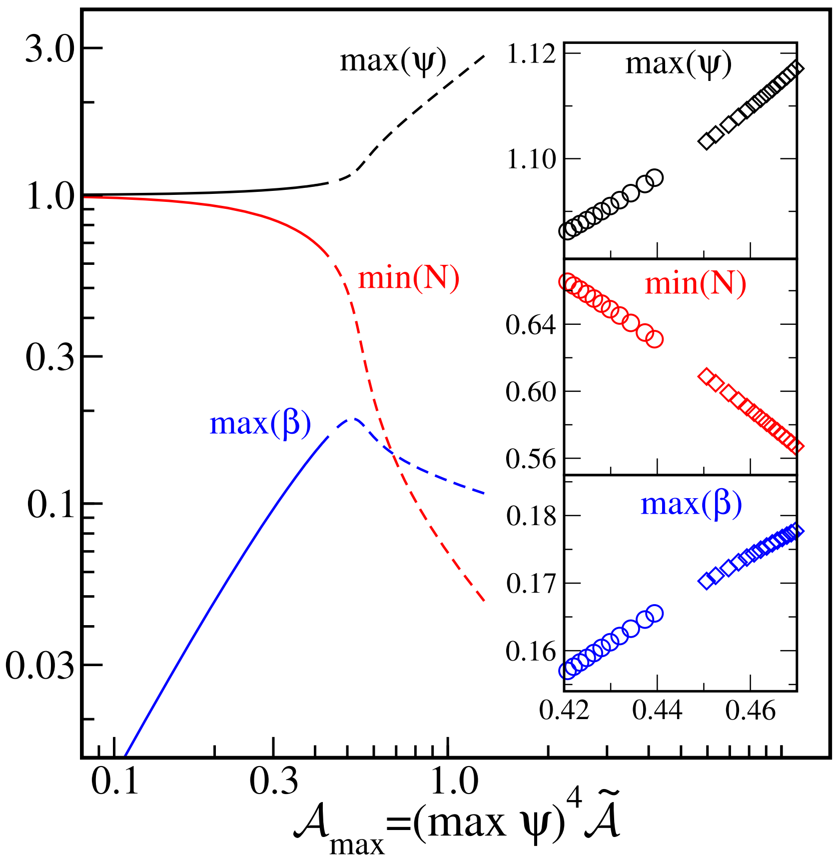


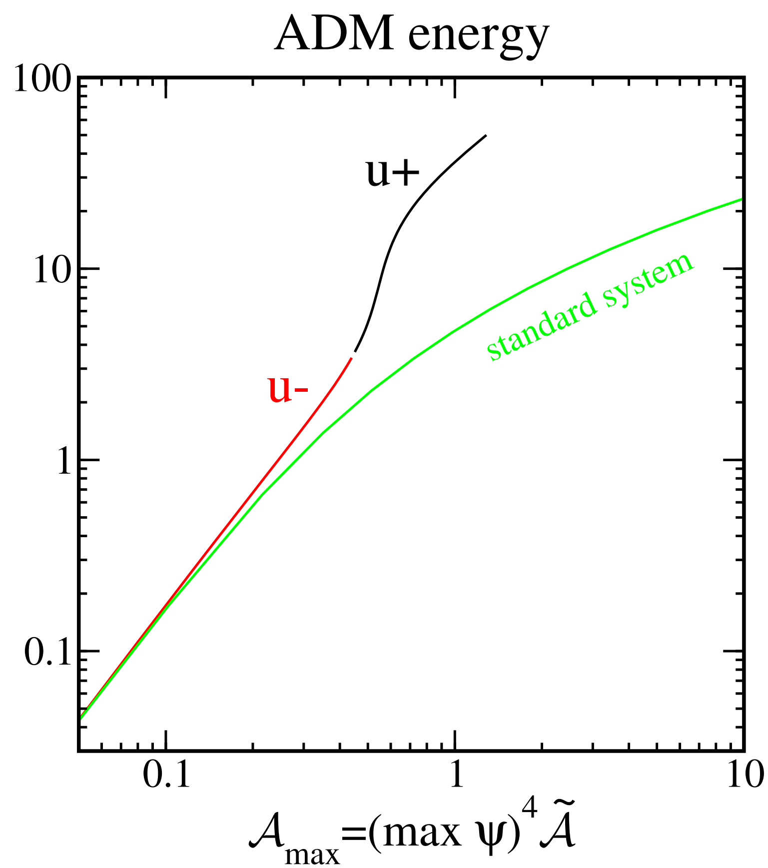
Apparent



Unique solutions, nevertheless?

- *Physics* is determined by g_{ij}, K_{ij}. For example $g_{ij} = \psi^4 \tilde{g}_{ij} = \psi^4 \delta_{ij} + \psi^4 \tilde{\mathcal{A}}\ h_{ij}$
- *Physical* amplitude of perturbation is $\mathcal{A} = \psi^4 \tilde{\mathcal{A}}$
- Question: For given *physical* amplitude $\mathcal{A}$, how many solutions exist?





Lessons from journey into mathematical wonderland

- Life is more interesting than expected (HP & York, gr-qc/0504142).
- Similar issues possible in any system containing the extended conformal thin sandwich equations ($\longrightarrow$ Jerome Novak's talk?).
- Non-unique data sets are very different from BBH initial data. "Just solving" works, so **let's go!**

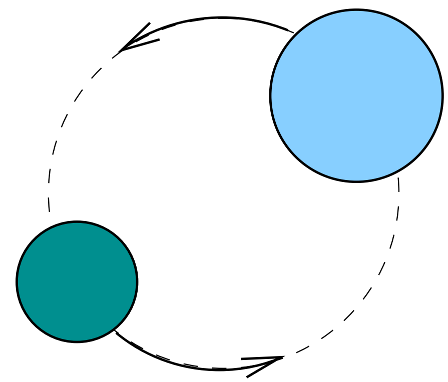
Astrophysically realistic BBH initial data

Question: How to choose free data and boundary conditions for a binary black hole in a circular orbit?

Traditional approach (conformally flat Bowen-York) has many problems:
