Probing the N-Body Problem for Fun & Profit

Piet Hut

Institute for Advanced Study, Princeton

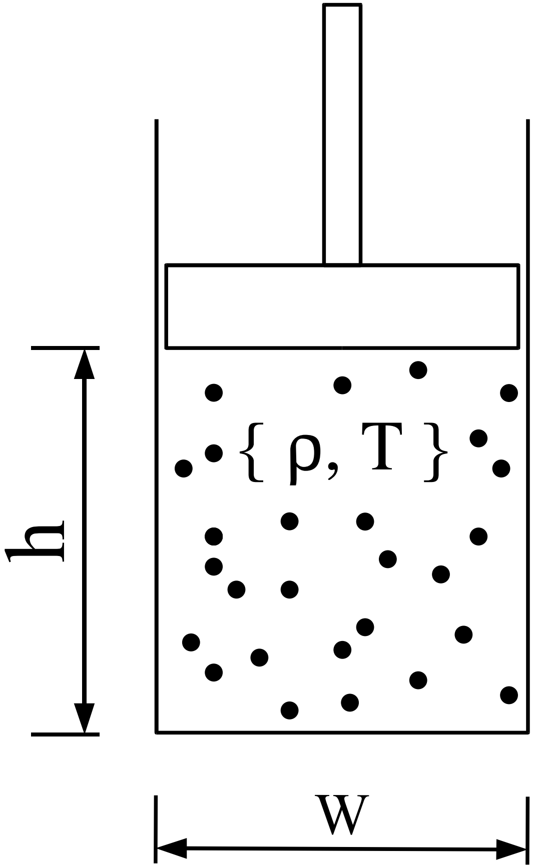
Workshop on “N-Body Problems in Astrophysics”, 18-22 April 2005
Grand Challenge Problems in Computational Astrophysics II
Institute for Pure and Applied Mathematics, UCLA

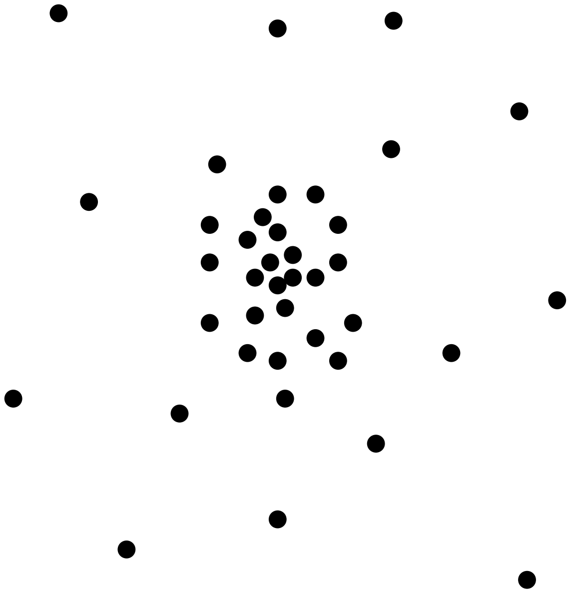
The Gravitational N-body Problem

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathbf{r}_i &= \nabla\mathbf{r}_i \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{G\ M_j}{|\mathbf{r}_j - \mathbf{r}_i|} \\
&= G\sum_{\substack{j=1\\ j\neq i}}^{N} M_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}
\end{aligned}
$$

For $N>2$: the oldest unsolved problem in mathematical physics

Near-Equilibrium Processes



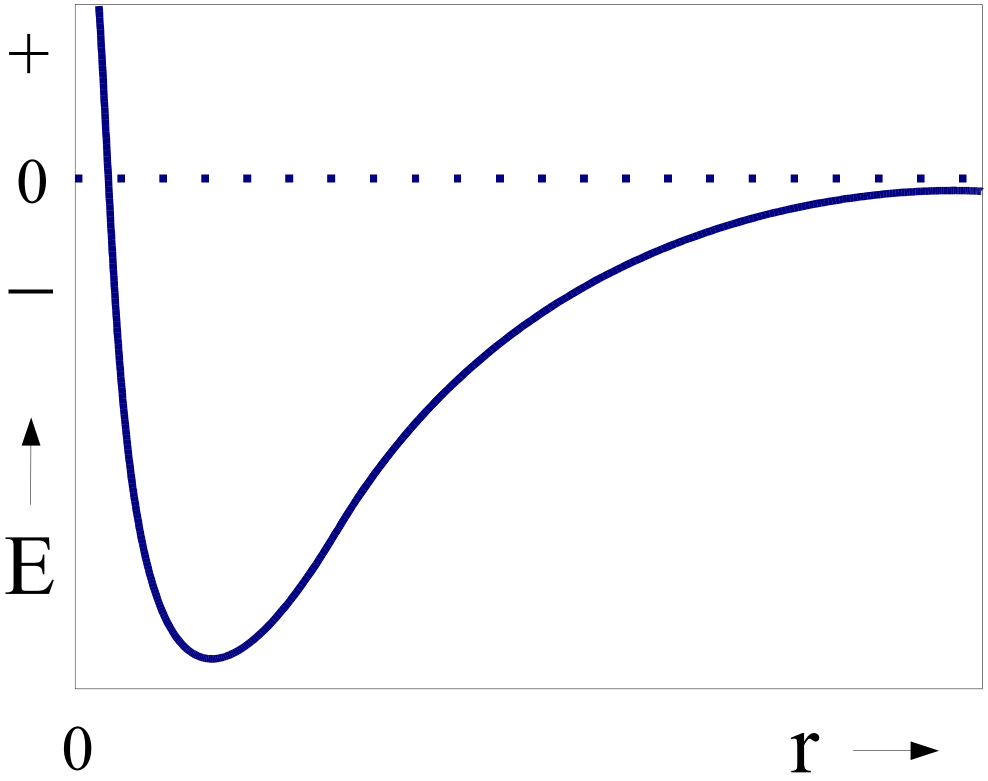


In a gravitational laboratory, experiments with point masses: your equipment has no “dials”, no free parameters.
When you specify the particle number N, everything is fixed (M = R = G = 1 for mass, half-mass radius, grav. constant)

Gravitational Thermodynamics

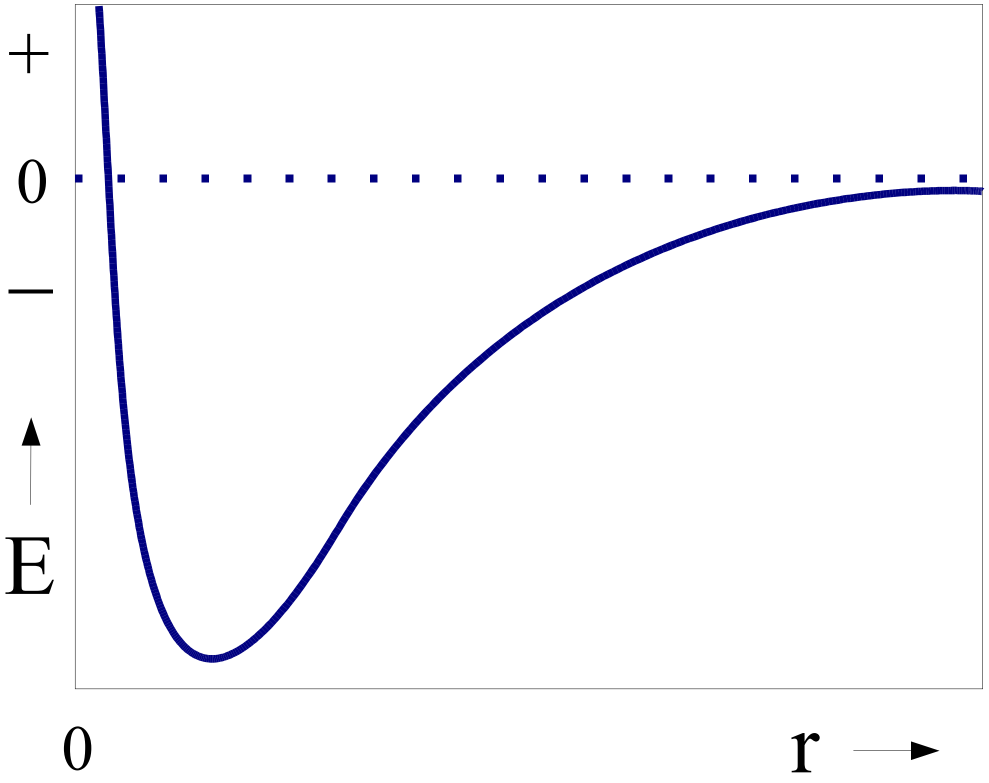
Problem: Gravitational Potential Energy
has no length scale ==>
divergence on short and long scales

Inter-particle potential

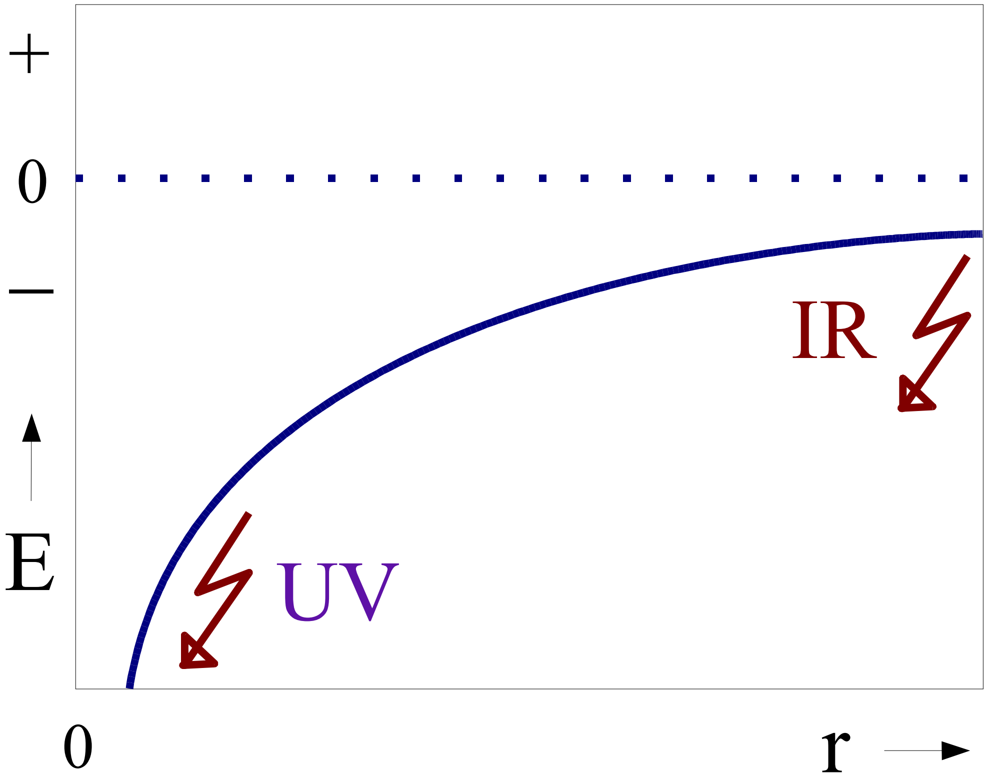


molecular dynamics

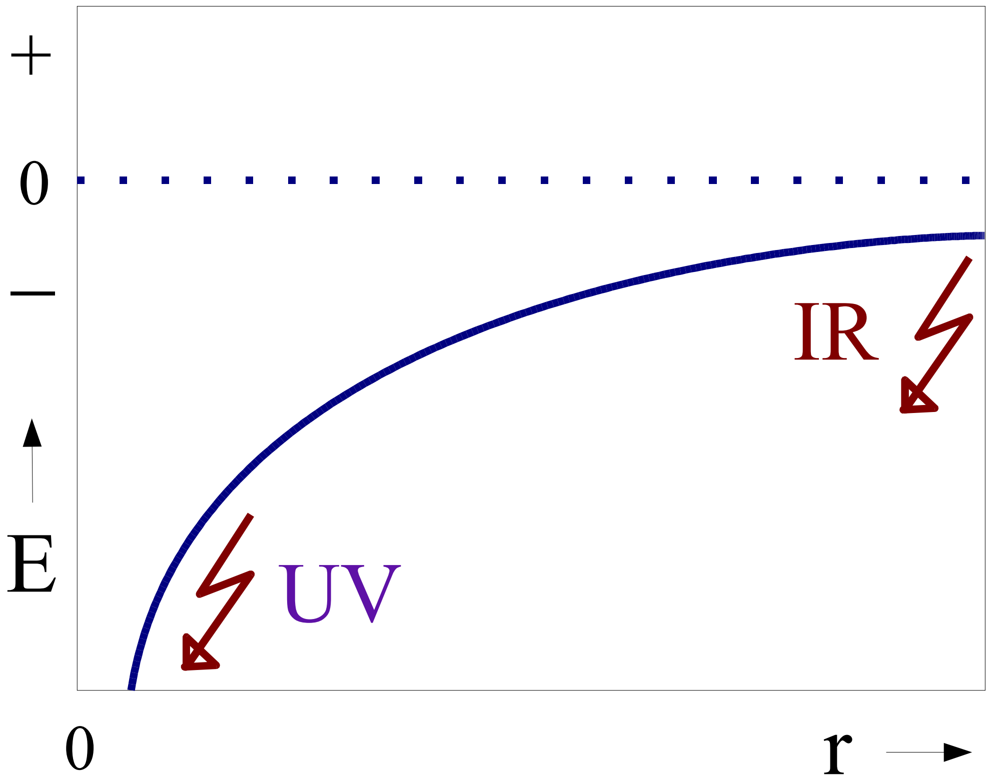
Two length scales:
- -- hard core
- -- finite range



molecular dynamics



gravitational dynamics



Gravitational thermodynamics has two problems:

