Direct N-Body Implementations

Sverre Aarseth

Institute of Astronomy, Cambridge

• Hermite integration	$\Delta t_n = \Delta t_1 / 2^{n-1}$
• Neighbour scheme	$\mathbf{F}_i = \mathbf{F}_d + \Sigma \mathbf{F}_{ij}$
• Two-body encounters	$\mathbf{u}'' = \frac{1}{2} h \, \mathbf{u} + \frac{1}{2} R \, \mathcal{L}^{\mathrm{T}}(\mathbf{u}) \, \mathbf{P}$
• Compact subsystems	$\Gamma^* = t' \, (H(\mathbf{Q}, \mathbf{P}) - E_0), \quad N \leq 10$
• Hierarchical systems	$a_{\mathrm{out}} \, (1 - e_{\mathrm{out}}) > \Psi(m, e_{\mathrm{in}}, i) \, a_{\mathrm{in}}$
• Stellar evolution	$m^*, \, r^*, \, L^*$ as $f(m_0, Z, t)$
• Binary astrophysics	$\dot{a}_{\mathrm{GR}} \propto a^{-4}, \quad r^* > r_{\mathrm{RL}}$
• Super-massive binary	GRAPE + TTL + GR

Seppo Mikkola	KS, chain & TTL
Rosemary Mardling	Circularization & stability
J. Hurley & C. Tout	Stellar evolution

http://www.ast.cam.ac.uk/~sverre http://sverre.com

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,t + \frac{1}{2}\mathbf{F}_0^{(2)}\,t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\,t^3 \\ \mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\,t + \frac{1}{2}\mathbf{F}_0^{(3)}\,t^2\end{aligned}$$

Higher derivatives

$$\begin{aligned}\mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{6}{t^3} \\ \mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{2}{t^2}\end{aligned}$$

Corrector

$$\begin{aligned}\Delta\mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5 \\ \Delta\mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4\end{aligned}$$

Quantized time-steps

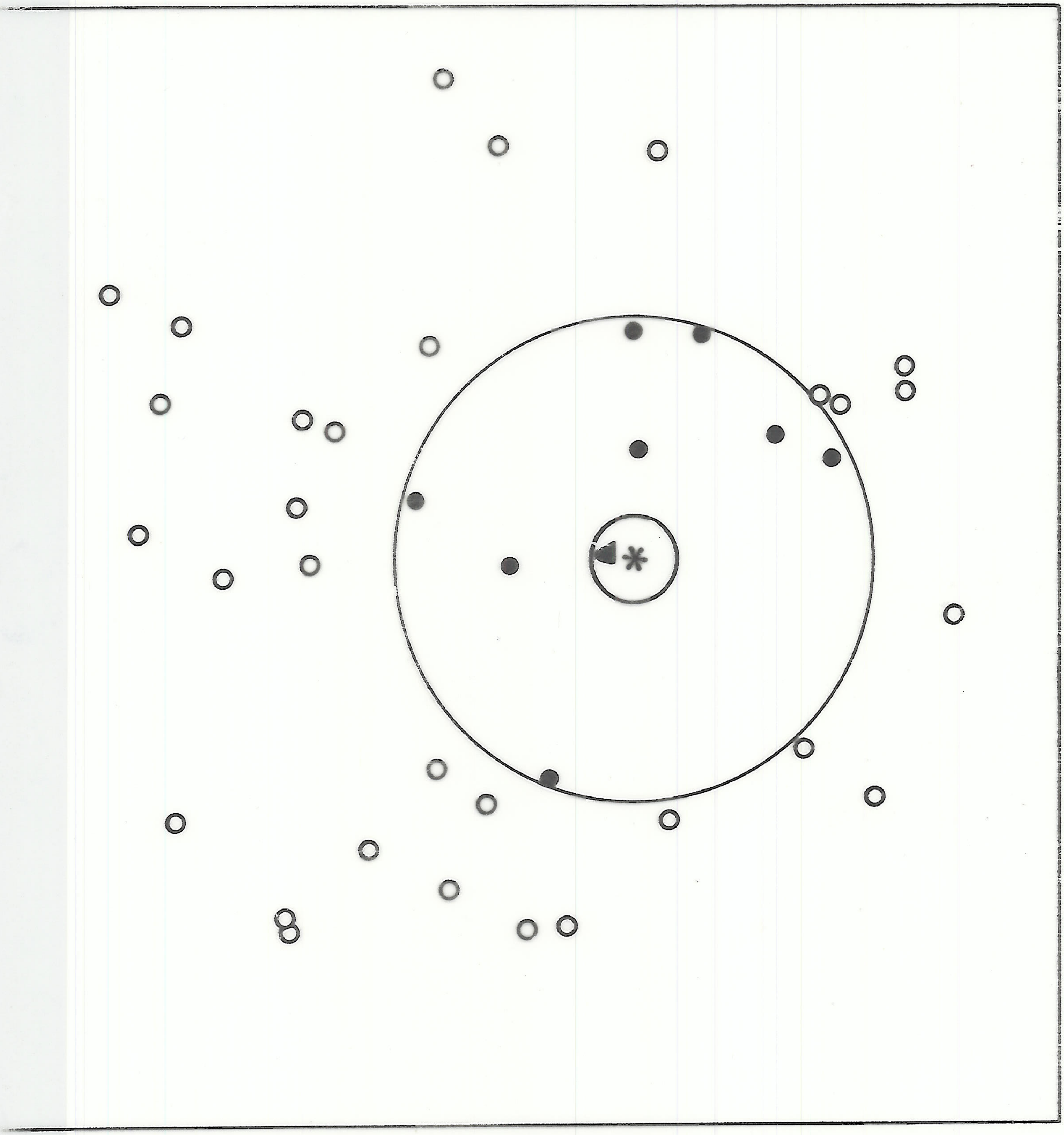
$$\Delta t_n = \left(\frac{1}{2}\right)^{n-1}$$

Prediction

$$\begin{aligned}\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}^{(1)}\,\delta t'_j + \frac{1}{2}\mathbf{F})\,\delta t'_j + \mathbf{v}_0)\,\delta t'_j + \mathbf{r}_0 \\ \dot{\mathbf{r}}_j &= ((\frac{1}{2}\mathbf{F}^{(1)}\,\delta t'_j + \mathbf{F})\,\delta t'_j + \mathbf{v}_0\,; \quad \delta t'_j = t - t_j\end{aligned}$$

Neighbour scheme

Total force	$\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$
Prediction scheme	$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$
	$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$
Time-scales	$\Delta t_n << \Delta t_d, \quad n << N$
Individual time-steps	$\Delta t_i = \left(\frac{\eta \lvert\mathbf{F}\rvert}{\lvert\ddot{\mathbf{F}}\rvert}\right)^{1/2}, \quad \eta \simeq 0.02$
Neighbour sphere	$R_s^{\rm new} = R_s^{\rm old}\left(\frac{n_p}{n}\right)^{1/3}, \quad n_p \simeq N^{1/2}$
Neighbour selection	$\lvert\mathbf{r}_i - \mathbf{r}_j\rvert < R_s$, Full N loop
Derivative corrections	$\ddot{\mathbf{F}}_{ij}$, $\mathbf{F}_{ij}^{(3)}$, Explicit differentiation
Performance	Break-even for $N \simeq 50$
Integration scheme	Divided differences or Hermite