ISCO wrong, BHs plunge too quickly, ...
Need better method!

Quasi-equilibrium method

Basic idea:
Approx. time-independence in corotating frame
Approx. helical Killing vector
(both concepts essentially equivalent,
both useful depending on context)



History:

- Wilson & Matthews 1985: Binary neutron stars
- Gourgoulhon, Grandclement & Bonazzola, 2002a,b
  BBH ID with inner boundary conditions
  (basically right, but various details deficient)
- Cook & HP, 2002, 2003, 2004
  General quasi-equilibrium method with isolated horizon BCs

Quasi-equilibrium method (the easy pieces)

- **Time-independence in corotating frame**

  $\Rightarrow$ natural choice: vanishing time derivatives

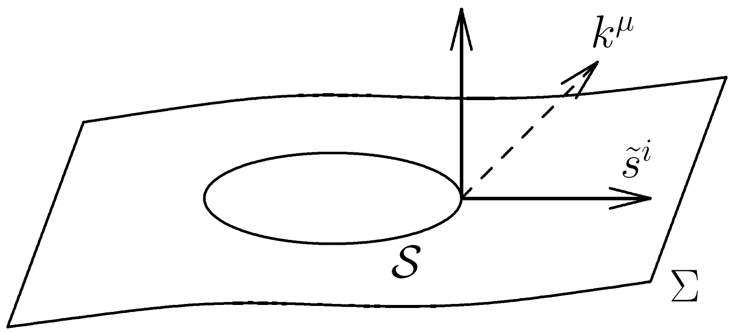
- **Extented conformal thin sandwich formalism**
  1. $\partial_t \tilde{g}_{ij} = 0 = \partial_t K$
  2. $\tilde{g}_{ij}$ and K still undetermined

- **Boundary conditions at infinity** from asymptotic flatness & corotation:

$$\psi = 1$$
$$\beta^i = (\vec{\Omega}_{\rm orbital} \times \vec{r})^i$$
$$N = 1$$

- **New contribution**: *inner boundary conditions* (next slide...)