IR catastrophe: no thermodynamic limit

UV catastrophe: run-away hardening of binaries

Gravitation: no thermodynamic limit

Extensive quantities: Energy, Volume, Entropy $\propto$ Mass
Intensive quantities: Density, Temperature $\propto$ constant

$$E_{tot} \propto E_{pot} \propto G\frac{M^2}{R} \propto G\rho^2 R^5 \propto G\rho^2 M^{5/3} \gg (\propto M)$$

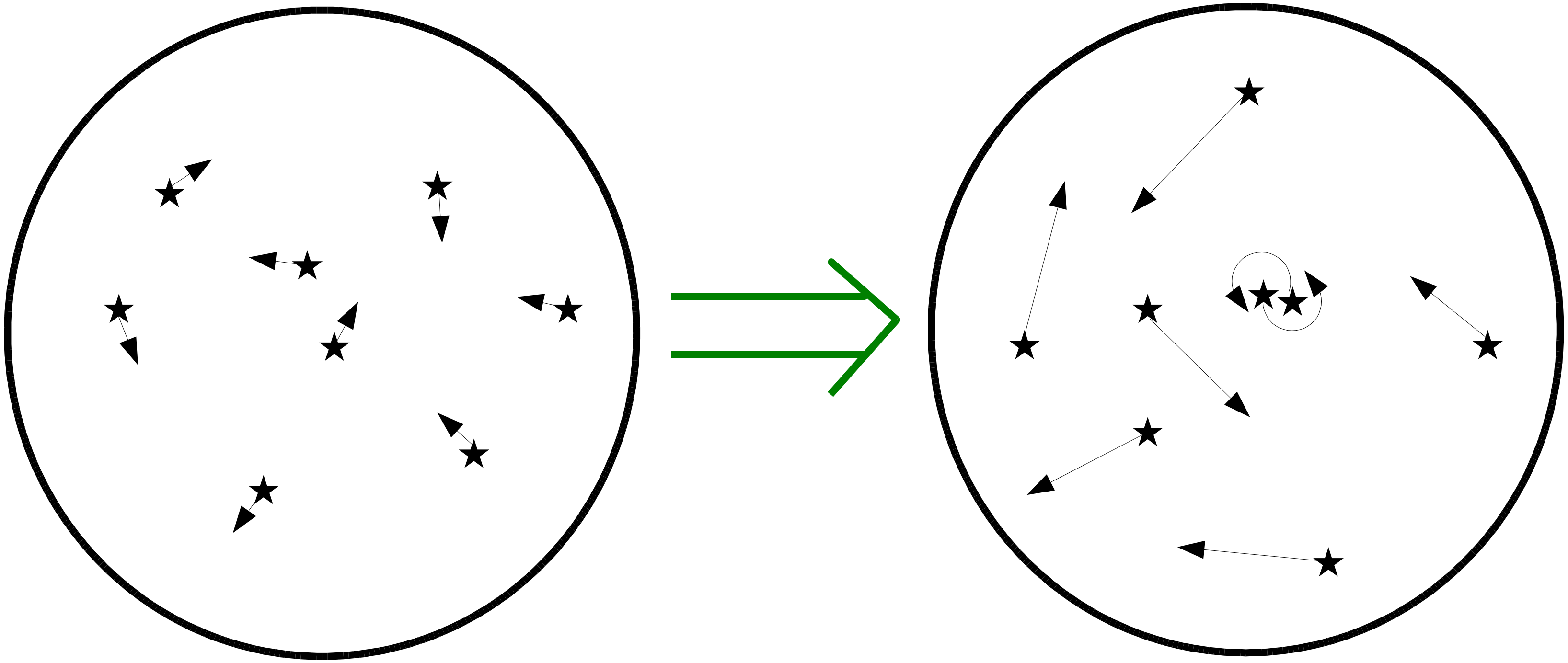
$$T \propto \frac{E_{kin}}{M} \propto G\frac{M}{R} \propto G\rho R^2 \propto G\rho M^{2/3} \gg (\propto \text{cst.})$$

$$\{\rho, T\} \Rightarrow R_{crit} \propto \sqrt{\frac{T}{\rho}}$$

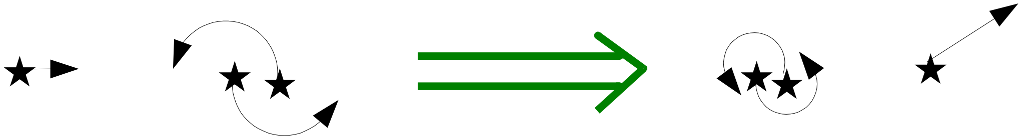
"droplet" formation: clusters of galaxies, galaxies,
molecular clouds, star clusters,
stars, planets, moons

IR catastrophe: structure formation in the Universe

UV catastrophe: run-away heating of an enclosed system:



because



Equipartition of energy:

hard binaries get harder

The distribution function *f(E)* for binary stars with binding energy *E* can be found quickly using the correspondence principle.

Starting with the Hydrogen atom:

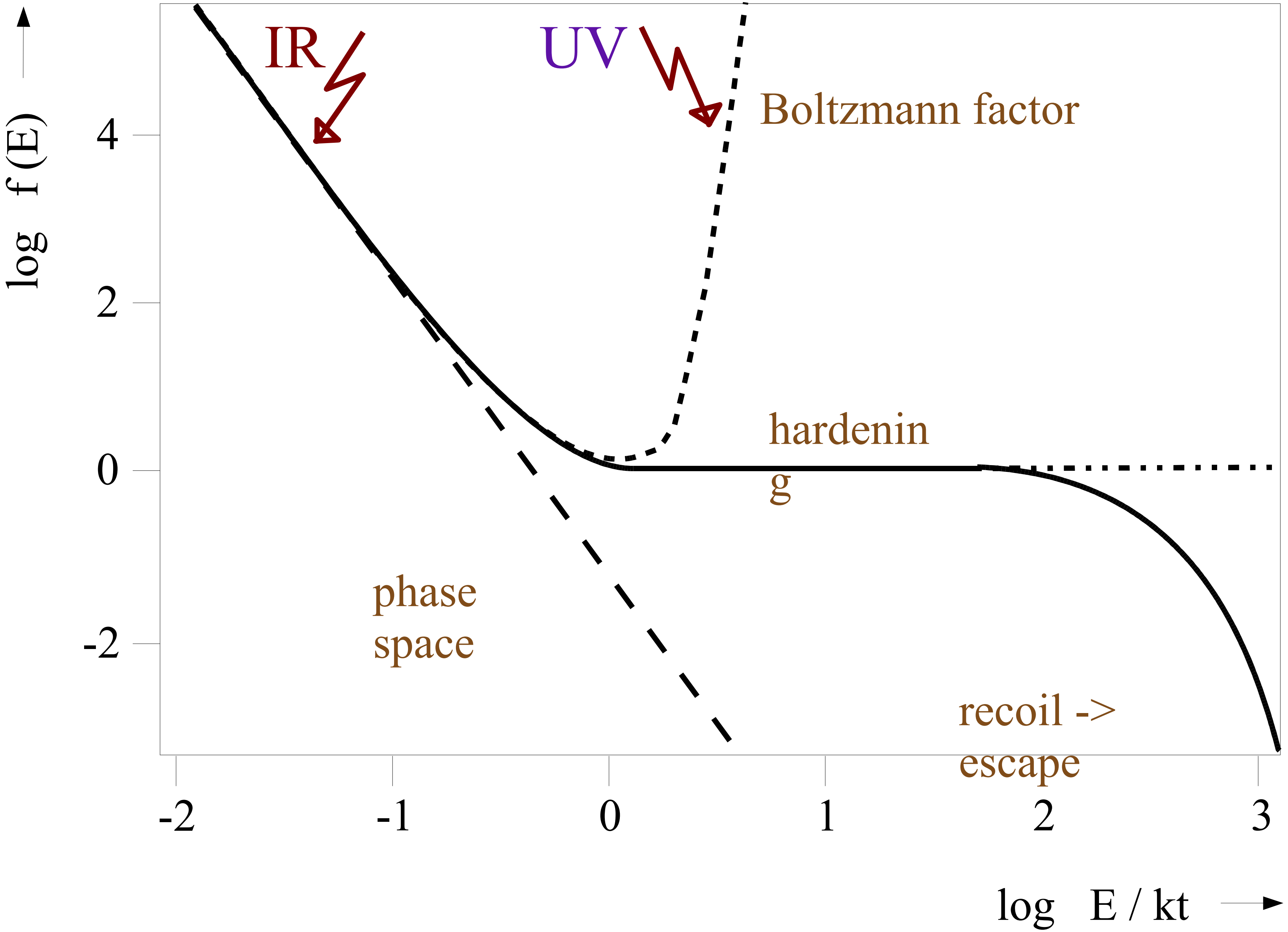
$$f(E) \;=\; w(E)\;\rho(E)\;e^{-E/kT}$$

$$E \propto \frac{1}{n^2} \;\;\Rightarrow\;\; dE \propto \frac{1}{n^3}dn \;\;\Rightarrow\;\; \frac{dn}{dE} \propto n^3 \;\;\Rightarrow\;\; \rho(E) = \frac{dn}{dE} \propto E^{-3/2}$$

$$l{=}0,\ldots,n{-}1\;\;;\;\; m{=}{-l},\ldots,{+l} \;\;\Rightarrow\;\; w(n) = \sum_{0}^{n-1}(2l+1) = n^2 \;\;\Rightarrow\;\; w(E) = E^{-1}$$

$$f(E) \;=\; \underset{\substack{\downarrow \\ \text{IR}}}{E^{-5/2}}\,\underset{\substack{\downarrow \\ \text{UV}}}{e^{-E/kT}}$$

Dynamics of binary stars

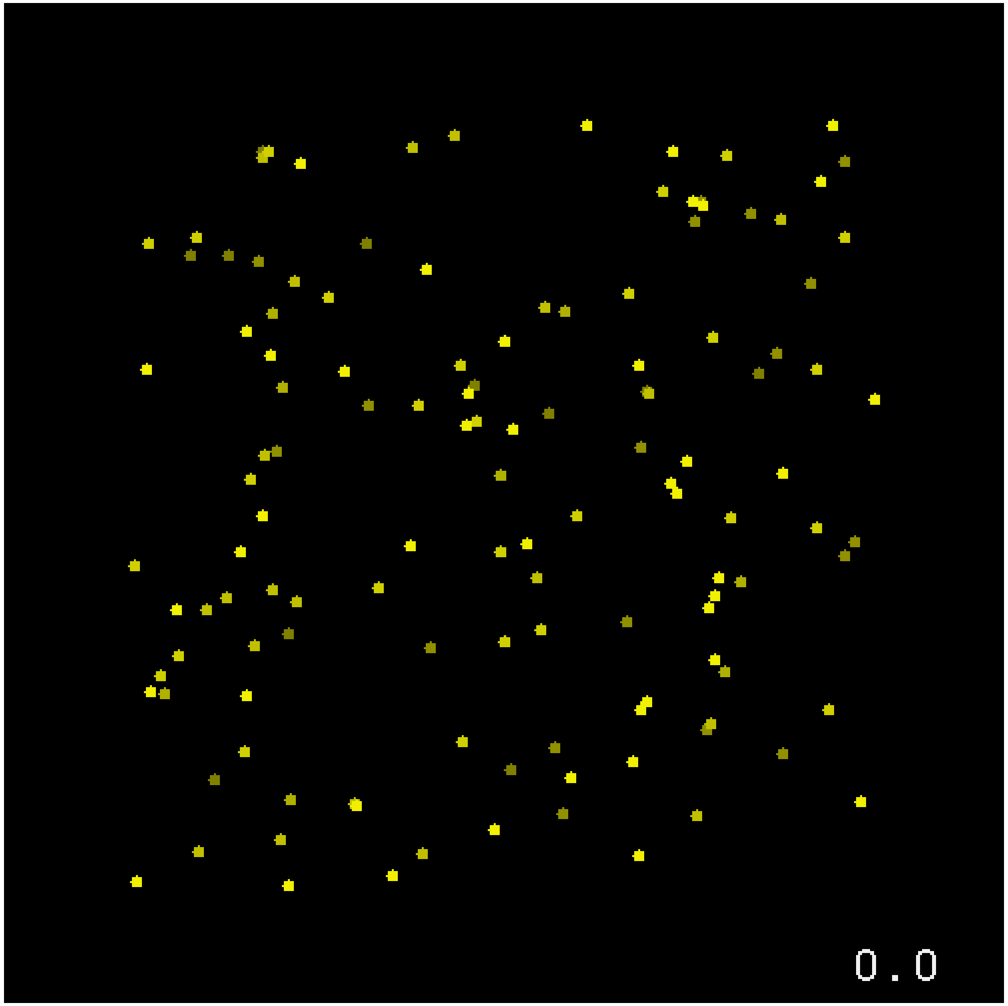


Three ways to study star clusters:

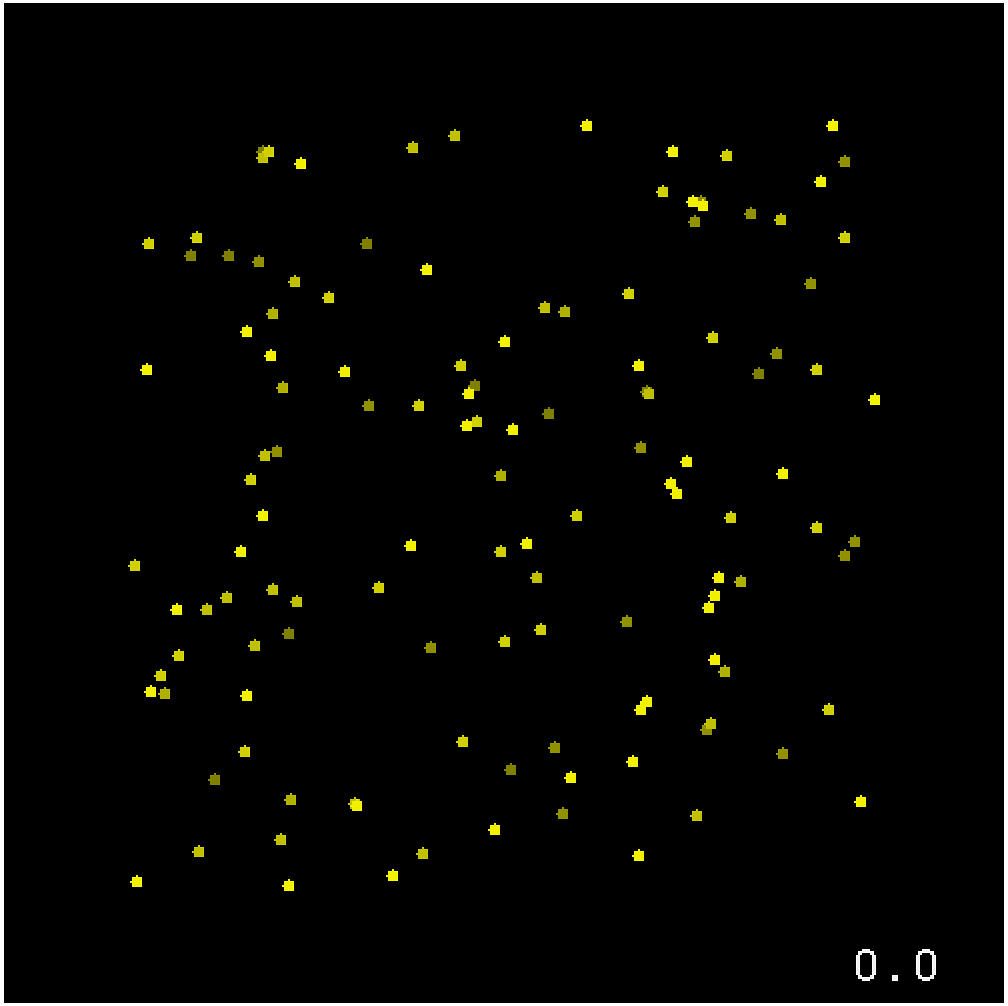
- Realistic: Tidally truncated
- Isolated: One cluster per universe
- Microscopic: A universe probing a cluster

Gravitylab: take a chunk of stars from a cluster,
multiply gravity with a finite-range kernel,
put the stars in a small periodic universe

(2005, with Toshi Fukushige, Jun Makino, Douglas Heggie)

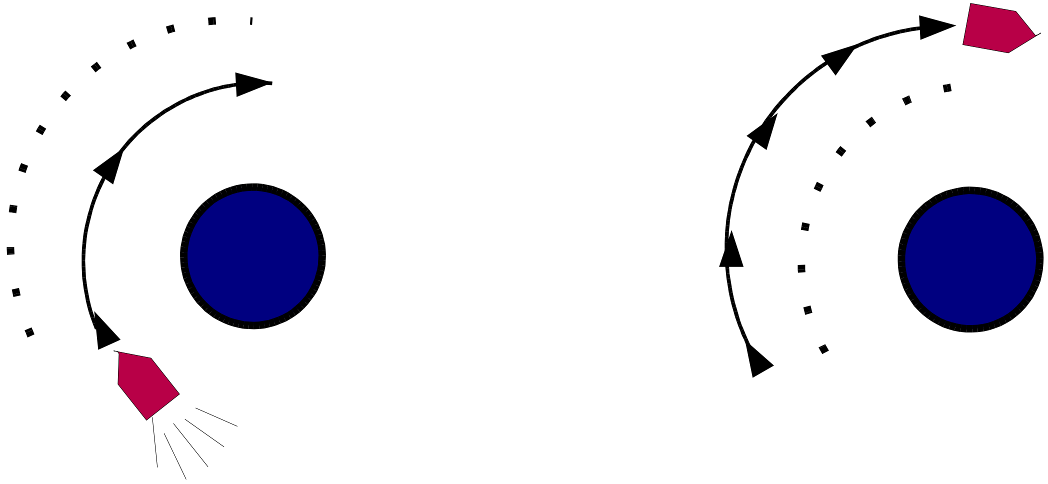


Hot

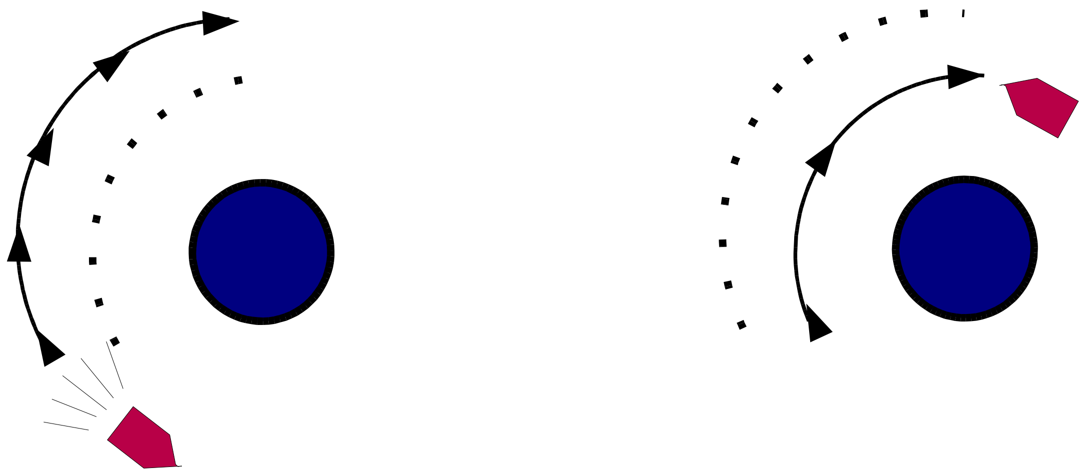


Cold

Satellite orbit paradox: add energy -> decrease kinetic energy

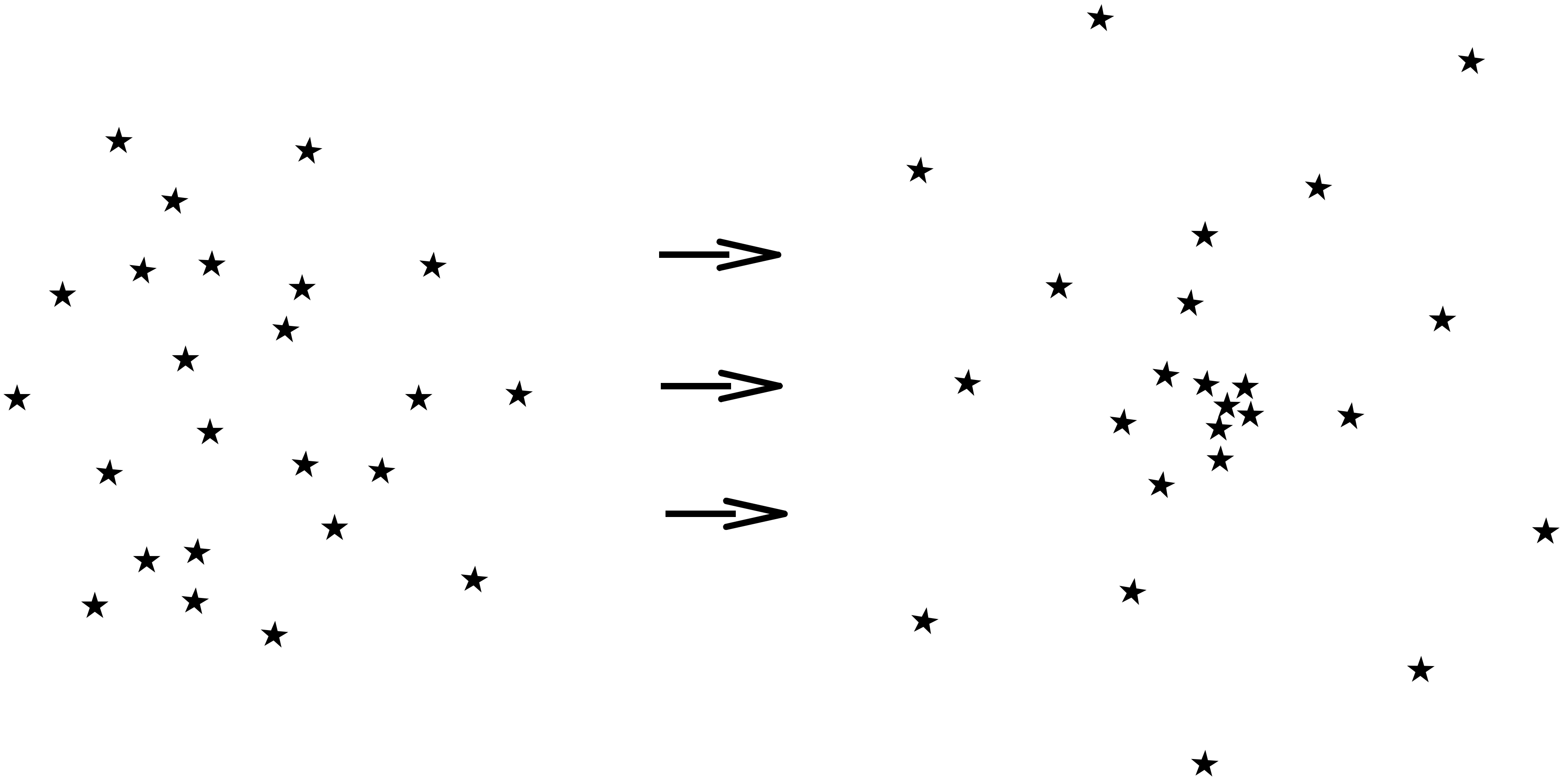


similarly: loose energy -> increase kinetic energy



For an ensemble of self-gravitating objects:

a type of “negative heat capacity”



Two-body relaxation:

the star gaining energy will “cool” and move outward to the halo;

the star loosing energy will “heat” and sink to the core of the cluster.