Principles of regularization

Two-body motion $\ddot{\mathbf{R}} = -(m_k + m_l)\mathbf{R}/R^3 + (\mathbf{F}_k - \mathbf{F}_l)$

Time transformation $dt = Rd\tau$

$$\begin{aligned} \frac{d}{dt} &= \frac{1}{R}\frac{d}{d\tau} \\ \frac{d^2}{dt^2} &= \frac{1}{R^2}\frac{d^2}{d\tau^2} - \frac{R'}{R^3}\frac{d}{d\tau} \end{aligned}$$

Motion in 1D $x'' = x'^2/x - (m_k + m_l)$

Binding energy $h = \frac{1}{2}\dot{x}^2 - (m_k + m_l)/x$

Substitution $\dot{x} = x'/x$

Displaced harmonic oscillator $x'' = 2hx + (m_k + m_l)$

Coordinate transformation $u^2 = x; \quad x' = 2uu'$

Final equation of motion $u'' = \frac{1}{2}hu$

Hermite KS formulation

Equations of motion

$$
\begin{aligned}
\mathbf{F}_u &= \frac{1}{2} h\, \mathbf{u} + \frac{1}{2} R\, \mathbf{Q}, \qquad \mathbf{Q} = \mathcal{L}^{\mathcal{T}}\, \mathbf{P} \\
\mathbf{F}'_u &= \frac{1}{2}(h'\mathbf{u} + h\mathbf{u}' + R'\mathbf{Q} + R\mathbf{Q}') \\
h' &= 2\mathbf{u}' \cdot \mathbf{Q} \\
h'' &= 2\mathbf{F}_u \cdot \mathbf{Q} + 2\mathbf{u}' \cdot \mathbf{Q}' \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}
$$

Low-order prediction $\mathbf{r}_j,\ \dot{\mathbf{r}}_j$ for perturbers and c.m.

Taylor series prediction $\mathbf{u}, \mathbf{u}'$ to order $\mathbf{F}_u^{(2)}$

KS transformation $\mathbf{r}_k = \mathbf{r}_{\rm cm} + \mu\mathbf{R}/m_k,\ \ \mathbf{r}_l = \mathbf{r}_{\rm cm} - \mu\mathbf{R}/m_l$

Physical perturbation $\mathbf{P},\ \dot{\mathbf{P}};\quad \mathbf{P}' = R\dot{\mathbf{P}}$

Taylor series corrector $\mathbf{u}, \mathbf{u}'$ to order $\mathbf{F}_u^{(3)}$

Physical time interval $\Delta t = \Sigma_{k=1}^{n} \frac{1}{k!}\, t_0^{(k)} \Delta\tau^k$

$$
\begin{aligned}
t''_0 &= 2\mathbf{u}' \cdot \mathbf{u} \\
t_0^{(3)} &= 2\mathbf{F}_u \cdot \mathbf{u} + 2\mathbf{u}' \cdot \mathbf{u}
\end{aligned}
$$

Inversion to regularized time $\delta\tau = \Sigma_{k=1}^{3} \frac{1}{k!} \tau_0^{(k)} \delta t^k$

$$
\tau_0^{(2)} = -t_0^{(2)}/R^3
$$

Practical Aspects of KS

Regular equations	Perturbed harmonic oscillator, $\gamma < 1$
Constant time-step	$\Delta\tau = \eta \left(\frac{1}{2\lvert h \rvert}\right)^{1/2}$ vs $\Delta t \propto R^{3/2}$
Linearized equations	Higher accuracy per step
Faster force calculation	Tidal perturbation, $P \propto 1/r^3$
Unperturbed motion	$\gamma < 10^{-6}, \quad \Delta t > t_{\rm K}$
Slow-down procedure	Adiabatic invariance, $\tilde{P} = \kappa P$
Energy rectification	Improve $\mathbf{u}, \mathbf{u}'$ from integration of h'
C.m. approximation	$d > 100\, a\,(1 + e)$
Transformations	$\mathbf{R} = \mathcal{L}\mathbf{u}, \qquad \mathbf{r}_j = \mathbf{r}_{\rm cm} \pm \mu \mathbf{R}/m_j$
	$\dot{\mathbf{R}} = 2\mathcal{L}\mathbf{u}'/R, \quad \dot{\mathbf{r}}_j = \dot{\mathbf{r}}_{\rm cm} \pm \mu \dot{\mathbf{R}}/m_j$
Two-body elements	$a, \mathbf{J}, e$ for averaging & circularization

N-body interface

Centre of mass acceleration

$$\ddot{\mathbf{r}}_{cm} = (m_k \mathbf{F}_k + m_l \mathbf{F}_l)/(m_k + m_l)$$

Global coordinates

$$\begin{aligned} \mathbf{r}_k &= \mathbf{r}_{cm} + \mu \mathbf{R}/m_k \\ \mathbf{r}_l &= \mathbf{r}_{cm} - \mu \mathbf{R}/m_l \end{aligned}$$

Relative perturbation

$$\gamma = |\mathbf{F}_k - \mathbf{F}_l| R^2/(m_k + m_l)$$

Tidal approximation

$$r_\gamma = R[2\tilde{m}/(m_k + m_l)\gamma_{\min}]^{1/3}, \qquad \gamma_{min} \simeq 10^{-6}$$

Perturber selection

$$r_{ij} < r_\gamma, \qquad R = a(1+e)$$

Regularized time-step

$$\Delta\tau = \eta_u (1/2|h|)^{1/2} 1/(1 + 1000\gamma)^{1/3}$$

Physical time-step

$$\Delta t = \sum_{k=1}^{n} \frac{1}{k!} t_0^{(k)} \Delta\tau^k, \qquad n = 6$$

Time derivatives

$$\begin{aligned} t_0'' &= 2\mathbf{u}' \cdot \mathbf{u} \\ t_0^{(3)} &= 2\mathbf{u}'' \cdot \mathbf{u} + 2\mathbf{u}' \cdot \mathbf{u} \end{aligned}$$

Unperturbed motion

KS Decision-Making

Close encounter	$R_{\rm cl} = \dfrac{4\,r_{\rm h}}{N\,C^{1/3}}, \quad \Delta t_{\rm cl} = \beta \left(\dfrac{R_{\rm cl}^3}{\bar{m}}\right)^{1/2}$
Time-step criterion	$\Delta t_k < \Delta t_{\rm cl}$
Neighbour list search	list all $r_{kj}^2, \quad \Delta t_j < 2\,\Delta t_{\rm cl}$
Two-body selection	$R_{kl} < R_{\rm cl}, \quad \dot{R}_{kl} < 0$
Dominant motion	$\dfrac{m_k + m_l}{R_{kl}^2} > \dfrac{m_k + m_j}{R_{kj}^2}$
KS initialization	$\mathbf{F}_U, \mathbf{F}'_U, \Delta\tau \,\&\, t^{(n)} \;\Rightarrow\; \Delta t$
Initialization of c.m.	$\mathbf{r}_{\rm cm} = \dfrac{m_k\,\mathbf{r}_k + m_l\,\mathbf{r}_l}{m_k + m_l}$
Perturber search	$r_{\rm p} < \left(\dfrac{2m_{\rm p}}{m_{\rm b}\gamma_{\rm min}}\right)^{1/3} a\,(1+e)$
Slow-down adjustment	$\gamma < \gamma_0, \quad \Delta\tau \Rightarrow \kappa\,\Delta t$
Termination test	$R > R_0, \quad \gamma > \gamma^*$
Delayed termination	$T_{\rm block} - t > \Delta t_i$
Final iteration	$\Delta\tau$ from $\dot{\tau}, \ddot{\tau}, \ldots$ and δt
Polynomial initialization	$\mathbf{F}_j, \dot{\mathbf{F}}_j, \Delta t_j, \quad j = k, l$

Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} = \dfrac{m_k m_3}{m_k + m_3}$

$$H = \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS coordinate transformation $\mathbf{Q}_k^2 = R_k\,, \quad (k = 1, 2)$

Time transformation $dt = R_1 R_2\, d\tau$

Regularized Hamiltonian $\Gamma^* = R_1 R_2\,(H - E_0)$

$$\Gamma^* = \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_l\, \mathbf{P}_k^2 + \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2 \mathbf{P}_2$$

$$- m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}, \qquad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \to 0 \quad \text{or} \quad R_2 \to 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| > \max(R_1,\ R_2)$

Three-Body Transformations

Coordinates & momenta $\mathbf{q}_k = \tilde{\mathbf{q}}_k - \tilde{\mathbf{q}}_3, \quad \mathbf{p}_k = \tilde{\mathbf{p}}_k$

Regularized coordinates ($q_1 \geq 0$)

$$\begin{aligned} Q_1 &= [\tfrac{1}{2}(|\mathbf{q}_1| + q_1)]^{1/2} \\ Q_2 &= \tfrac{1}{2}q_2/Q_1 \\ Q_3 &= \tfrac{1}{2}q_3/Q_1 \\ Q_4 &= 0 \end{aligned}$$

Regularized momenta $\mathbf{P}_k = \mathbf{A}_k \, \mathbf{p}_k$

Basic matrix

$$\mathbf{A}_1 = 2 \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \\ -Q_2 & Q_1 & Q_4 & -Q_3 \\ -Q_3 & -Q_4 & Q_1 & Q_2 \\ Q_4 & -Q_3 & Q_2 & -Q_1 \end{bmatrix}$$

KS transformations $\mathbf{q}_k = \tfrac{1}{2}\mathbf{A}_k^{\mathrm{T}}\mathbf{Q}_k$

Physical momenta $\mathbf{p}_k = \tfrac{1}{4}\mathbf{A}_k^{\mathrm{T}}\mathbf{P}_k/R_k$

Coordinates & momenta

$$\begin{aligned} \tilde{\mathbf{q}}_3 &= -\sum_{k=1}^{2} m_k \mathbf{q}_k / M \\ \tilde{\mathbf{q}}_k &= \tilde{\mathbf{q}}_3 + \mathbf{q}_k \\ \tilde{\mathbf{p}}_k &= \mathbf{p}_k \\ \tilde{\mathbf{p}}_3 &= -(\mathbf{p}_1 + \mathbf{p}_2) \qquad (k = 1, 2) \end{aligned}$$

Wheel-Spoke Regularization

Initial conditions $\quad m_i,\ \tilde{\mathbf{q}}_i,\ \tilde{\mathbf{p}}_i,\quad i = 0,\ldots,N,\quad n \le N$

Generating function $\quad W(\mathbf{q}_i, \tilde{\mathbf{p}}_i) = \sum_{i=1}^{N} \tilde{\mathbf{p}}_i \cdot \mathbf{q}_i + \left(\sum_{i=0}^{N} \tilde{\mathbf{p}}_i\right) \cdot \mathbf{q}_0$

Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2\mu_i} + \frac{1}{m_0} \sum_{i<j}^{N} \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{p}_j - m_0 \sum_{i=1}^{N} \frac{m_i}{R_i} - \sum_{i<j}^{N} \frac{m_i m_j}{R_{ij}}$$

Canonical variables $\quad W(\mathbf{p}_i, \mathbf{Q}_i) = \sum_{i=1}^{N} \mathbf{p}_i^{\mathrm{T}} \cdot \mathbf{f}_i(\mathbf{Q}_i)$

Regularized momenta $\quad \mathbf{P}_i = \mathbf{A}_i \mathbf{p}_i\,,\quad (i = 1,\ldots,n)$

Coordinates and momenta $\quad \mathbf{q}_i = \tilde{\mathbf{q}}_i - \tilde{\mathbf{q}}_0,\quad \mathbf{p}_i = \tilde{\mathbf{p}}_i$

Inverse transformations $\quad \mathbf{q}_i = \frac{1}{2}\mathbf{A}_i^{\mathrm{T}}\mathbf{Q}_i,\quad \mathbf{p}_i = \frac{1}{4}\mathbf{A}_i^{\mathrm{T}}\mathbf{P}_i / R_i$

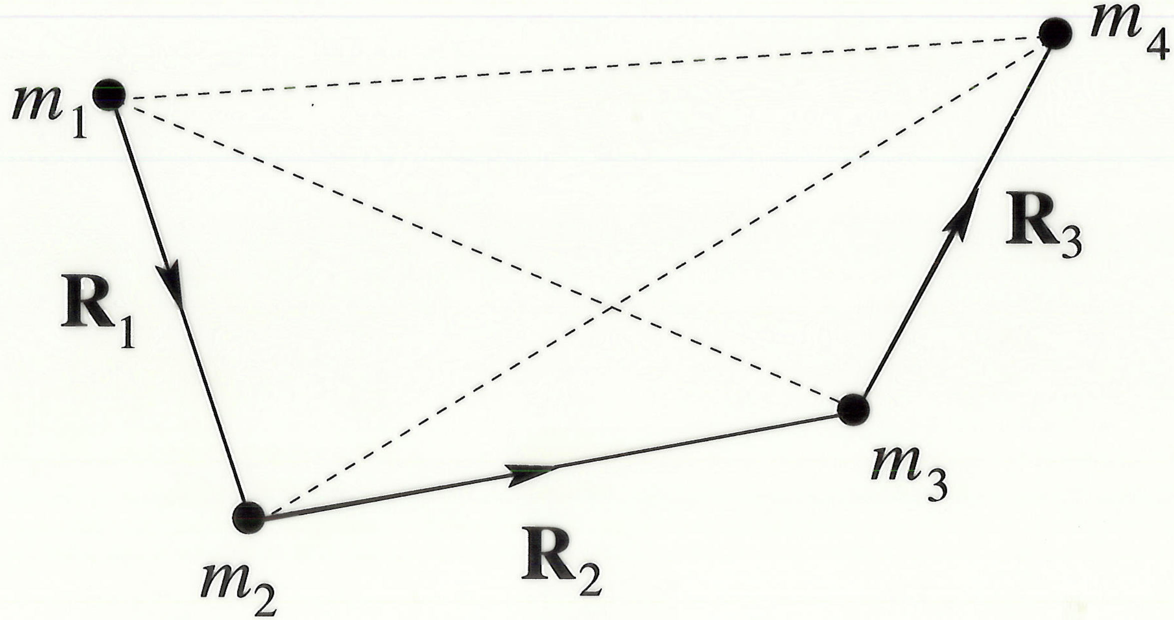
Local coordinates & momenta

$$\begin{aligned}
\tilde{\mathbf{q}}_0 &= -\sum_{i=1}^{n} m_i \mathbf{q}_i \Big/ \sum_{i=0}^{n} m_i \\
\tilde{\mathbf{q}}_i &= \tilde{\mathbf{q}}_0 + \mathbf{q}_i \\
\tilde{\mathbf{p}}_i &= \mathbf{p}_i\,, \qquad\qquad (i = 1,\ldots,n) \\
\tilde{\mathbf{p}}_0 &= -\sum_{i=1}^{n} \mathbf{p}_i
\end{aligned}$$