Quasi-equilibrium excision boundary conditions



k^μ – outward pointing null normal to $\mathcal{S}$
$\tilde{s}^i$ – (conformal) spatial normal to $\mathcal{S}$

- **Excise** topological sphere(s) $\mathcal{S}$

- **Require**
  1. $\mathcal{S}$ be apparent horizon(s)
  2. When evolved, the coordinate locations of the AH's remain stationary
  3. The shear $\sigma_{\mu\nu}$ of k^μ vanishes

- Item **3** is an **isolated horizon condition**. It implies for the expansion θ

$$\mathcal{L}_k\theta = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} = 0 \quad \text{on } \mathcal{S}$$

$\Rightarrow$ AH moves along k^μ, and its area is constant (initially)

Quasi-equilibrium excision boundary conditions cont'd

- **Rewrite** in variables of conformal thin sandwich

$$\partial_r\psi = \dots \qquad \text{on } \mathcal{S} \tag{1}$$

$$\beta^i = \psi^2 N\tilde{s}^i + \beta^i_{\|} \qquad \text{on } \mathcal{S} \tag{2}$$

Boundary conditions for ψ and β^i

- **Rotating black holes**
"Vanishing shear" restricts $\beta^i_{\|}$. Freedom to specify spin of BH remains.

- Lapse boundary condition not fixed by IH (also Jaramillo *et al*, 2004).

- Determine orbital frequency by $E_{\rm ADM} = M_K$.

Numerical solutions: Single black holes I

- $\tilde{g}_{ij}$, K, $\mathcal{S}$ not determined. and lapse-BC.
- For now **arbitrary choices:** Conformal flatness, $\mathcal{S}$ =sphere
- **Do not** use knowledge of single BH solutions – use single BHs to **test** method