The time scale for the rate of change of the central density ρ is determined by the central two-body relaxation time t_r:

$$\frac{d\rho}{dt} \propto \frac{\rho}{t_r} \propto \frac{\rho}{\frac{1}{\rho}} \propto \rho^2 \quad \Rightarrow$$

$$\rho^{-2} d\rho \propto dt \quad \Rightarrow$$

$$d\rho^{-1} \propto -dt \quad \Rightarrow$$

$$\rho^{-1} \propto t_{cc} - t \quad \Rightarrow$$

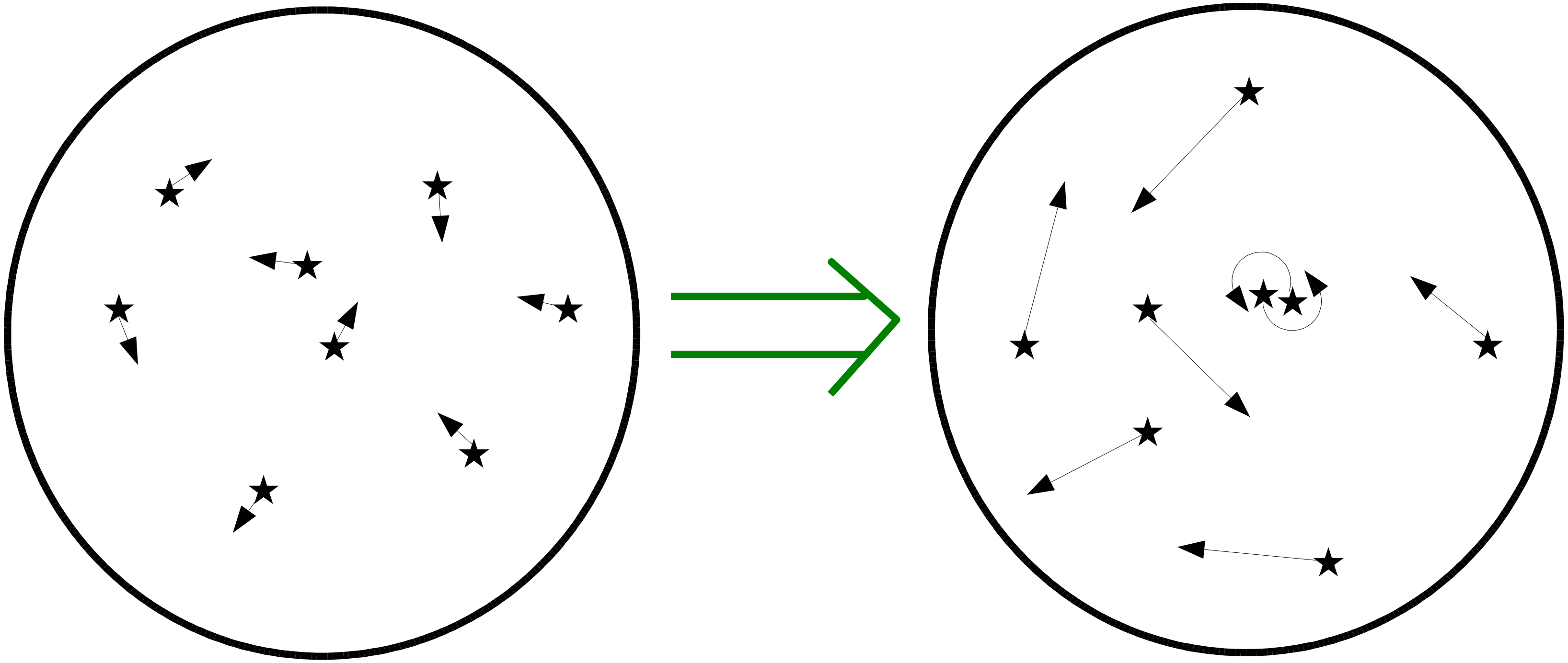
$$\rho \propto \frac{1}{t_{cc} - t} \quad \Rightarrow$$

$$\rho(t) = \rho(0) \left(1 - \frac{t}{t_{cc}}\right)^{-1}$$

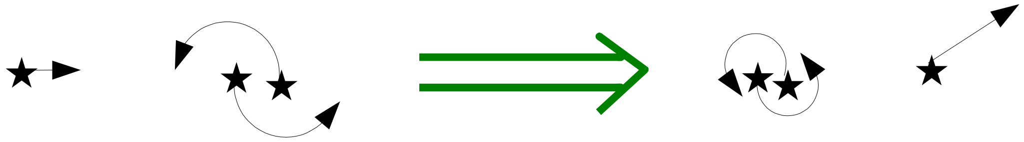
where t_{cc} is the time of core collapse.

We already saw:

UV catastrophe: run-away heating of an enclosed system:



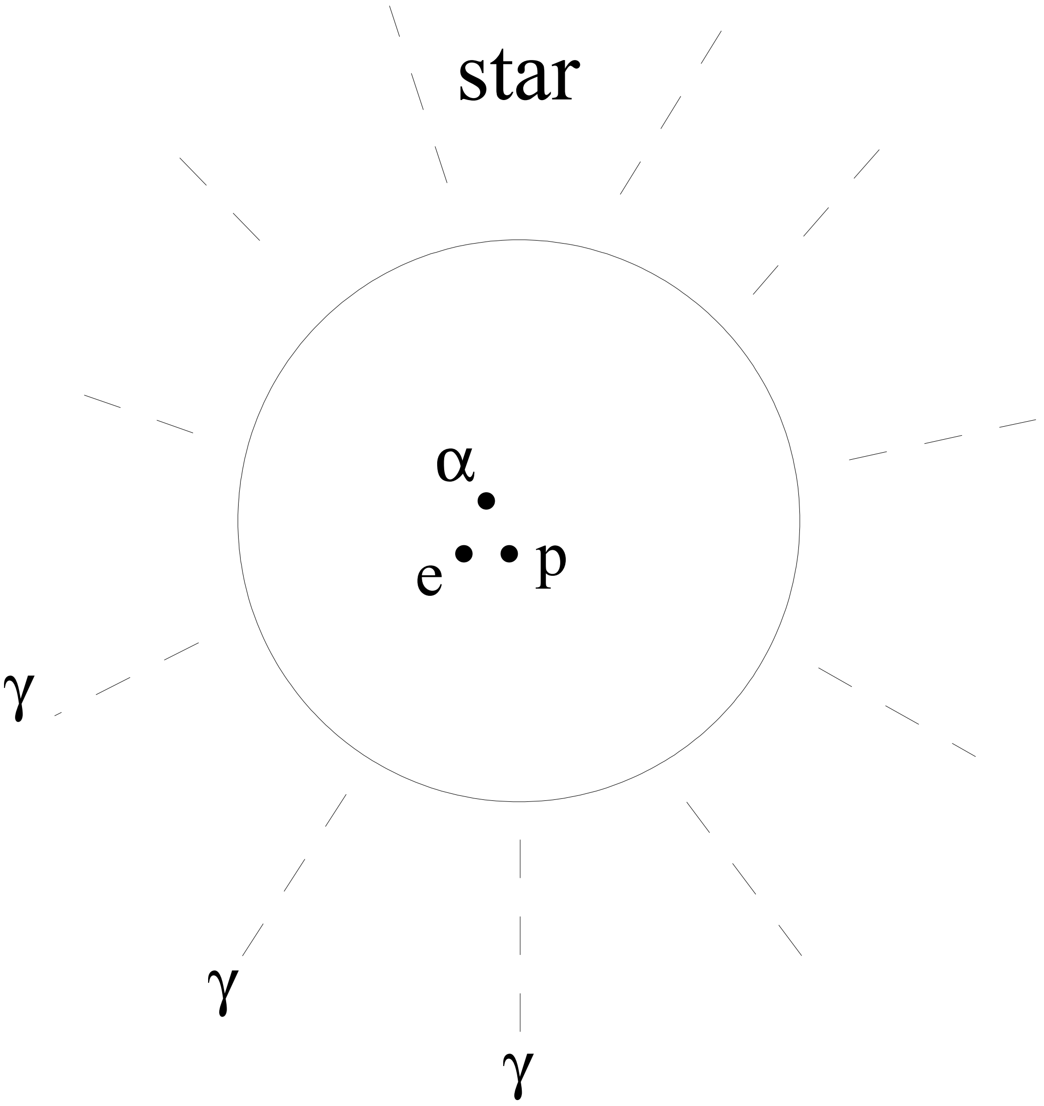
because

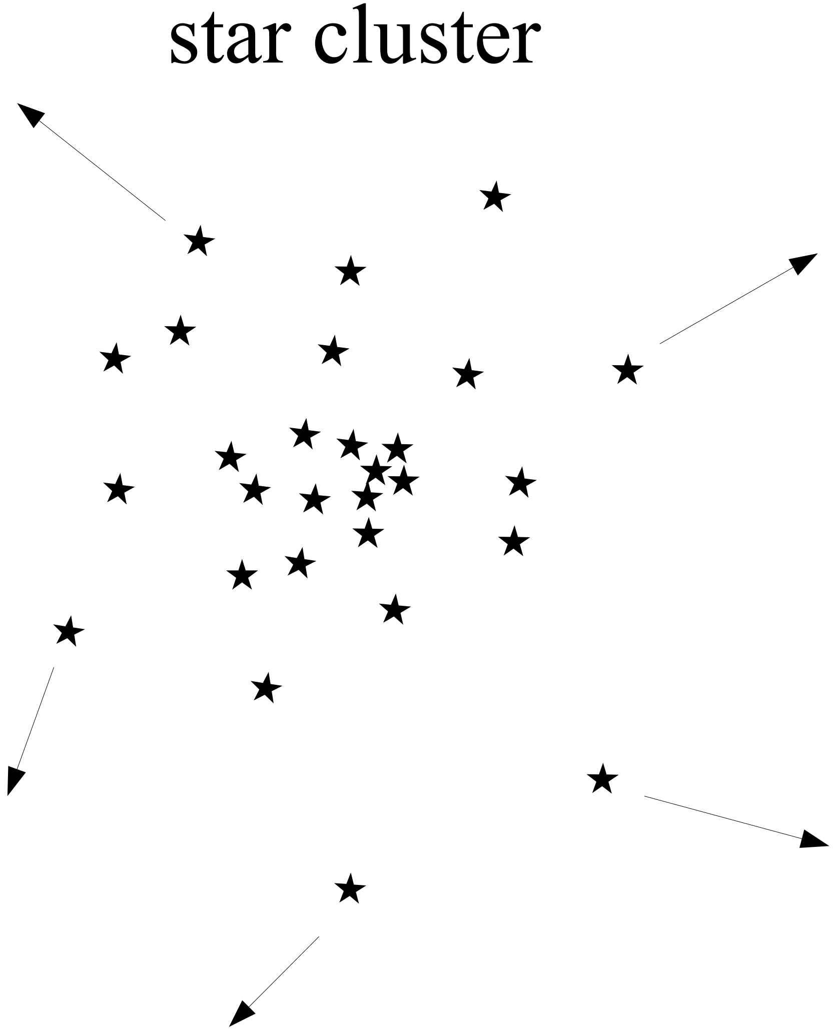


Equipartition of energy:

hard binaries get harder

in an open system:





Energy generation by

nuclear *gravitational*

fusion

We have seen what two-body relaxation predicts:

negative heat capacity $\Rightarrow$ core contraction $\Rightarrow$

core collapse at finite time t_{cc}:

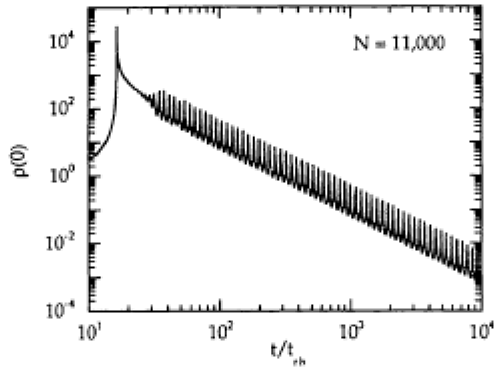
$$\rho(t) = \rho(0) \left(1 - \frac{t}{t_{cc}}\right)^{-1}$$

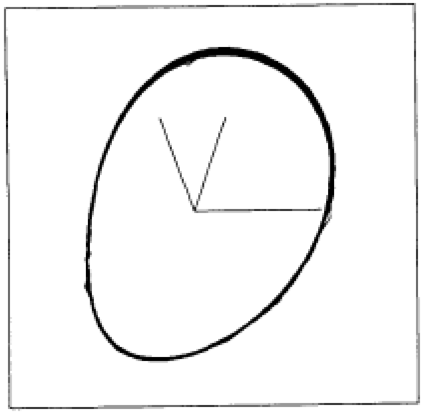
In practice, three-body interactions form binaries,

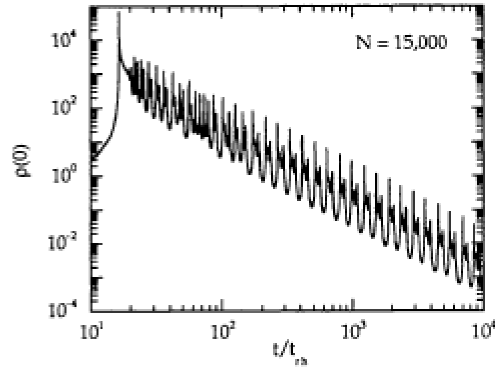
which generate energy, prevent total collapse,

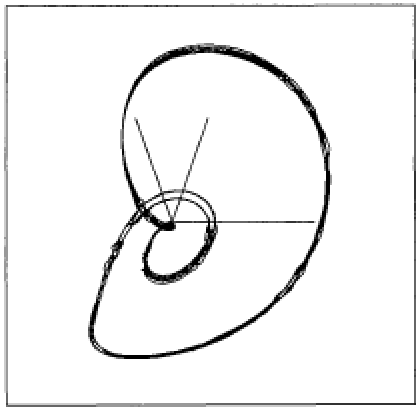
and power the slow evaporation of the system.

