Fully-constrained formulation of Einstein's field equations using Dirac gauge

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Outline

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   - Motivation for a fully-constrained scheme
2. Description of the formulation and strategy
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   - Einstein equations in Dirac gauge and maximal slicing
   - Spherical coordinates and tensor components
   - Integration strategy
3. Numerical implementation and results
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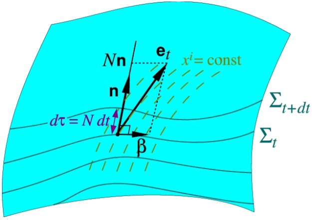
   - Evolution of gravitational wave spacetimes
   - Models of rotating neutron stars
   - Current limitations and new strategy

Decomposition of spacetime and of Einstein equations



Evolution equations:

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2 N K_{ik} K^k_{\ j} +$$
$$N \left[K K_{ij} + 4\pi((S - E)\gamma_{ij} - 2 S_{ij}) \right]$$
$$K^{ij} = \frac{1}{2N} \left(\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i \right).$$

Constraint equations:

$$R + K^2 - K_{ij} K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\, (dx^i + \beta^i dt)\, (dx^j + \beta^j dt)$$

Constraint violation

If the constraints are verified for initial data, evolution should preserve them. Therefore, one could in principle solve Einstein equations without solving the constraints

$$\Downarrow$$

Appearance of constraint violating modes

However, some cures have been (are) investigated :

- solving the constraints at (almost) every time-step ...
- constraints as evolution equations (Gentle *et al.* 2004)
- constraint-preserving boundary conditions (Lindblom *et al.* 2004 & presentation by M. Scheel)
- relaxation (Marronetti 2005)
- constraint projection (presentation by L. Lindblom)

Some reasons not to solve constraints

computational cost of usual elliptic solvers ...

few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems

definition of boundary conditions at finite distance and at black hole excision boundary

Motivations for a fully-constrained scheme

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Summary

"Alternate" approach (although most straightforward)

- partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
- fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams *et al.* (1994), Choptuik *et al.* (2003)

⇒Rather popular for 2D applications, but disregarded in 3D
Still, many advantages:

- constraints are verified!
- elliptic systems have good stability properties
- easy to make link with initial data
- evolution of only two scalar-like fields ...

Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura formalism)

Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

Problem

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$
not always easy to deal with tensor densities... not *really* covariant!

Introduction of a flat metric

We introduce f_{ij} (with $\dfrac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Define:

$$\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij}$$

with

$$\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$$f := \det f_{ij}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

$\Rightarrow$*no more tensor densities: only tensors.*

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

Generalized Dirac gauge

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

Divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.

Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}_{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Generalized Dirac gauge properties

- h^{ij} is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, h^{ij} is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of h^{ij}
- ${}^3R$ does not contain any second-order derivative of h^{ij}
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[(\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the Conformal Flat Condition (CFC) verifies the Dirac gauge $\Rightarrow$possibility to easily use initial data for binaries now available

Einstein equations

Dirac gauge and maximal slicing ($K = 0$)

Hamiltonian constraint

$$\begin{aligned}\Delta(\Psi^2 N) \quad = \quad & \Psi^6 N \left(4\pi S + \frac{3}{4}\tilde{A}_{kl}A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2\Bigg[N\Big(\frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} \\ & -\frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k \ln\Psi\Big) + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N\Bigg]\end{aligned}$$

Momentum constraint

$$\begin{aligned}\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) \quad = \quad & 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12 N A^{ij}\mathcal{D}_j \ln\Psi - 2\Delta^i{}_{kl} N A^{kl} \\ & -h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l\end{aligned}$$

Trace of dynamical equations

$$\Delta N = \Psi^4 N\left[4\pi(E+S) + \tilde{A}_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N$$

Einstein equations

Dirac gauge and maximal slicing ($K = 0$)

Evolution equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_{\beta}\frac{\partial h^{ij}}{\partial t} + \pounds_{\beta}\pounds_{\beta} h^{ij} = \mathcal{S}^{ij}$$

6 components - 3 Dirac gauge conditions - $\left(\det \tilde{\gamma}^{ij} = 1\right)$

2 degrees of freedom

$$-\frac{\partial^2 \chi}{\partial t^2} + \Delta\chi = S_\chi$$

$$-\frac{\partial^2 \mu}{\partial t^2} + \Delta\mu = S_\mu$$

Spherical coordinates and components

Choice for f_{ij} : spherical polar coordinates

- stars and black holes are of spheroidal shape
- compactification made easy (only r)
- use of spherical harmonics
- grid boundaries are smooth surfaces