Spherical symmetry

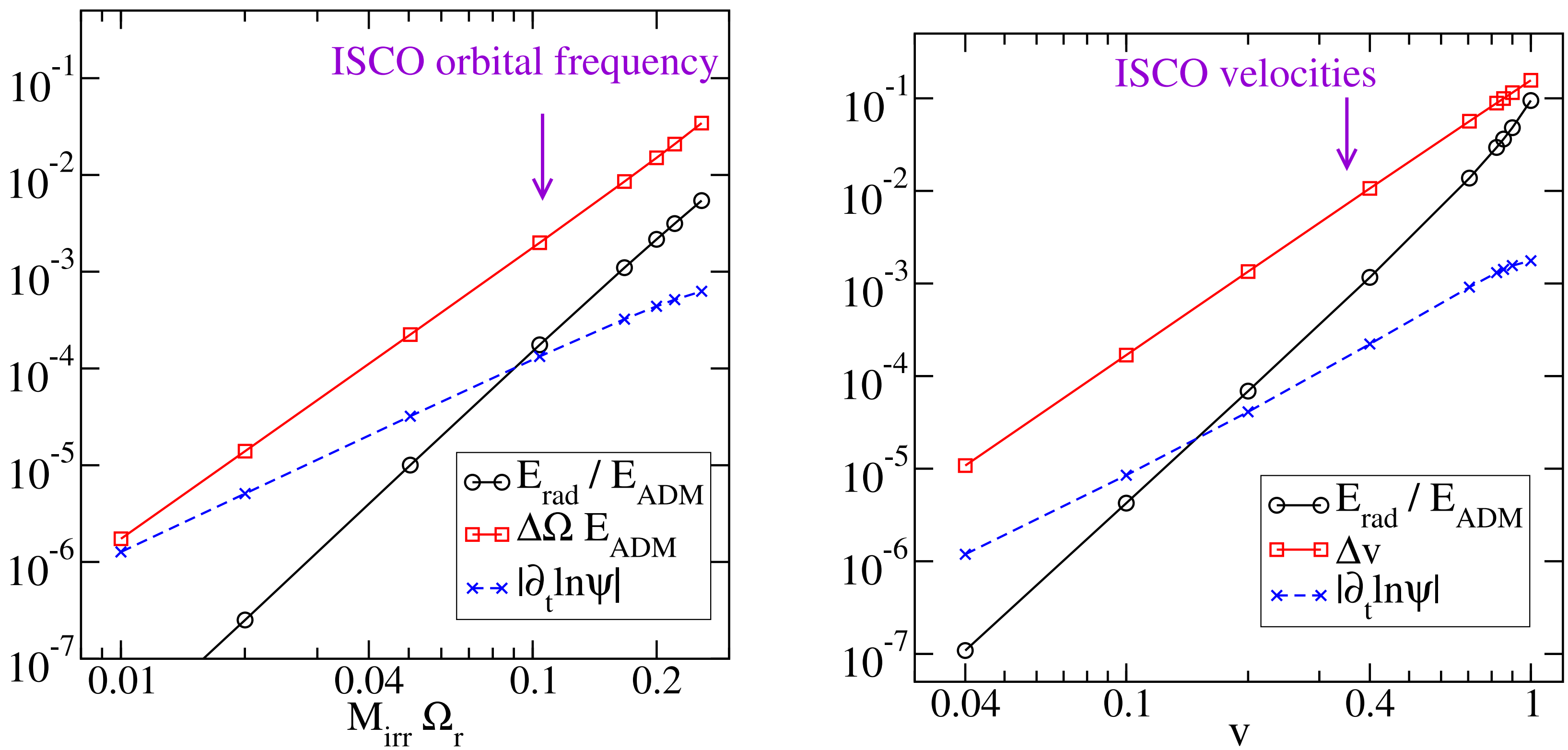
1. w.l.o.g. conformally flat
2. Try different choices for K and lapse boundary condition
3. *any* spherically symmetric K and *any* spherically symmetric lapse-BC yield:
   – exact slice through Schwarzschild
   – totally vanishing time-derivatives $\partial_t g_{ij} = \partial_t K_{ij} = 0$
4. **Full success**: Recover Schwarzschild independent of arbitrary choices.