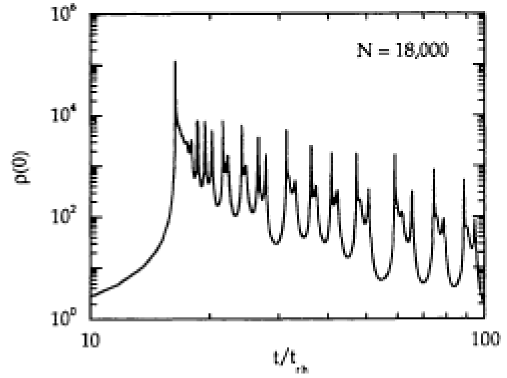
While this is going on, what will happen after core collapse?

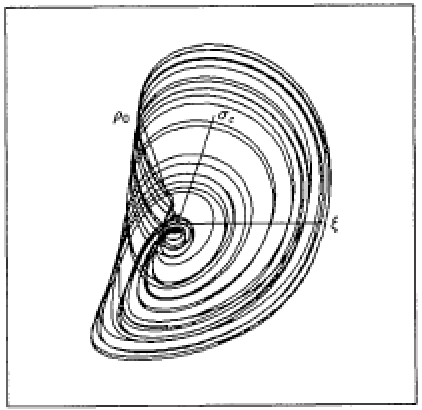

N = 11,000
ρ(0)
t/t_rh
10^4
10^2
10^0
10^-2
10^-4
10^1
10^2
10^3
10^4

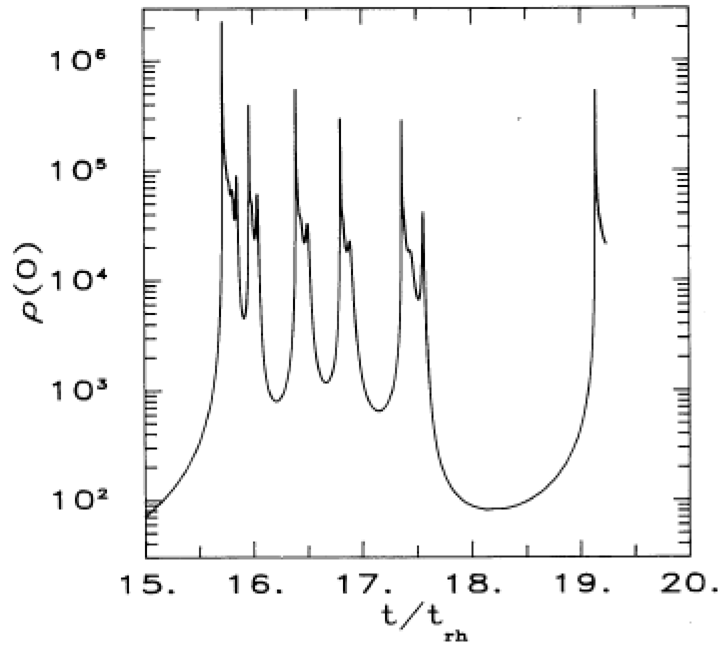



N = 15,000
ρ(0)
t/t_rh
10^4
10^2
10^0
10^-2
10^-4
10^1
10^2
10^3
10^4

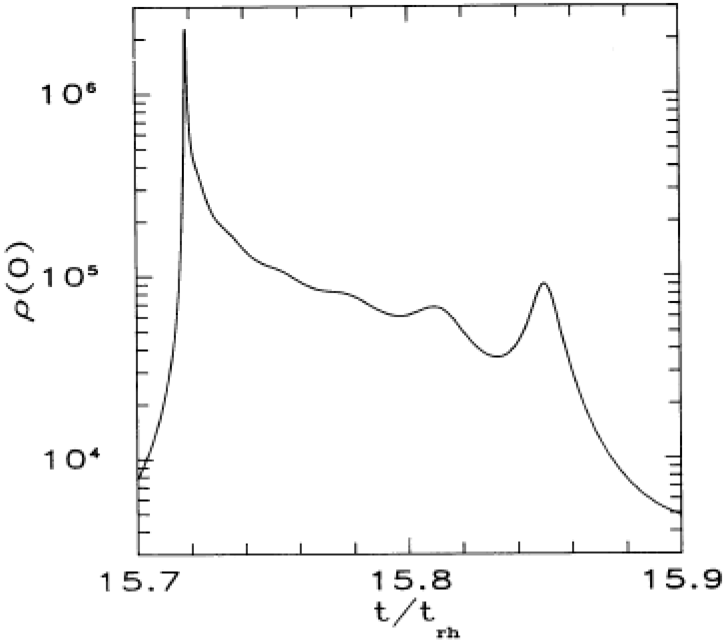



N = 18,000
ρ(0)
t/t_rh
10^6
10^4
10^2
10^0
10
100

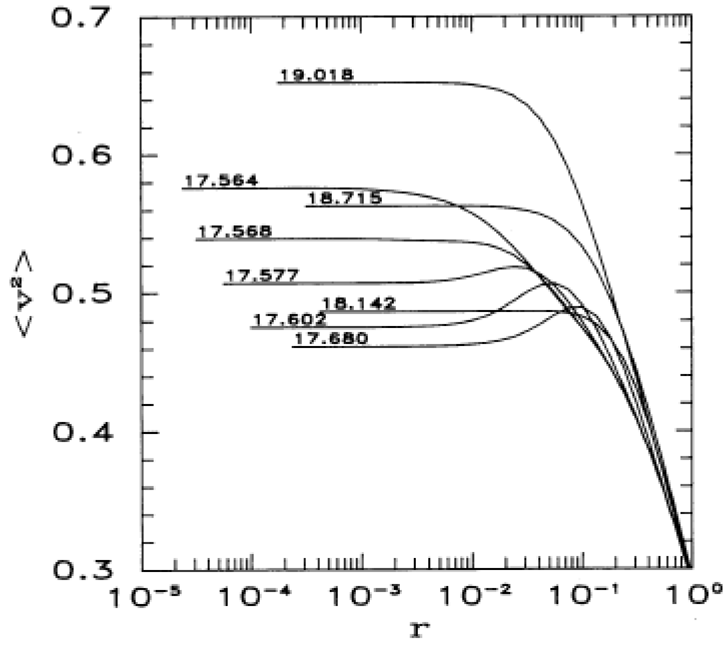

ρ0
σc
ξ



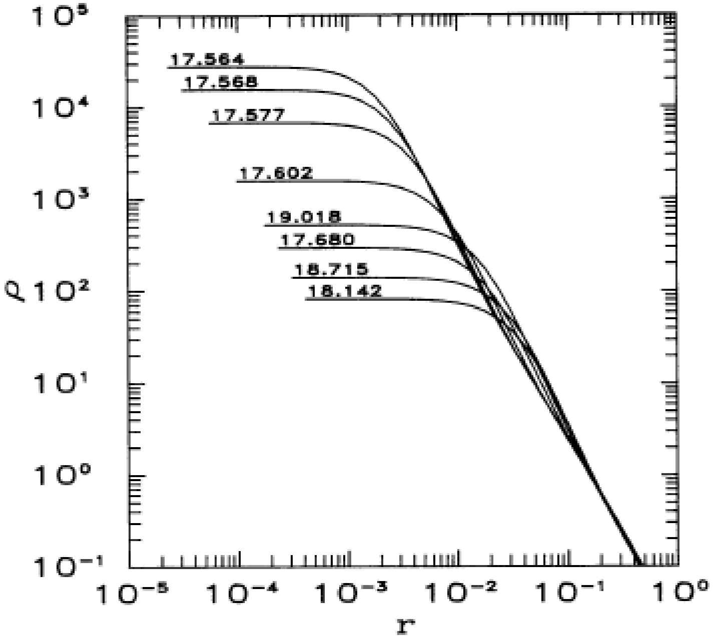
$\rho(O)$
10^6
10^5
10^4
10^3
10^2
15.
16.
17.
18.
19.
20.
t/t_{rh}



$\rho(O)$
10^6
10^5
10^4
15.7
15.8
15.9
t/t_{rh}



$<v^2>$
0.7
0.6
0.5
0.4
0.3
19.018
17.564
18.715
17.568
17.577
18.142
17.602
17.680
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
r



ρ
10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}
17.564
17.568
17.577
17.602
19.018
17.680
18.715
18.142
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
r