Use of spherical orthonormal triad (tensor components)

- Dirac gauge can easily be imposed
- boundary conditions for excision might be better formulated
- asymptotically, it is easier to extract gravitational waves

Representation of h^{ij}

Introduction of $\bar{h}^{ij}$ as the transverse-traceless part of h^{ij}, with only two degrees of freedom:

First Dirac condition $\mathcal{D}_i h^{ir} = 0$

$$\frac{\partial \bar{h}^{rr}}{\partial r} + \frac{3\bar{h}^{rr}}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0$$

with

$$\bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi}\right)$$

$$\bar{h}^{r\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta}\right)$$

Knowing $\bar{h}^{rr}$ and μ, it is possible to deduce $\bar{h}^{r\theta}$ and $\bar{h}^{r\varphi}$ from the first Dirac condition.
$\Rightarrow \bar{h}^{\theta\varphi}$ and $\bar{h}^{\varphi\varphi}$ from the other two gauge conditions
$\Rightarrow \bar{h}^{\theta\theta}$ from the trace-free condition.

Integration procedure

Everything is know on slice Σ_t

$\Downarrow$

Evolution of $\chi = r^2 \bar{h}^{rr}$ and μ to next time-slice Σ_{t+dt} (+ hydro)

$\Downarrow$

Deduce $\bar{h}^{ij}(t+dt)$ from Dirac and trace-free conditions

$\Downarrow$

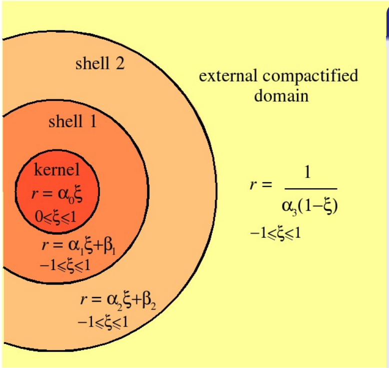
Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.

$\Downarrow$

Iterate on the system of elliptic equations for $N, \Psi^2 N$ and β^i on Σ_{t+dt}

Multidomain 3D decomposition

numerical library Lorene (http://www.lorene.obspm.fr)



Decomposition:

Chebyshev polynomials for ξ, Fourier or Y_l^m for the angular part (θ, ϕ),

- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs $\Rightarrow$boundary conditions are well imposed

Drawback: Gibbs phenomenon!

Constrained formulation of Einstein equations

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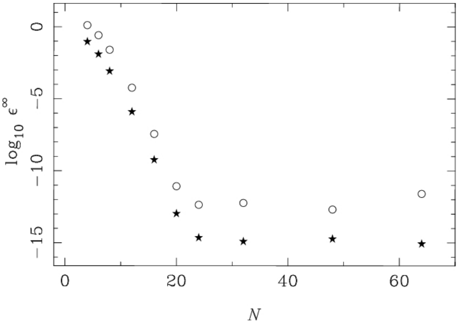
Summary

$f(x) = \cos^3\left(\frac{\pi}{2}x\right) - \frac{1}{8}(x+1)^3$ over $[-1;1]$, represented by a series of N Chebyshev polynomials ($T_n(x) = \cos(n \arccos x)$)



Error decaying as e^{-N}; as for comparison, more than 10^5 points are necessary with a third-order finite-difference scheme ...

Solutions of Poisson and wave equations

The angular part of any field ϕ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta, \varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

$\Delta \phi = \sigma$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ (exponential decay)

$\Box \phi = \sigma$

$$\left[1 - \frac{\delta t^2}{2} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2} \right) \right] \phi_{\ell m}^{J+1} = \sigma_{\ell m}^J$$

Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell, m)$ the operator inversion $\iff$ inversion of a $\sim 30 \times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

Boundary conditions

Poisson-type PDEs

Thanks to the compactification of type $u = 1/r$, it is possible to impose asymptotic flatness "exactly"

Wave-type PDEs

Standard compactification cannot apply ⇒choice of transparent boundary conditions for $\ell \leq 2$ at finite distance, in the linear regime.

$$\forall \ell, \ \left. \frac{\partial \phi_{\ell m}}{\partial r} + \frac{\partial \phi_{\ell m}}{\partial t} + \frac{\phi_{\ell m}}{r} \right|_{r=R} = \zeta_{\ell m}(t)$$

with $\zeta(t, \theta, \varphi)$ being a function verifying a wave-like equation on the outer-boundary sphere