Numerical solutions: Single black holes II

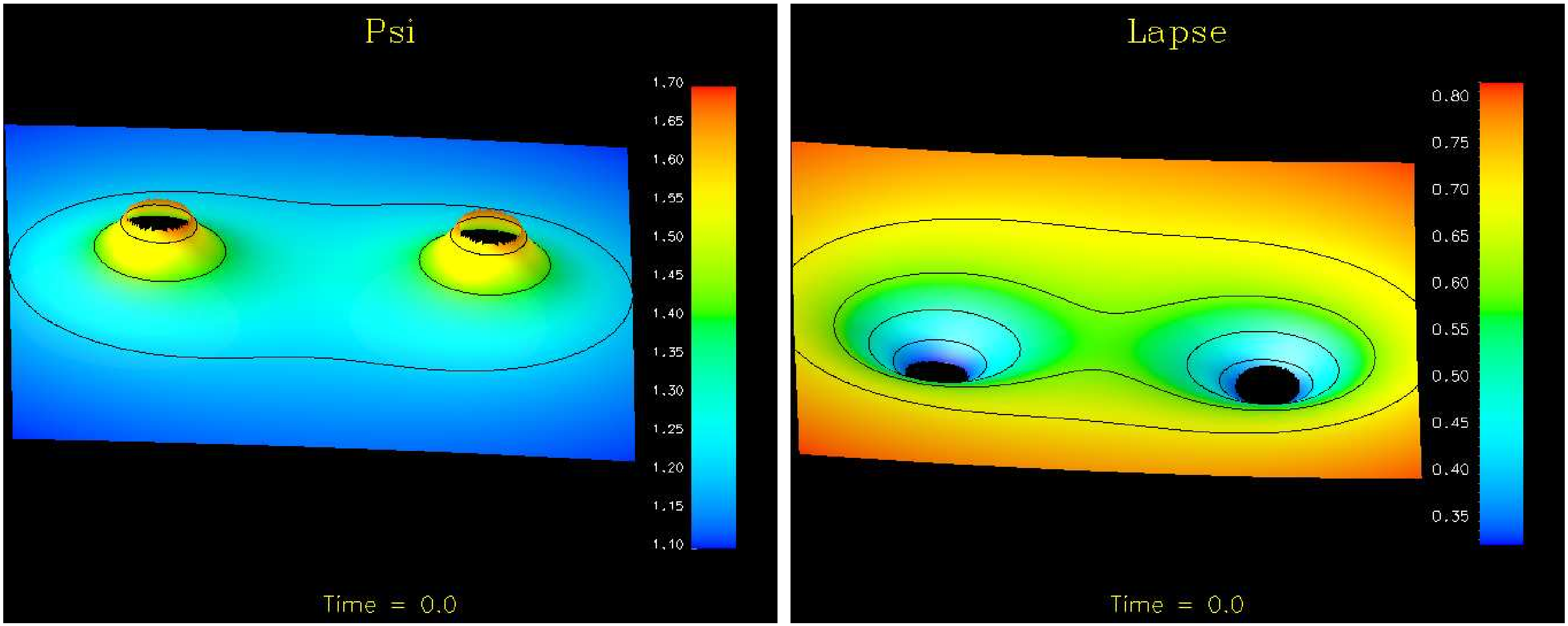
- **Spinning/Boosted black holes**
  Compute quantities that vanish for Kerr:

$$E_{rad} \equiv \sqrt{E_{\rm ADM}^2 - P_{\rm ADM}^2} - \sqrt{M_{\rm irr}^2 + J_{\rm ADM}^2/(4M_{\rm irr}^2)} \qquad (3)$$

$$\Delta\Omega \equiv \Omega_r - \frac{J_{\rm ADM}/E_{\rm ADM}^3}{2 + 2\sqrt{1 - J_{\rm ADM}^2/E_{\rm ADM}^4}}, \qquad \Delta v \equiv v - \frac{P_{\rm ADM}}{E_{\rm ADM}} \qquad (4)$$



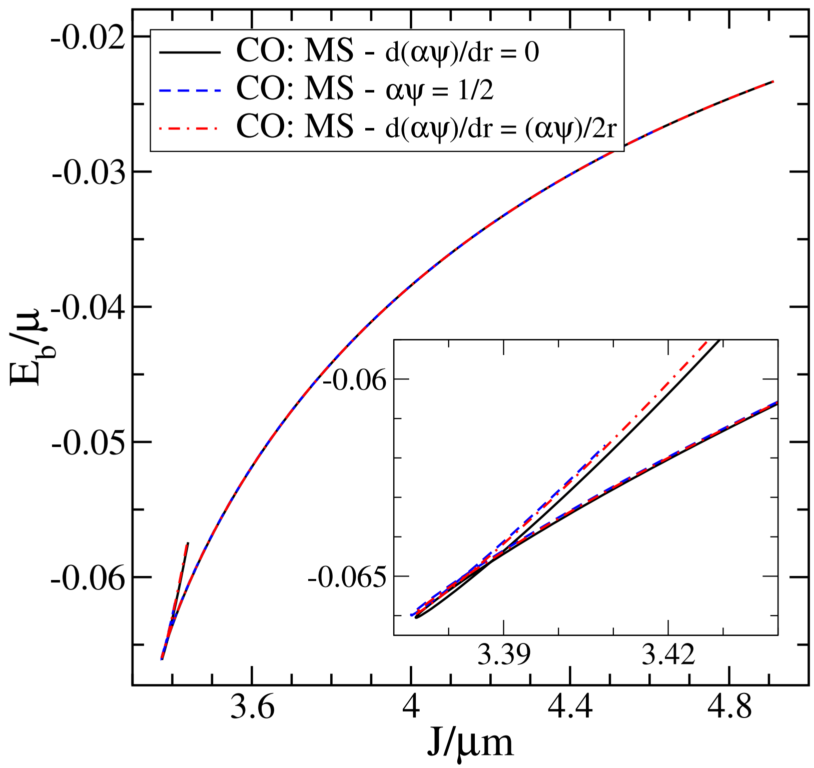
Binary black hole solutions (corotating, $K = 0$)