Wheel-Spoke Implementation

Select members	$\Delta t_{\rm cm} < \Delta t_{\rm cl}, \quad R = a\,(1+e)$
Initialize in local c.m.	$\sum m_i\,\mathbf{r}_i, = 0, \quad \sum m_i\,\dot{\mathbf{r}}_i = 0$
Chain indices & vectors	$\mathbf{Q},\,\mathbf{P}, \quad N_{\rm eq} = 8\,(N-1)$
Define useful quantities	$T_{\rm cr},\, R_{\rm grav},\, \Delta\tau_0$
Softening of singularities	$\epsilon = f\,R_{\rm grav}, \quad \Rightarrow\; E = {\rm const}$
Form perturber list	$d < \left(\frac{2m}{M_{\rm ch}\gamma_0}\right)^{1/3} R_{\rm grav}$
Check time-step	$\Delta\tau = \int L\,dt, \quad L = T - \Phi$
B-S integration step	$\mathbf{r}_i = ((\frac{1}{6}\dot{\mathbf{F}}_i\delta t_i + \frac{1}{2}\dot{\mathbf{F}}_i)\delta t_i + \dot{\mathbf{r}}_i)\delta t_i$
Physical variables	$\mathbf{R}_k = \frac{1}{2}\,\mathbf{A}_k\,\mathbf{Q}_k, \quad \mathbf{p}_k = \frac{1}{4}\,\frac{\mathbf{A}_k\,\mathbf{P}_k}{\mathbf{Q}_k^2}$
Addition of member	$\gamma > 0.05, \quad r_p \le \sum R_k$
Termination test	$\dot{R}^2 > 2M/R, \quad R > R_{\rm cl}$
Continue N-body integration	$t > t_{\rm max} = t_{\rm blk}$

Chain Regularization



Chain Regularization

Chain vectors $\mathbf{R}_k = \mathbf{r}_{k+1} - \mathbf{r}_k; \quad k = 1, ..., N-1$

Physical momenta $\mathbf{p}_k = m_k \mathbf{v}_k; \quad k = 1, ..., N$

Relative momenta $\mathbf{W}_k = \mathbf{W}_{k-1} - \mathbf{p}_k; \quad k = 2, ..., N-2$

Hamiltonian

$$H = \sum_{k=1}^{N-1} \frac{1}{2}\left(\frac{1}{m_k} + \frac{1}{m_{k+1}}\right)\mathbf{W}_k^2 - \sum_{k=2}^{N} \frac{1}{m_k}\mathbf{W}_{k_1} \cdot \mathbf{W}_k - $$
$$- \sum_{k=1}^{N-1} \frac{m_k m_{k+1}}{R_k} - \sum_{1 \le i \le j-2}^{N} \frac{m_i m_j}{R_{ij}}$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}; \qquad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

KS relations $\mathbf{R}_k = \mathcal{L}_k\,\mathbf{Q}_k; \quad \mathbf{W}_k = \mathcal{L}_k\,\mathbf{P}_k / 2\mathbf{Q}_k^2$

Time transformation $dt = g d\tau\,; \quad g = 1/L$

Regularized Hamiltonian $\Gamma^* = g(H - E)$

Regular solutions $R_k \to 0; \qquad k = 1, ..., N-1$

Time-transformed leapfrog

Standard leapfrog

$$\mathbf{r}_{\frac{1}{2}} = \mathbf{r}_0 + \frac{h}{2}\mathbf{v}_0$$

$$\mathbf{v}_1 = \mathbf{v}_0 + h\,\mathbf{F}(\mathbf{r}_{\frac{1}{2}})$$

$$\mathbf{r}_1 = \mathbf{r}_{\frac{1}{2}} + \frac{h}{2}\mathbf{v}_1$$

$$t_1 = t_0 + h$$

Time transformation

$$ds = \Omega(\mathbf{r})\,dt; \quad W = \Omega$$

Equations of motion

$$\mathbf{r}' = \mathbf{v}/W, \quad t' = 1/W, \quad \mathbf{v}' = \mathbf{F}/\Omega$$

Auxiliary function

$$\dot{W} = \mathbf{v} \cdot \frac{\partial \Omega}{\partial \mathbf{r}}, \qquad W' = \frac{\dot{W}}{\Omega}$$

New leapfrog

$$\mathbf{r}_{\frac{1}{2}} = \mathbf{r}_0 + \frac{h}{2}\frac{\mathbf{v}_0}{W_0}$$

$$t_{\frac{1}{2}} = t_0 + \frac{h}{2}\frac{1}{W_0}$$

$$\mathbf{v}_1 = \mathbf{v}_0 + h\,\frac{\mathbf{F}(\mathbf{r}_{\frac{1}{2}})}{\Omega(\mathbf{r}_{\frac{1}{2}})}$$

$$W_1 = W_0 + h\frac{\mathbf{v}_0 + \mathbf{v}_1}{2\Omega(\mathbf{r}_{\frac{1}{2}})} \cdot \frac{\partial \Omega(\mathbf{r}_{\frac{1}{2}})}{\partial \mathbf{r}_{\frac{1}{2}}}$$

$$\mathbf{r}_1 = \mathbf{r}_{\frac{1}{2}} + \frac{h}{2}\frac{\mathbf{v}_1}{W_1}$$

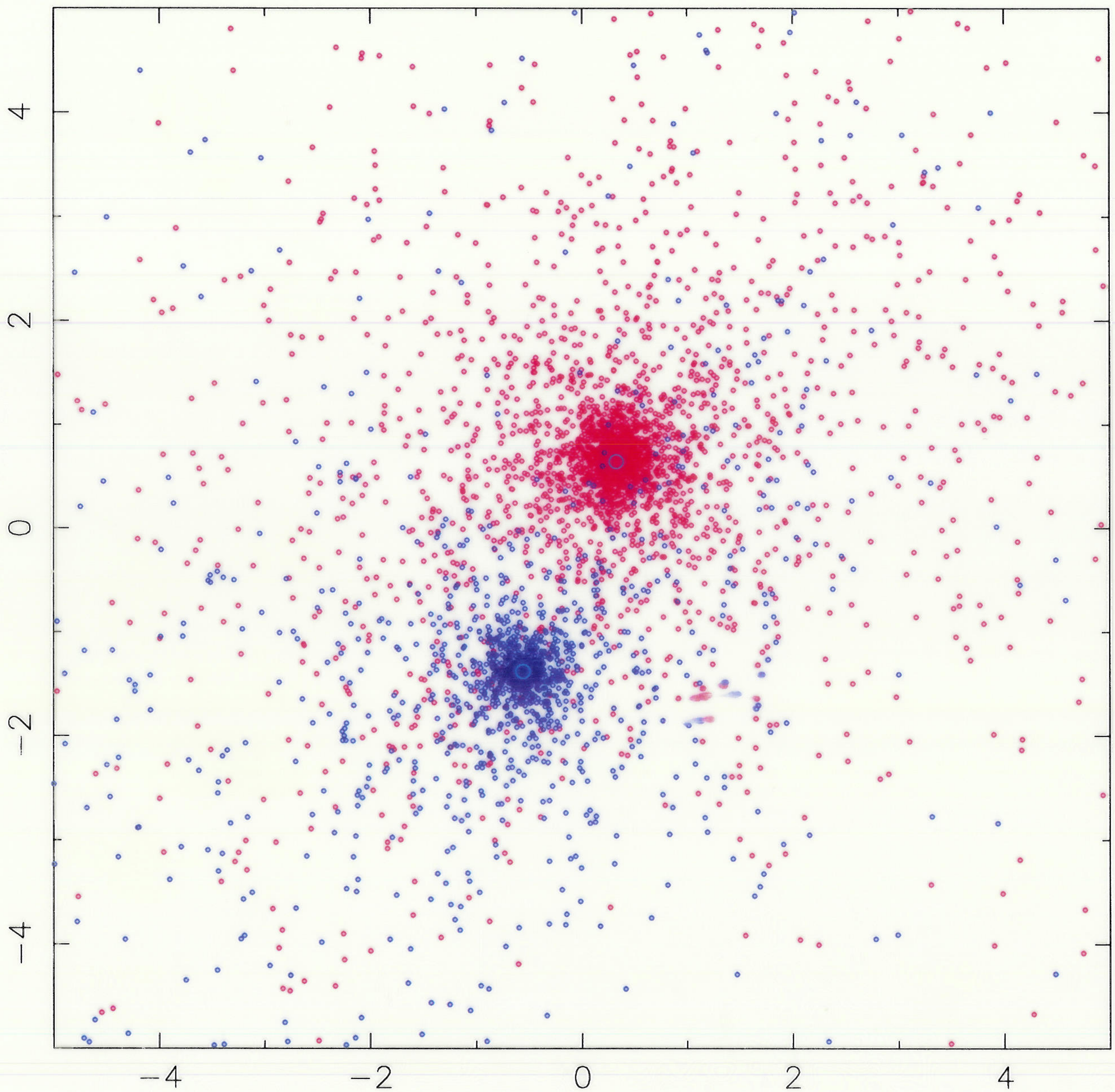
$$t_1 = t_{\frac{1}{2}} + \frac{h}{2}\frac{1}{W_1}$$