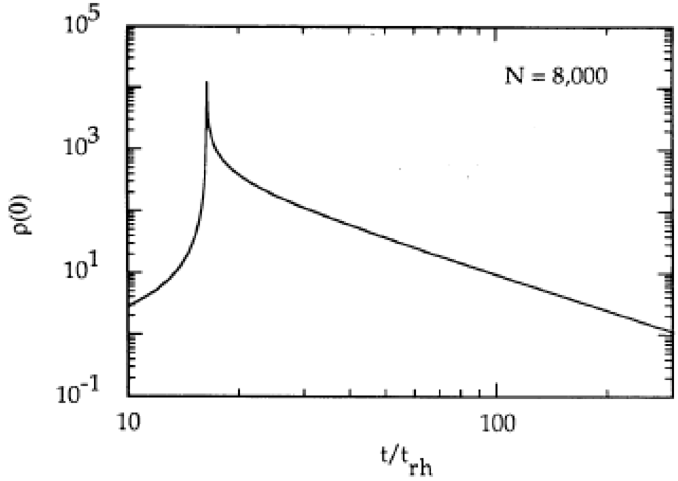


FIG. 1*a*

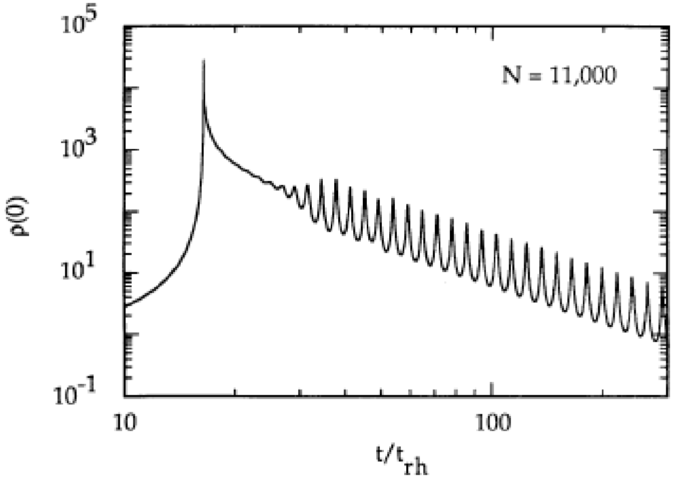


FIG. 1*b*

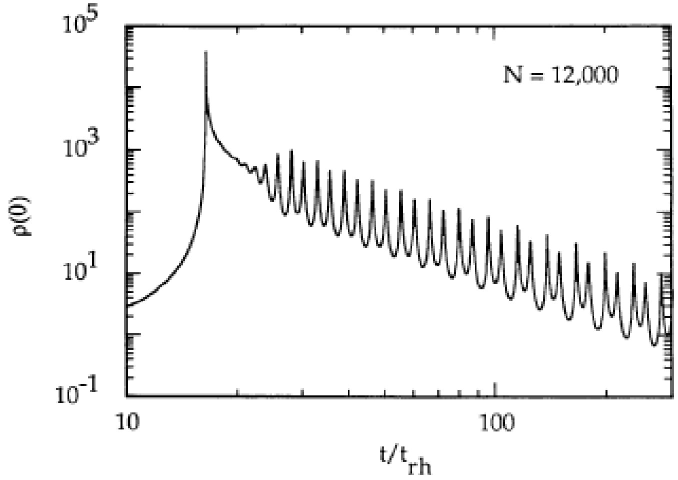


FIG. 1*c*

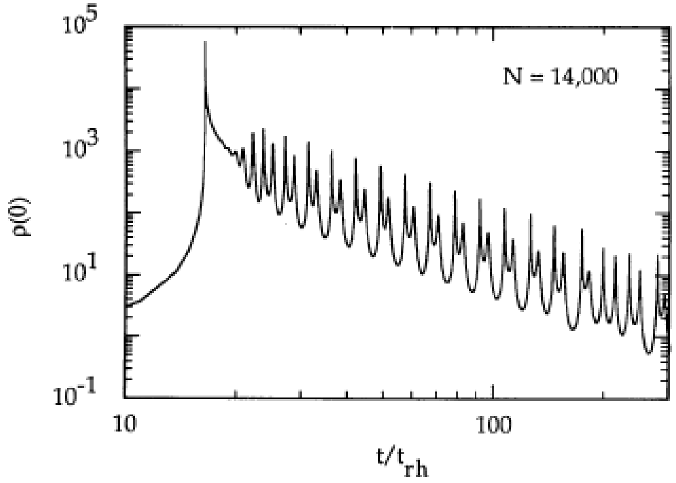
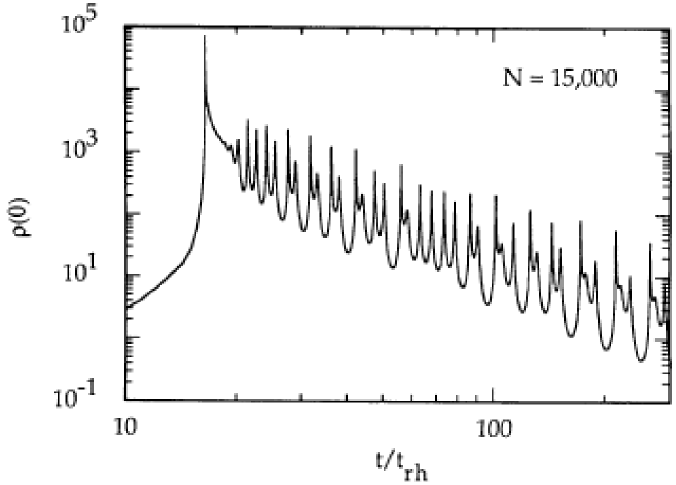