Lapse positive through horizon

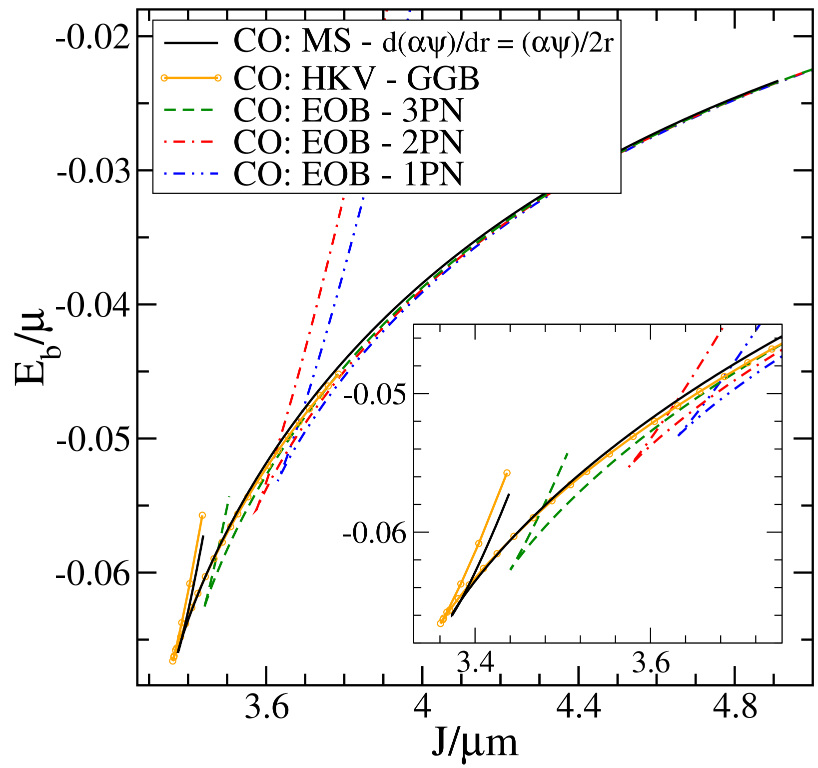
Sequences of quasi-circular orbits (corotating, $K = 0$)