TTL treatment

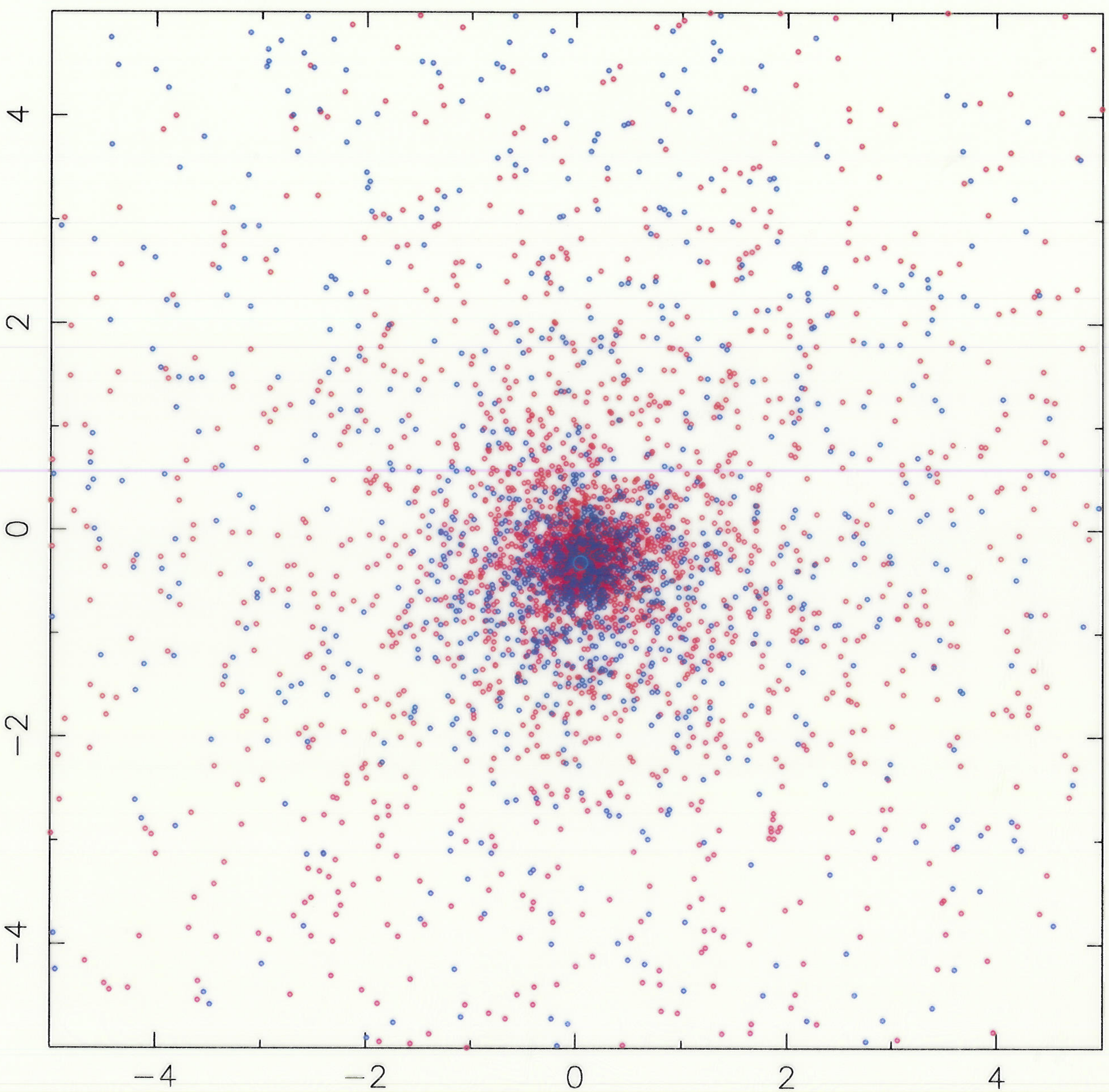
Auxiliary function	$\Omega = \sum\sum_{i<j} \frac{\Omega_{ij}}{r_{ij}}, \quad \Omega_{ij} = m_i m_j$ or 1
Gradient	$\frac{\partial\Omega}{\partial \mathbf{r}_k} \equiv \mathbf{G}_k = \sum_{j\neq k} \Omega_{jk} \frac{\mathbf{r}_j - \mathbf{r}_k}{r_{jk}^3}$
Equations of motion	$\mathbf{r}'_k = \frac{\mathbf{v}_k}{W}, \quad t' = \frac{1}{W}$
	$\mathbf{v}'_k = \frac{\mathbf{F}_k}{\Omega}, \quad W' = \frac{1}{\Omega}\sum_k \mathbf{v}_k \cdot \mathbf{G}_k$
External forces	$\mathbf{v}' = \frac{\mathbf{F} + \mathbf{f}(\mathbf{v})}{\Omega}$
Velocity jump	$\mathbf{v}_1 = \frac{\mathbf{v}_0 + h(\mathbf{F} + \mathbf{f}(\mathbf{v}_a))}{\Omega}$
Midpoint rule	$\mathbf{v}_a = \frac{\mathbf{v}_0 + \mathbf{v}_1}{2}$, 2 iterations
Energy equation	$E'_{\rm int}(\mathbf{v}_a) = \sum_k m_k \mathbf{v}_k \cdot \frac{\mathbf{f}_k(\mathbf{v})}{\Omega}$
Conservation	$E_{\rm tot} = E_{\rm ext} + E_{\rm int}, \quad \Delta E_{\rm int} = h\, E'_{\rm int}(\mathbf{v}_a)$
GR	$\mathbf{F} = \mathbf{F}_0 + c^{-2}\,\mathbf{F}_2 + c^{-4}\,\mathbf{F}_4 + c^{-5}\,\mathbf{F}_5$