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{3}{4R^2} \Delta_{\theta\varphi} \zeta + \frac{3}{R} \frac{\partial \zeta}{\partial t} + \frac{3\zeta}{2R^2} = \frac{1}{2R^2} \Delta_{\theta\varphi} \left(\frac{\phi}{R} - \left. \frac{\partial \phi}{\partial r} \right|_{r=R} \right)$$

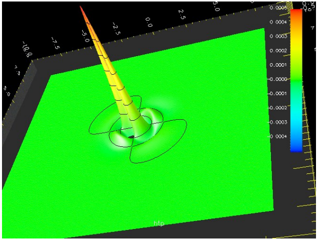
Results with a pure gravitational wave spacetime

Initial data

Similar to Baumgarte & Shapiro (1999), namely a momentarily static ($\partial \tilde{\gamma}^{ij} / \partial t = 0$) Teukolsky (1982) wave $\ell = 2$, $m = 2$:

$$\begin{cases} \chi(t=0) = \dfrac{\chi_0}{2} r^2 \exp\left(-\dfrac{r^2}{r_0^2}\right) \sin^2\theta \sin 2\varphi \\ \mu(t=0) = 0 \end{cases} \quad \text{with } \chi_0 = 10^{-3}$$

Preparation of the initial data by means of the *conformal thin sandwich* procedure



Evolution of $h^{\phi\phi}$ in the plane $\theta = \frac{\pi}{2}$

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How well are equations solved?

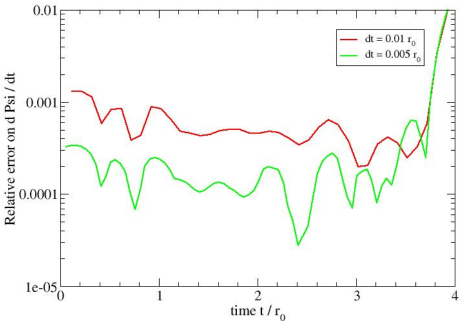
Constraints

- Imposed numerically at every time-step!
- depends on spectral resolution & number of iterations
- keep the error below 10^{-6}

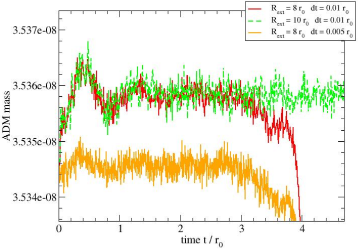
Evolution equations

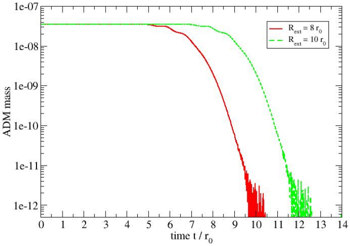
- only two out of six are solved
- check on the others: equation for Ψ

$$\frac{\partial \Psi}{\partial t} = \beta^k \mathcal{D}_k \Psi + \frac{\Psi}{6} \mathcal{D}_k \beta^k$$



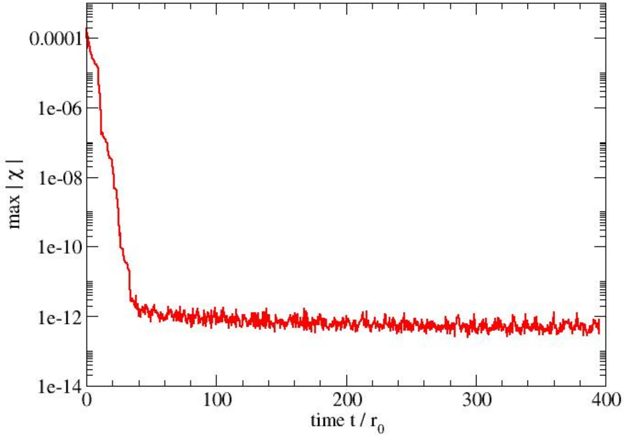
Evolution of ADM mass
Are the boundary conditions efficient?