Three different lapse boundary conditions



No difference – solution robust

Compare to GGB and post-Newtonian results

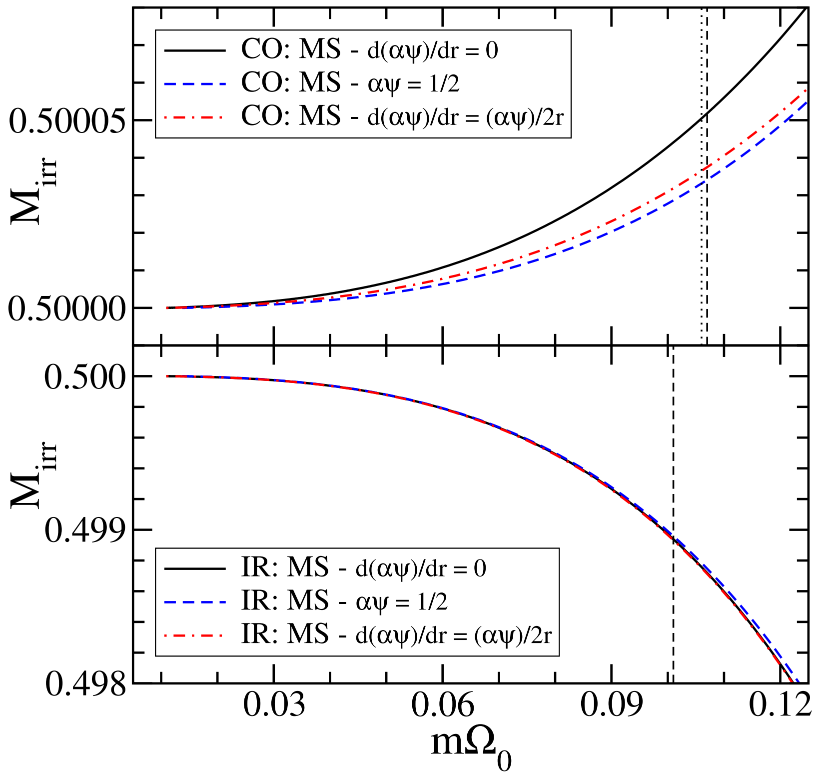


Excellent agreement

Testing the 2nd law

Normalize sequences such that $dE_{\text{ADM}} = \Omega_0 \, dJ_{\text{ADM}}$

Irreducible mass along these sequences



⇐ corotating sequences
(three different lapse BC's)
M_{irr} slightly increasing during inspiral – ok

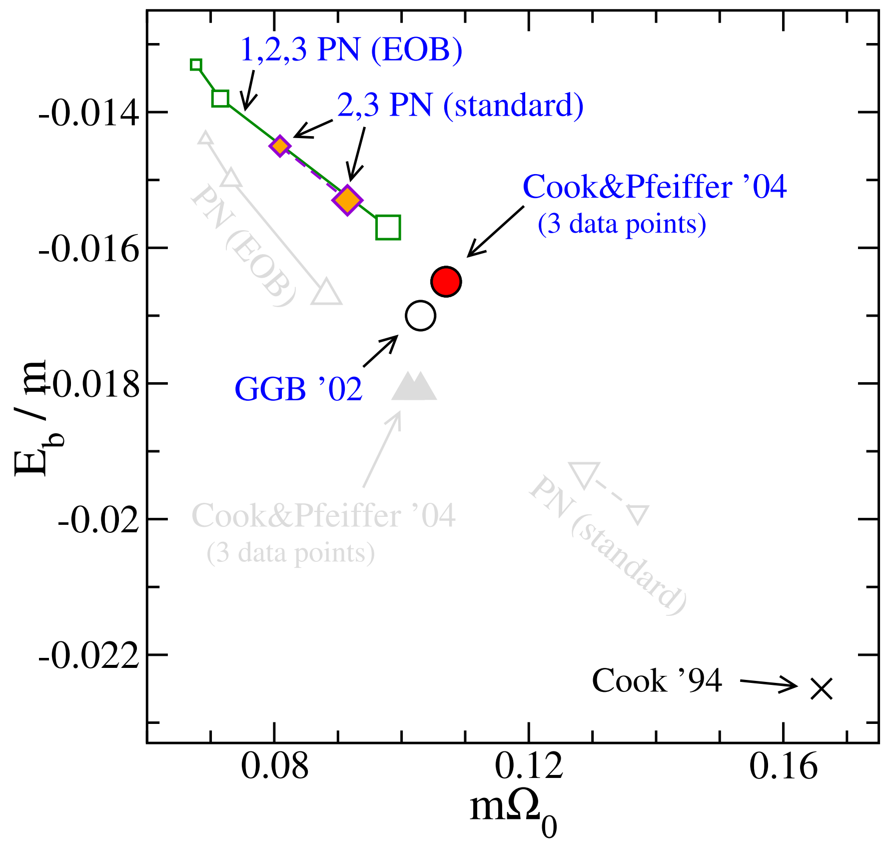
⇐ irrotational sequences
(three different lapse BC's)
M_{irr} decreasing during inspiral
– Normalization of sequences wrong?
– Remaining free data insufficient?
– Rigidity theorem?