TTL implementation

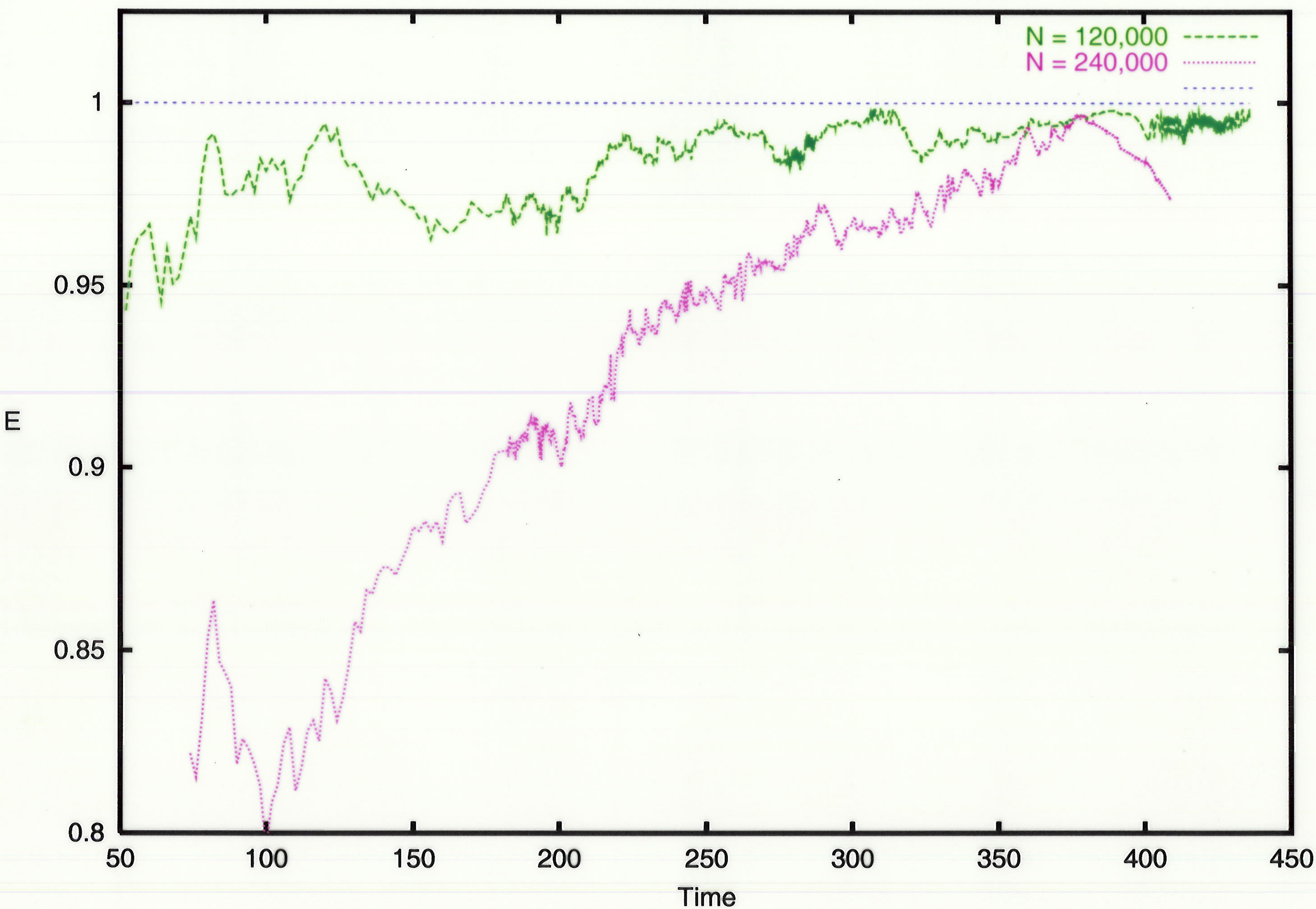
Coordinate system	chain vectors or binary c.m.
Perturbing force	$\mathbf{P}_i = \mathbf{F}_i - \frac{1}{M_{\rm cm}} \sum_{i=1}^{N_{\rm bh}} m_i \sum_{j=1}^{N_{\rm pert}} \frac{m_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}$
Particle force	GRAPE & $\Delta \mathbf{F}$, $\Delta \dot{\mathbf{F}}$
Energy check	$\Delta E_{\rm bh} = E_{\rm int} + E_{GR} + E_{\rm pert} - E_{\rm calc}$
Membership change	$d < 25\, a_{\rm bh}, \quad E_{\rm tot} = E_{\rm ext} + E_{\rm int}$
Updating	$\Delta E_{\rm corr} = \Delta T_1 + \Delta \Phi - \Delta T_2$
CPU cost	$T_{\rm comp} \simeq \frac{N_{\rm bh}}{a_{\rm bh}^{3/2}}(A + BN_{\rm pert})$
GR terms	$\dot{\mathbf{F}}$ not needed, cf. $\dot{a}$, $\dot{e}$ with KS
Accuracy	exact two-body motion
Early stages	KS and chain regularization
Switch to TTL	$a_{\rm bh} \simeq R_{\rm close}$, BS integrator
Final stages	host is bottleneck with GR



M14
T=28



M14
T = 56


N = 120,000
N = 240,000
E
1
0.95
0.9
0.85
0.8
50
100
150
200
250
300
350
400
450
Time

Hierarchical Systems

Hierarchical formation	$\mathrm{B} + \mathrm{B} \Rightarrow \mathrm{T} + \mathrm{S}$ or $\mathrm{B} + \tilde{\mathrm{B}}, \quad e_{\rm out} < 1$
Dynamical molecules	[B,S], [B,B], [[B,S],S], [[B,B],S]
Formation rate	binary fraction
Stability	$a_{\rm out}(1 - e_{\rm out}) > \Psi(m, e_{\rm in}, i)\, a_{\rm in}$
Chaos boundary	fuzzy region & holes
Inclination effect	prograde vs retrograde stability
Kozai cycles	$\cos^2 i\,(1 - e_{\rm in}^2) = \mathrm{const}$
Eccentricity modulation	orbit averaging
Instability	$\dot{e}_{\rm out} > 0 \quad \Rightarrow$ slingshot
Superfast particles	time-step reduction

Hierarchical Stability

Chaos theory: binary tides problem

Correspondence: inner binary acts as oscillator

Stability criterion: planar prograde motion

$$a_{out}(1 - e_{out}) \; > \; R_p^{out}$$

$$R_p^{out} \; = \; C \left[(1 + q_{out}) \frac{(1 + e_{out})}{(1 - e_{out})^{1/2}} \right]^{2/5} a_{in}$$