FIG. 1*d*



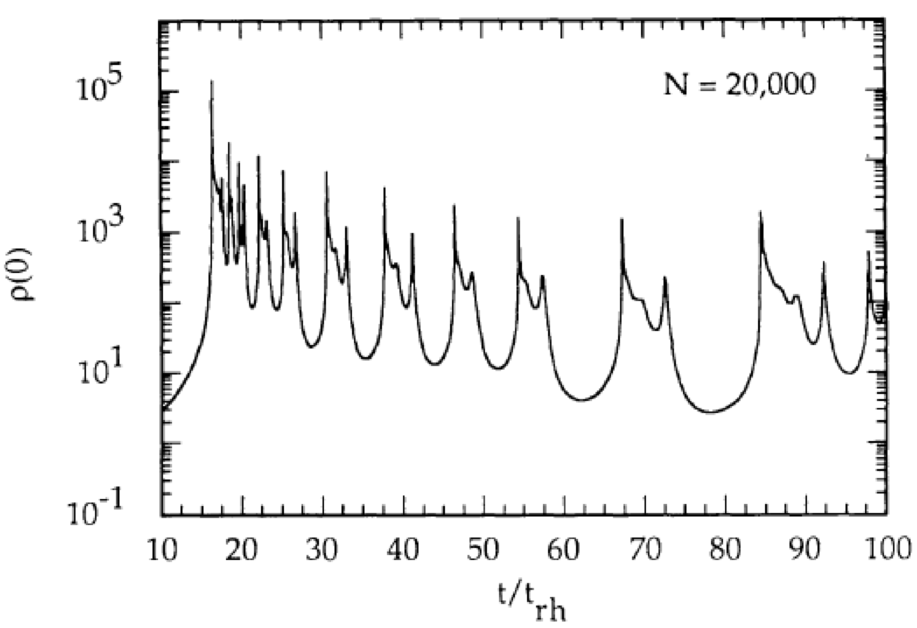


FIG. 3*a*

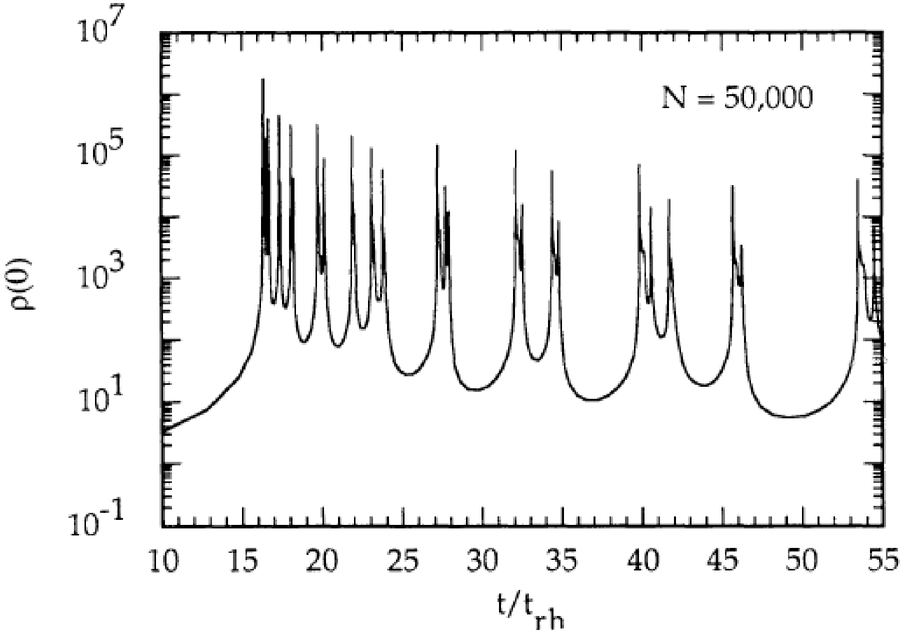


FIG. 3*b*

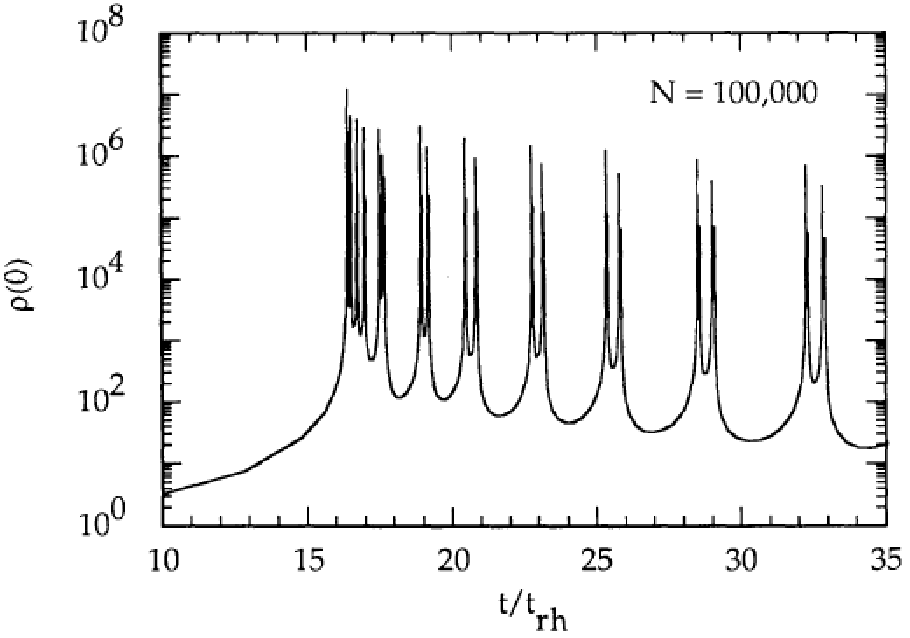


FIG. 3*c*

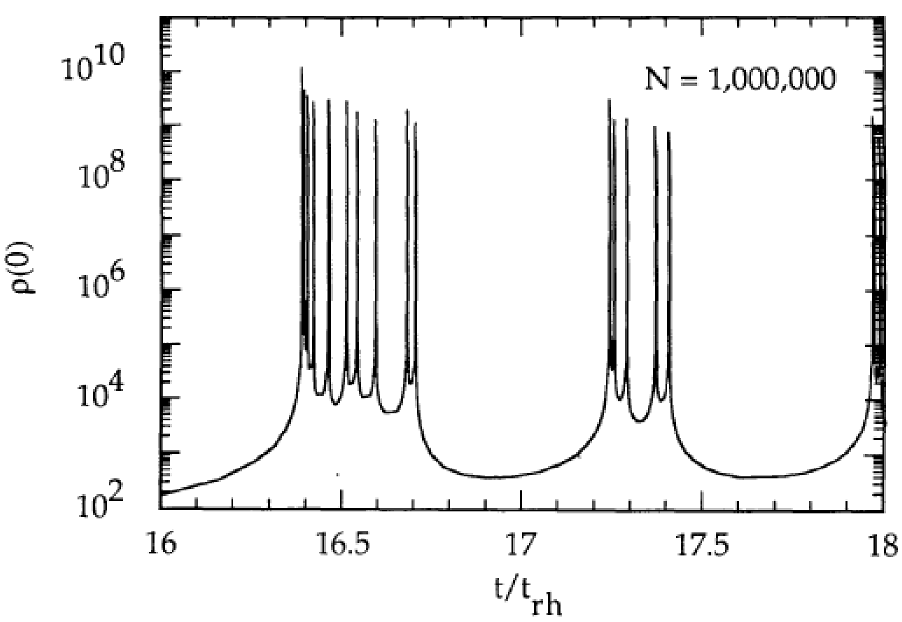
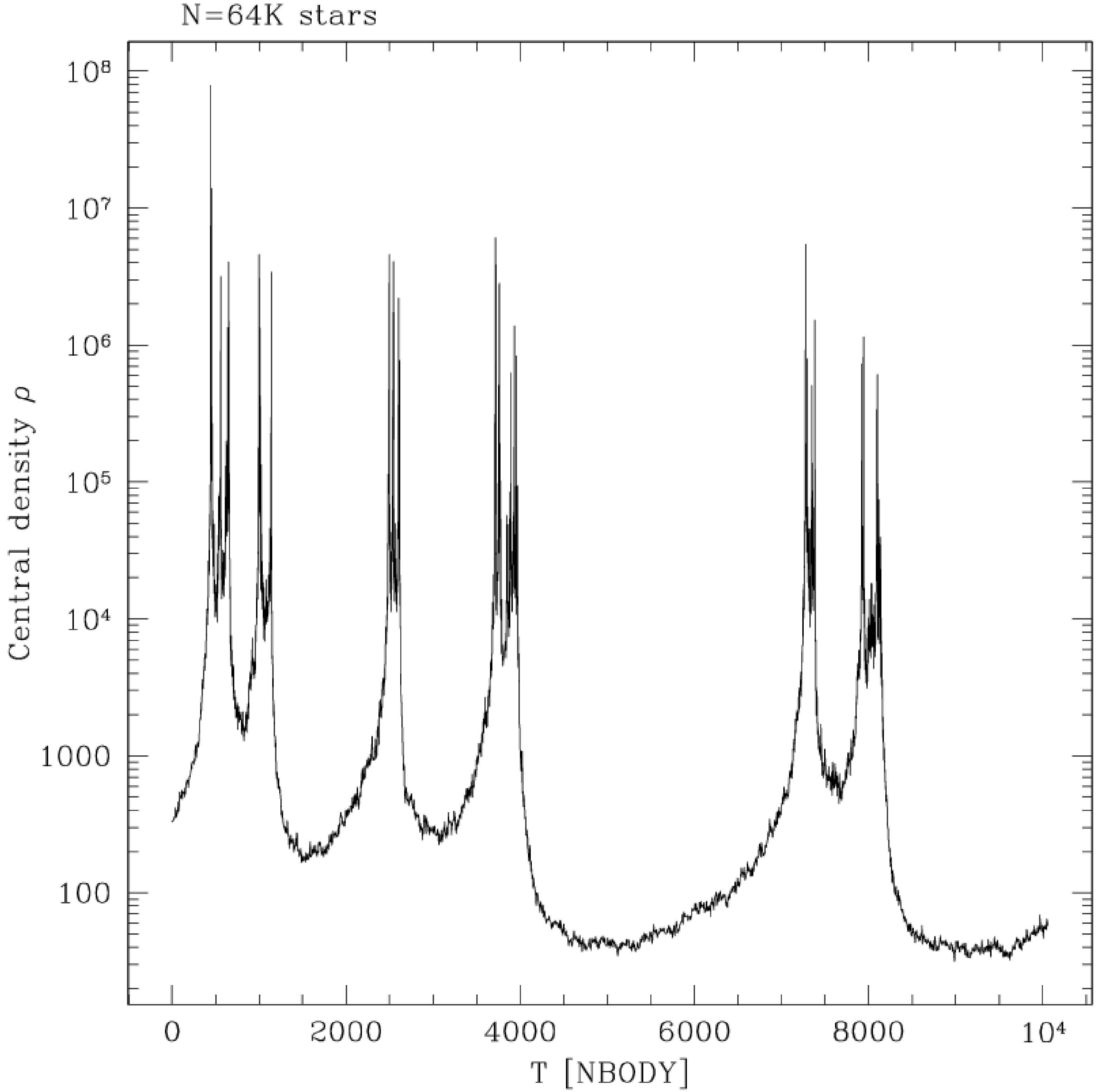
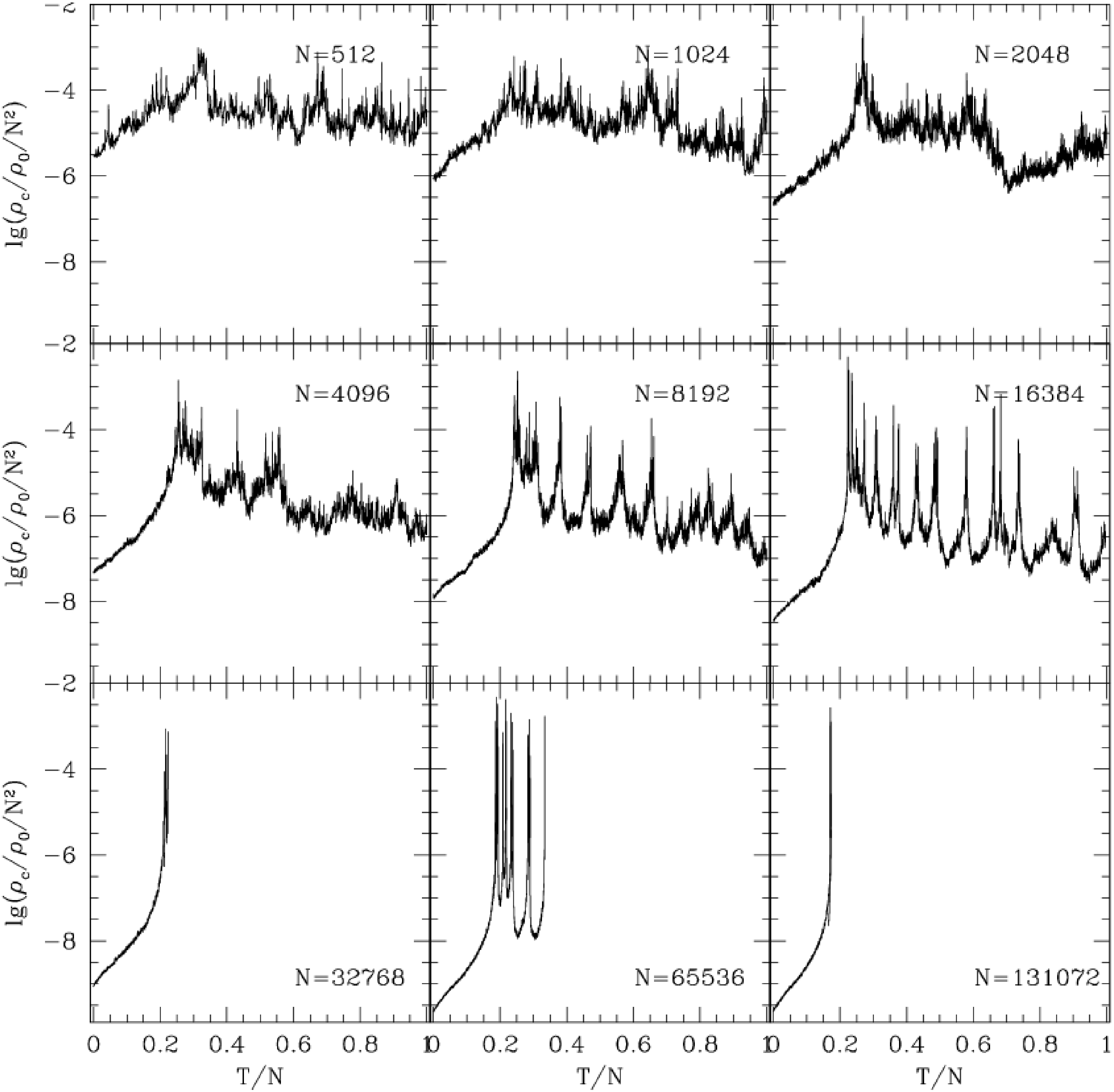


FIG. 3*d*


N=64K stars
Central density ρ
T [NBODY]
10^8
10^7
10^6
10^5
10^4
1000
100
0
2000
4000
6000
8000
10^4



N=512
N=1024
N=2048
N=4096
N=8192
N=16384
N=32768
N=65536
N=131072
lg(ρc/ρ0/N²)
−2
−4
−6
−8
0
0.2
0.4
0.6
0.8
1
T/N

Dense Stellar Systems as a Laboratory for Fundamental Physics

We can study *elementary particles* through their interactions:

- bound states
- scattering experiments

Other extreme forms of matter: *black holes* and *neutron stars*

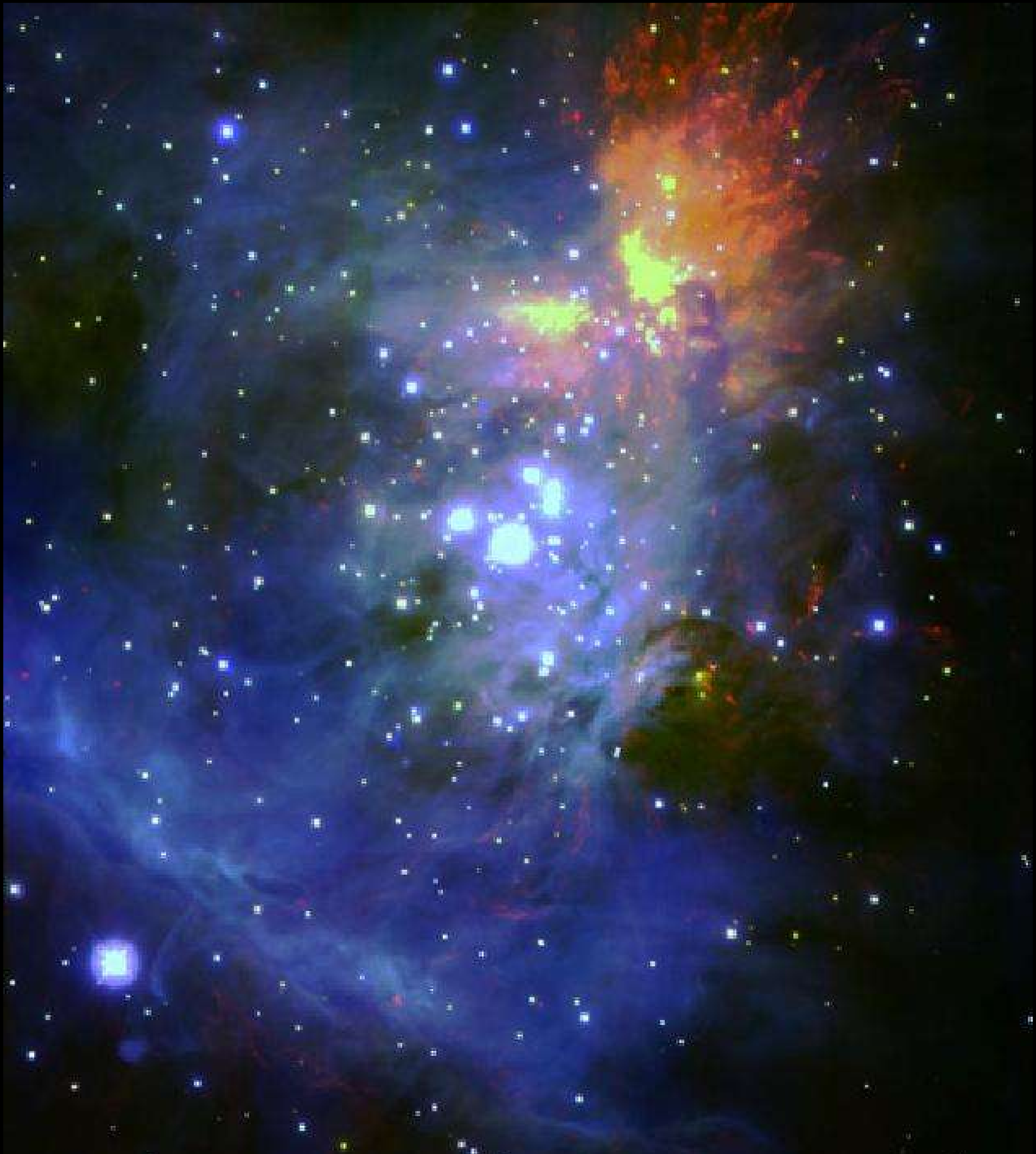
We can study these, too, through their interactions:

- bound states: double stars
- scattering experiments: collisions between stars

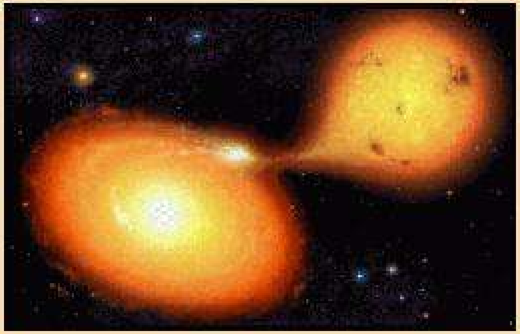
There is a natural laboratory: **dense stellar systems**

Dense Stellar Systems

- Interactions between individual stars important
  - Two-body relaxation time < Age of the system
  - binary--single-star encounters; physical collisions
- Locations:
  - star-forming regions
  - old open star clusters
  - globular clusters
  - galactic nuclei

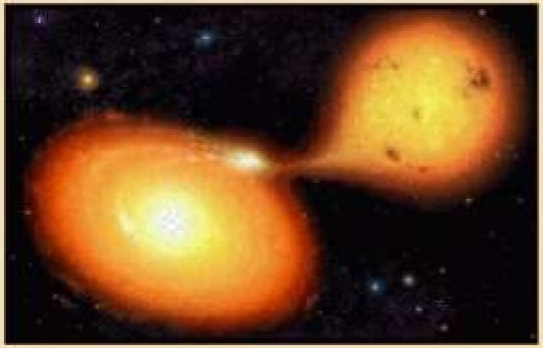


MODEST



MOdeling DEnse STellar systems

MODEST



MOdeling DEnse STellar systems

www.manybody.org

MODEST Workshops

MODEST -- 1:	New York, NY, USA	June 17-21, 2002
MODEST -- 2:	Amsterdam, Holland	Dec. 16-17, 2002
MODEST -- 3:	Melbourne, Australia	July 9-11, 2003
MODEST -- 4:	Lausanne, Switzerland	Jan. 12-14, 2004
MODEST -- 4a:	Strasbourg, France	March 19-22, 2004
MODEST -- 4b:	Amsterdam, Holland	June 7-8, 2004
MODEST -- 5:	Hamilton, ON, Canada	Aug. 11-14, 2004
MODEST -- 5a:	Edinburgh, Scotland, UK	Dec. 15-17, 2004
MODEST – 5b:	Prague, Czech Republic	Sep. 20-25, 2004
MODEST -- 5c:	Amsterdam, Holland	July 24-30, 2005
MODEST -- 5d:	Princeton, NJ, USA	April 7, 2005
MODEST -- 6:	Evanston, IL, USA	Aug. 29-31, 2005

MODEST Working Groups

1)	Star Formation	Ralf Klessen
2)	Stellar Evolution	Onno Pols
3)	Stellar Dynamics	Rainer Spurzem
4)	Stellar Collisions	Marc Freitag
5)	Sim. Obs. of Simulations	Simon Portegies Zwart
6)	Data Structures	Peter Teuben
7)	Validation	Douglas Heggie
8)	Literature	Melvyn Davies
9)	Observations	Giampaolo Piotto
10)	Outreach	James Lombardi

A Brief History of Science

~ 2000 years ago: Theory -- Greek mathematics

~ 400 years ago: Theory & Experiment -- Modern Science

~ 50 years ago: Theory, Experiment & Simulations -- ?