- ADM mass is conserved up to 10^{-4}
- main source of error comes from time finite-differencing
- the wave is let out at better than 10^{-4}

Long-term stability



40000 time-steps

Physical model of rotating neutron stars

Code developed for

- self-gravitating perfect fluid in general relativity
- two Killing vector fields (axisymmetry + stationarity)
- Dirac gauge
- equilibrium between matter and gravitational field
- equation of state of a relativistic polytrope $\Gamma = 2$

Considered model here:

- central density $\rho_c = 2.9\rho_{nuc}$
- rotation frequency $f = 641.47$ Hz $\simeq f_{\text{Mass shedding}}$
- gravitational mass $M_g \simeq 1.51 M_\odot$
- baryon mass $M_b \simeq 1.60 M_\odot$

Equations are the same as in the dynamical case, replacing time derivative terms by zero

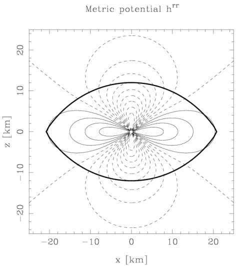
Comparison with quasi-isotropic gauge

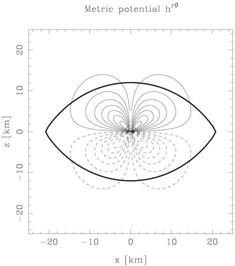
Other code using quasi-isotropic gauge has been used for a long time and successfully compared to other codes in Nozawa *et al.* (1998).

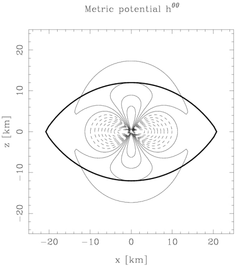
Global quantities

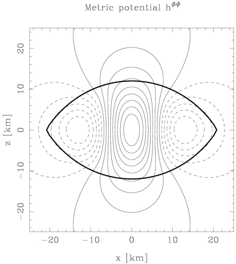
Quantity	q-isotropic	Dirac	rel. diff.
$N(r=0)$	0.727515	0.727522	10^{-5}
$M_g\ [M_\odot]$	1.60142	1.60121	10^{-4}
$M_b\ [M_\odot]$	1.50870	1.50852	10^{-4}
$R_{\rm circ}$ [km]	23.1675	23.1585	4×10^{-4}
$J\ \left[GM_\odot^2/c\right]$	1.61077	1.61032	3×10^{-4}
Virial 2D	1.4×10^{-4}	1.5×10^{-4}	
Virial 3D	2.5×10^{-4}	2.1×10^{-4}	

Virial identities (2 & 3D) are covariant relations that should be fulfilled by any stationary spacetime; they are not imposed numerically.









For all components (except $h^{r\varphi}$ and $h^{\theta\varphi}$, which are null), $h^{ij}_{\mathrm{max}} \sim 0.005$
$\Rightarrow$comparable with $\gamma_{\theta\theta} - \gamma_{\varphi\varphi}$ in quasi-isotropic gauge

Problems for large amplitudes

In both gravitational wave evolution and neutron star models, there are convergence problems for

$$h^{ij} \gtrsim 0.01$$

In the neutron star case, these are very compact (in practice on the unstable branch) models *and* close to mass-shedding limit

Possible reason:

1. if χ and μ are supposed regular (obtained from the PDE solver)
2. $h^{r\theta}, h^{r\varphi} \sim \partial_r \chi$
3. $h^{\theta\theta}, h^{\theta\varphi} \sim \partial_r^2 \chi$
4. S^{ij} (source of the equation for h^{ij}) $\sim \partial_r^4 \chi$!

New integration strategy

Taking tensor spherical harmonics from Zerilli (1970), it is possible to express the tensor wave/Poisson equation for h^{ij} and the Dirac gauge condition in terms of quantities expandable on Y_ℓ^m.

1. solve for potentials linked with $h^{\theta\theta}$ and $h^{\theta\varphi}$
2. from the θ- and φ- components of the Dirac gauge, integrate to get η and μ
3. from the remaining Dirac condition, integrate to get h^{rr}
4. the trace is obtained from the requirement that $\det \tilde{\gamma}^{ij} = 1$ as before

Works fine for the simplified case of divergence-free vector Poisson equation.

Summary

- We have developed and implemented a fully-constrained evolution scheme for solving Einstein equations, using a generalization of Dirac gauge and maximal slicing
- Easy to extract gravitational radiation (asymptotical TT gauge + spherical grid)
- Well tested for $h^{ij} \lesssim 0.01$, corresponding to most of relevant astrophysical scenarios without a black hole

- Ongoing work and outlook
  - Try to reduce the number of radial derivatives in our scheme and replace them by integrations
  - Already possible applications to core collapse ("Mariage Des Maillages" project) or study of oscillations of relativistic stars
  - Compatible with no-radiation approximations: *e.g.* Schäfer & Gopakumar (2004); useful for slow evolution studies of inspiralling compact binaries

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