No secular effects and neglect of short-term fluctuations

Temporary merger of inner binary: KS of outer component

Heuristic inclination effect

$$\tilde{R}_p^{out} = R_p^{out} \left(1 - \frac{0.3\psi}{180} \right)$$

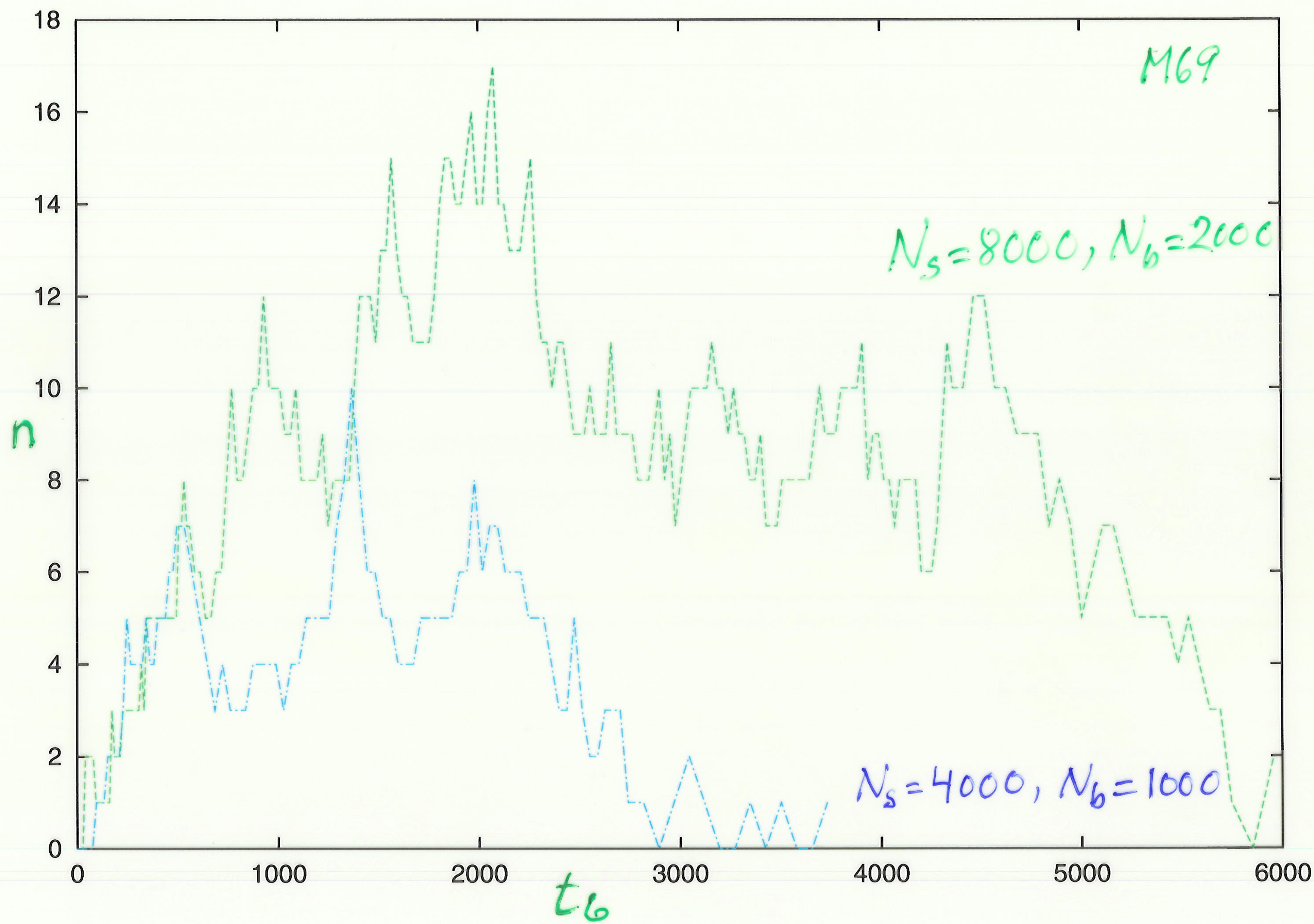
Higher-order systems: [B-B] & [[B-B]-B]

$$\tilde{C} \; = \; C(1 + 0.1 a_2/a_1)$$

Zare (1976) exchange criterion

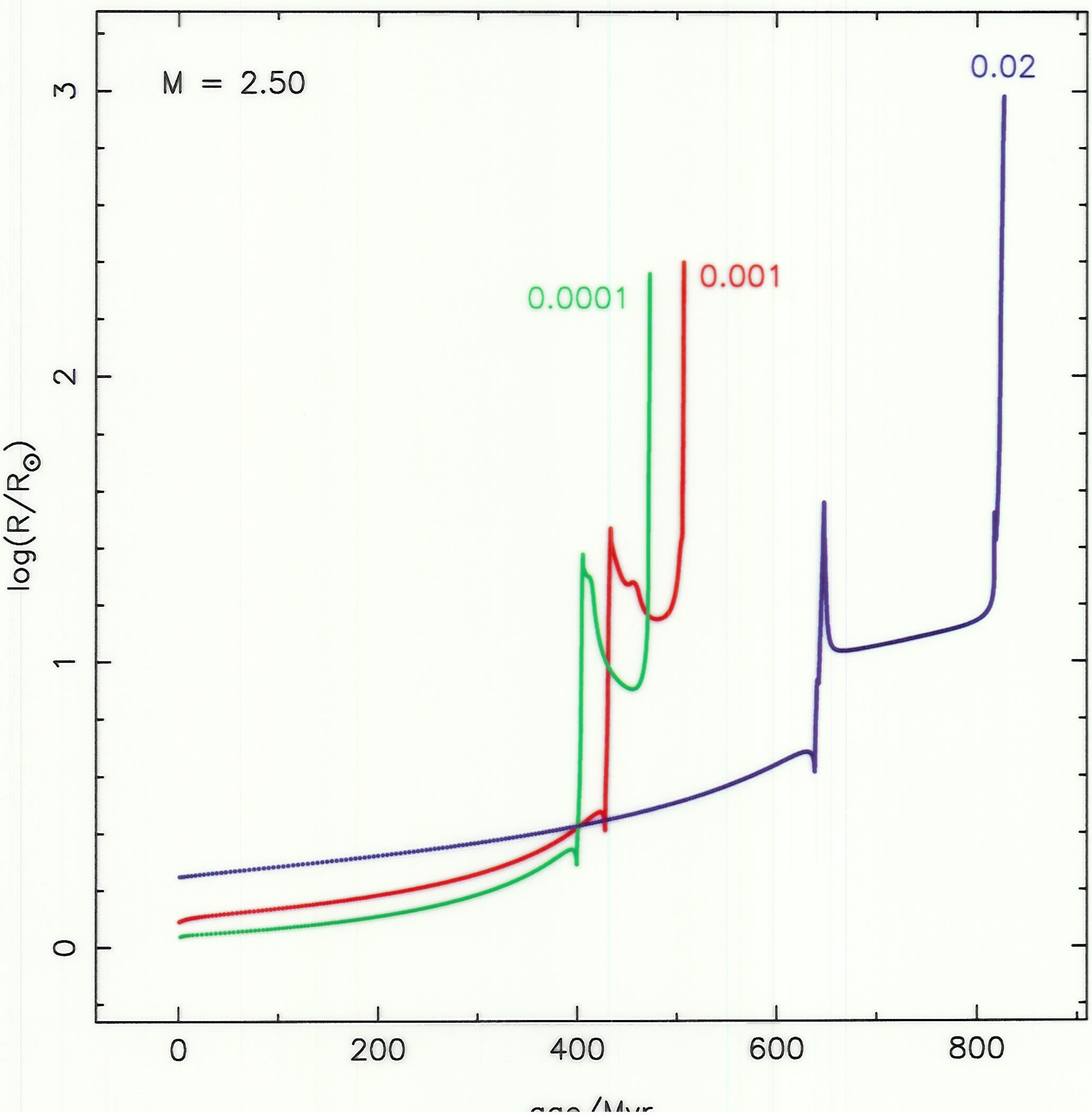
$$(c^2 E)_{crit} = -\frac{G^2 f^2(\rho) g(\rho)}{2(m_1 + m_2 + m_3)}$$