ISCO location

Caution: ISCO is not a sharp, well-defined concept! Anyway...

Color: Corotating BH's

Grey: Irrotational BH's



- Excellent agreement between NR and PN
- Superior to Bowen-York w/ effective potential (cf. Cook 1994)

Qualitity of todays QE-BBH initial data

Contains two black holes	of course
ISCO agrees with PN predictions	yes
ID-solve results in lapse & shift	yes
BHs at rest early in evolution	yes
Time-derivatives small	$\psi/\partial_t\psi \sim 50T_{\rm orbit}$ $50T_{\rm orbit} \gg T_{\rm inspiral}$ at large separation!
Spins incorporated exactly?	no
Tidal distortion of BHs correct?	no
Satisfies testmass limit?	no
Correct $\dot{r}$ (slightly negative)?	no
Contains wavetrain of earlier insipral?	no

$\Rightarrow$ superior to Bowen-York, but room for further improvement

How good is good enough?

Just some thoughts and conjectures — further opinions welcome!!

How good is good enough?

Answer depends on application and desired accuracy.

Short term – improving evolution codes:

- Testing evolution codes
  ID doesn't matter.
  (But smaller initial transient may well be advantageous for dynamic gauge conditions)
- Qualitative features of plunge wave-forms (⟶ Frans Pretorius' talk)
  My feeling is that ID won't matter (must be demonstrated, though!).
- The first evolution of two orbits and plunge
  ID must contain BHs in circular-ish orbits. I believe this is the case (no proof).
  Such evolutions must quantify initial transient caused by initial data.

Long term – gravitational wave science

1. If ID contains significant initial transients, then one must wait until they are beyond wave-extraction radius $R_{\rm extract}$

$$T_{\rm wait} \approx 2R_{\rm extract} \approx 100M \sim 10^5 \ldots 10^6 \text{CPU-hours}$$

   Expensive. But let's assume we're willing to pay...

2. Initial transient **changes** coordinatesystem, masses, spins, orbital elements.
   Accumulated phase-error very sensitive to (e.g.) eccentricity ε.
   How to measure ε in a dynamical spacetime in an unknown coordinatesystem?

This matters for

- Testing PN-results

- Parameter extraction
  May be limited by our knowledge of evolution-parameters.

- *Coherent* template PN-inspiral+numerical evolution+ringdown
  Most stringent test of strong field GR.
  Total phase error must be (much) smaller than QN-period. (LISA SMBH removal!)

Summary

- Non-unique solutions exist (and may bite us)
- Quasi-equilibrium initial data vast improvement over Bowen-York...
- ... and contains handles for next round of improvements $(\tilde{g}_{ij}, \mathcal{S})$
- Further improvements essential, especially when evolutions mature
- QE-initial data publicly available soon.

Summary of QE method

- **Framework** for BBH initial data in a kinematical setting (helical Killing vector)
- Explicitly displays the **remaining choices** $\tilde{g}_{ij}, K, \mathcal{S}$, Lapse-BC
- Close in spirit to GGB, but greatly improved:
  1. Constraints are satisfied
  2. Incorporates isolated horizon boundary conditions
  3. General spins possible
  4. Retains freedom to choose any $\tilde{g}_{ij}, K, \mathcal{S}$.
  5. Lapse is positive on horizon
- **Agrees** very well with **PN** (even with simple choices)
- Data sets will be **publicly available** soon.