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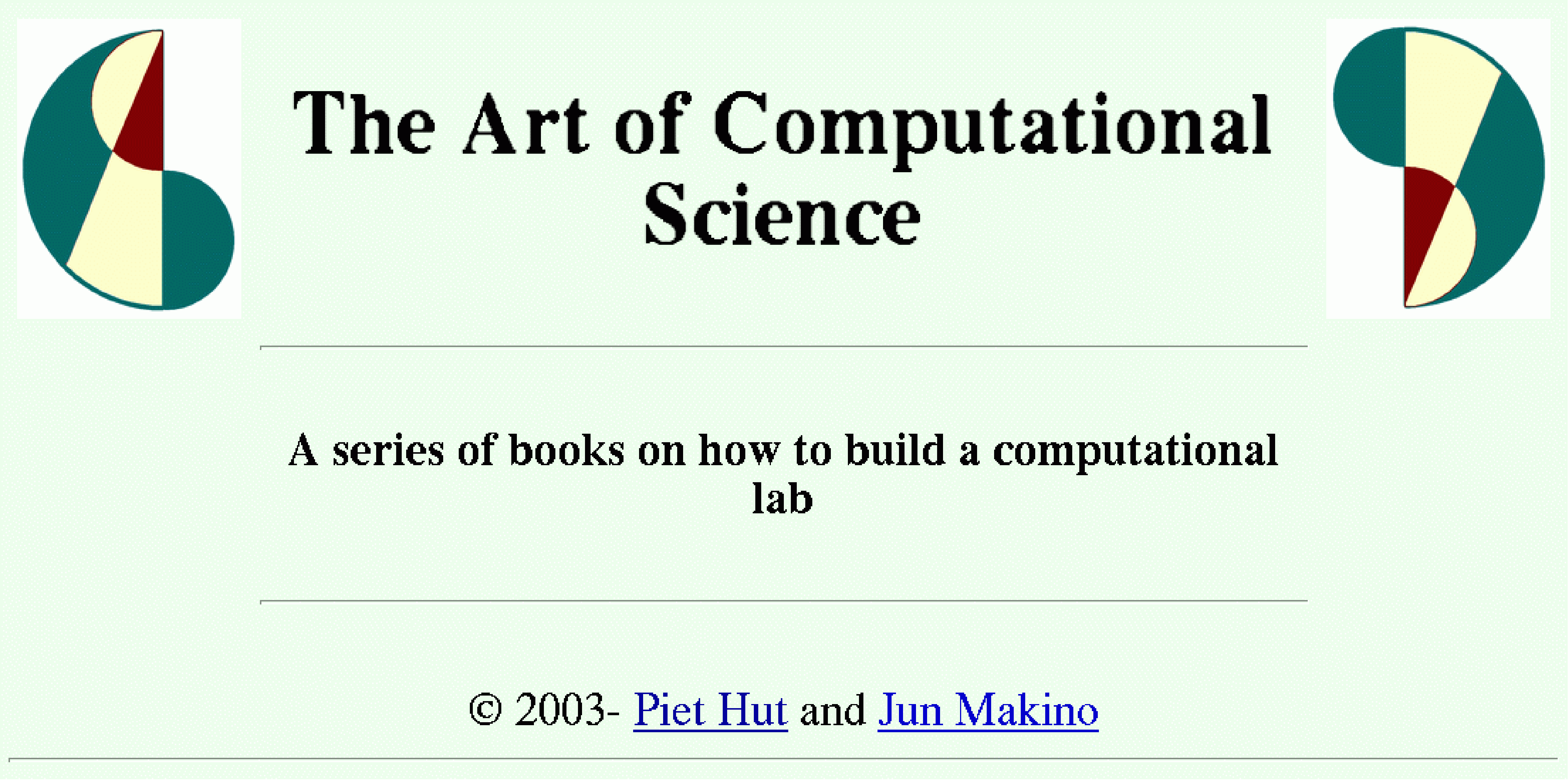
Science is the first “open source” project.

Experiments: a new lab culture had to be developed
-- make detailed lab notes, keep raw data
-- report failures as well as success

Simulations: a new `virtual lab' culture is now emerging
-- we don't yet have a good way to share code
-- we don't yet know how to share knowledge

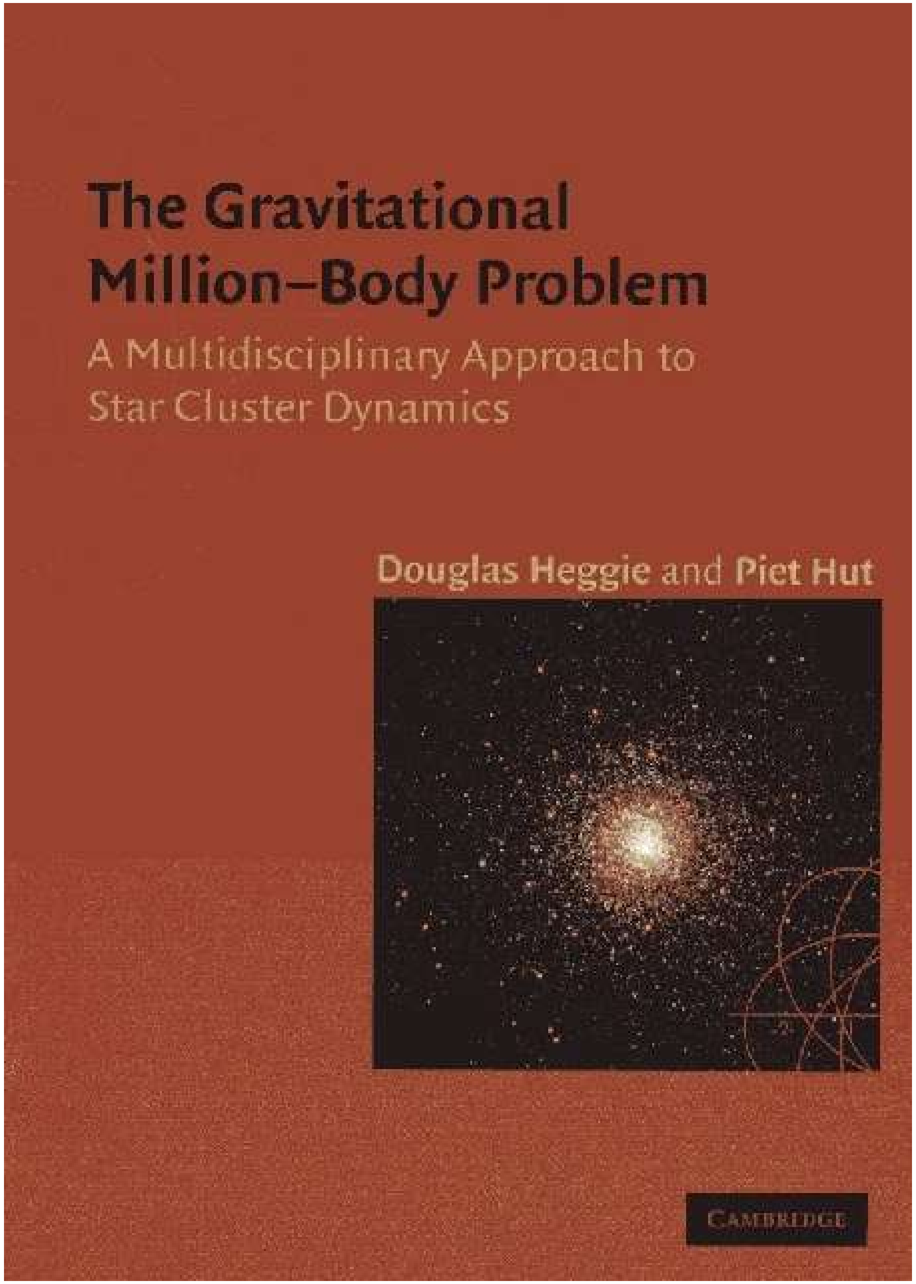


The Art of Computational Science

A series of books on how to build a computational lab

www.artcompsci.org

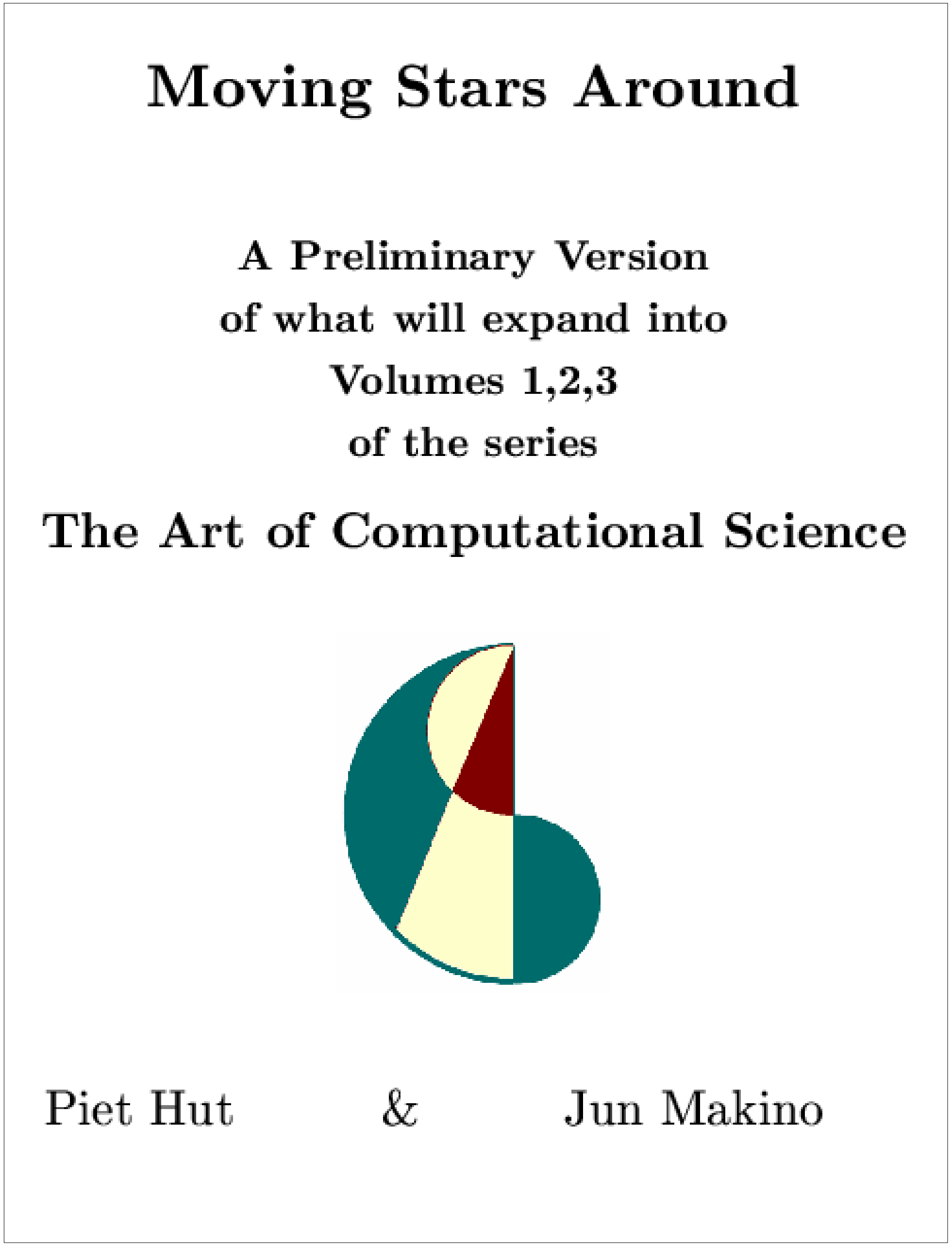
An open source project



The Gravitational
Million–Body Problem
A Multidisciplinary Approach to
Star Cluster Dynamics

Douglas Heggie and Piet Hut

CAMBRIDGE



Moving Stars Around

A Preliminary Version
of what will expand into
Volumes 1,2,3
of the series

The Art of Computational Science

Piet Hut & Jun Makino

4 Introductions: astrophysics
theoretical physics
computational physics
mathematics

3 themes: exploring N-body algorithms
writing N-body codes
performing N-body experiments
(www.artcompsci.org)

The Art of Computational Science

- code < comments < manual < primer < dialogue
  containing motivation, exploration, debugging sessions
- research = documentation = education
  cf.: Summer school Amsterdam, July 24-30, 2005
- mixing text, math, code (literate programming)
  Knuth, cf. James Quirk: amrita
- rapid prototyping (scripting language: Ruby)
  slow, of course; speed-up through C function calls
- extremely modular
  example: individual time steps -> individual algorithms

(www.artcompsci.org)