Boundary of exchange versus escape: $q_{out} \; \simeq \; 5$


M69
$N_s=8000, N_b=2000$
$N_s=4000, N_b=1000$
n
t_6
0
2
4
6
8
10
12
14
16
18
0
1000
2000
3000
4000
5000
6000

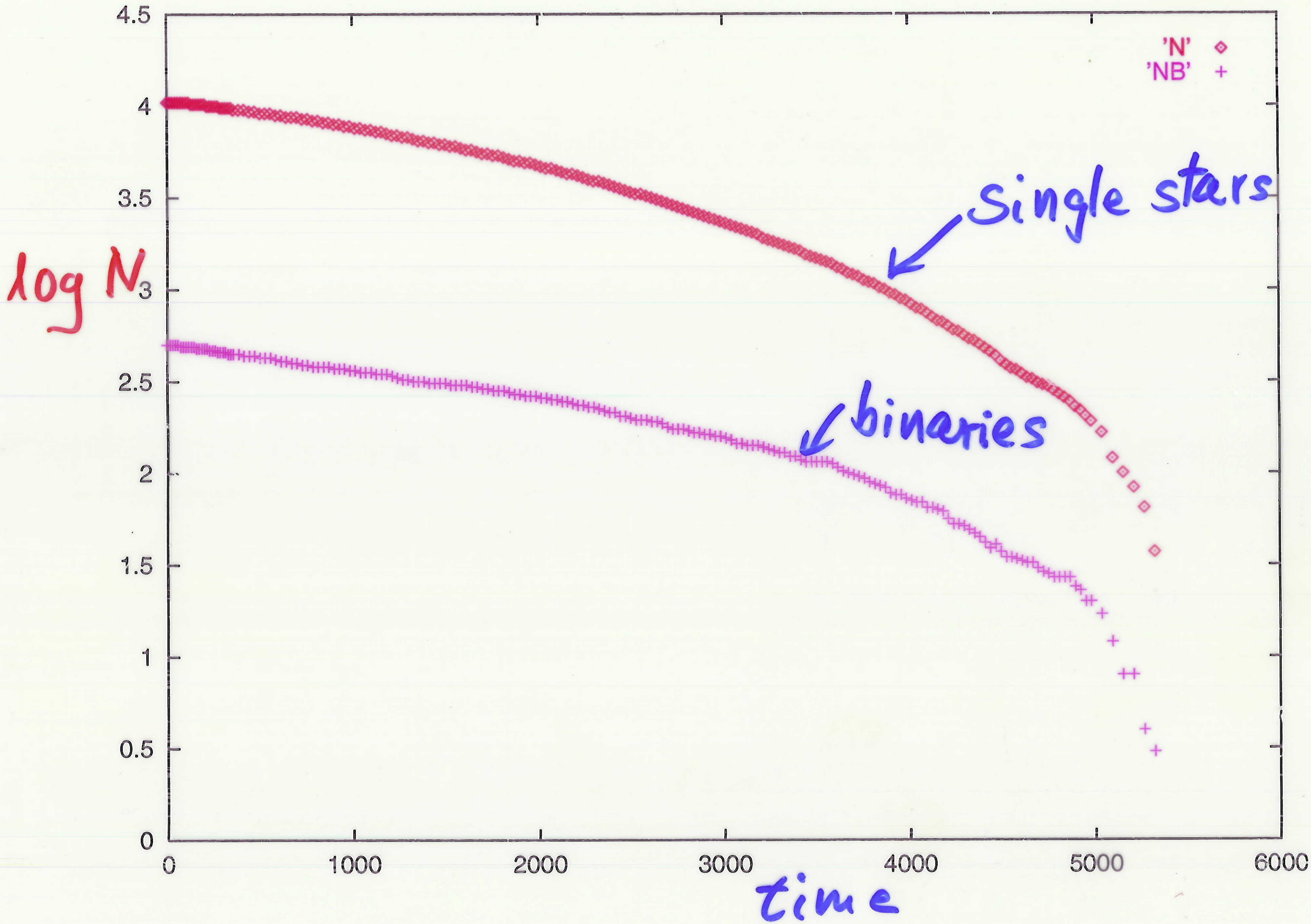
Stellar Evolution

Stellar HR types	$K^* = 0, ..., 15$
Fast look-up (Pop I & II)	$r^*(t)$, $m_c(t)$, $L^*(t)$, $K^*(t)$
Wind mass loss	$\dot{m} = -2 \times 10^{-13} r^* L^*/m$
Single stars	$\Delta m/m > 2\%$, new r^*
Updating times	$T_{\rm ev} = t + \min(\Delta t_{\rm ev}, \Delta t_{\rm rem})$
Stellar rotation	$\Delta J_{\rm spin} = 2\Delta m r^2 \Omega_{\rm rot}/3$
White dwarfs	cooling curves
Supernova outburst	$m_{\rm c} > m_{\rm chandra} \Rightarrow$ SN
NS velocity kick	$v >> v_\infty \sim 2\,km/s$
Binary mass loss	$m\,a =$ const
Synthetic HR diagram	binaries and single stars
Energy conservation	$\Delta E = \Delta m\,(\frac{1}{2}v^2 + \Phi)$


M = 2.50
0.02
0.0001
0.001
log(R/R⊙)
0
1
2
3
0
200
400
600
800

Binary Processes

Tidal circularization	$a\,(1 - e^2) = \mathrm{const} \;\; \Rightarrow \;\; \dot{a} < 0$
Roche-lobe mass transfer	$r^* > r_{\mathrm{RL}}, \;\; \Delta m_2 = -f \Delta m_1$
Common envelope evolution	$m_c > 0$, MS + giant
Magnetic braking	$\dot{a}_{\mathrm{MB}} \propto a^{-4}$
Gravitational radiation	$\dot{a}_{\mathrm{GR}} \propto a^{-3}$
Spin-orbit coupling	$J_{\mathrm{tot}} = J_{\mathrm{orb}} + J_{\mathrm{spin}}$
Stellar collisions	$a\,(1 - e) < 0.75\,(r_1^* + r_2^*)$
Blue stragglers	mass transfer or MS collisions
Cataclysmic variables	WD + giant
X-ray objects	WD + MS, NS + MS
Doubly degenerates	WD + WD, $P \simeq 10$ mins
Type Ia supernova	WD – WD collision or inspiral


'N'
'NB'
Single stars
binaries
log N
time
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
1000
2000
3000
4000
5000
6000

Direct N-Body Codes

NBODY1	ITS	ϵ	$3 - 100$
NBODY2	ACS	ϵ	$50 - 10^4$
NBODY3	ACS	KS & MREG	$3 - 100$
NBODY4	HITS	KS & MREG	$10 - 10^5$ x)
NBODY5	ACS	KS & MREG	$50 - 10^4$
NBODY6	HACS	KS & MREG	$50 - 10^4$
NBODY7	HACS	KS & BH	$50 - 10^4$

MREG: Three-body, four-body & chain regularization

http://www.ast.cam.ac.uk/~sverre

http://sverre.com

http://www.nbodylab.org

public micro-Grape