Fluid-structure interaction problems in the cardiovascular system

Optimal Transport in the Human Body Lungs and Blood,
UCLA, May 19 - 23, 2008

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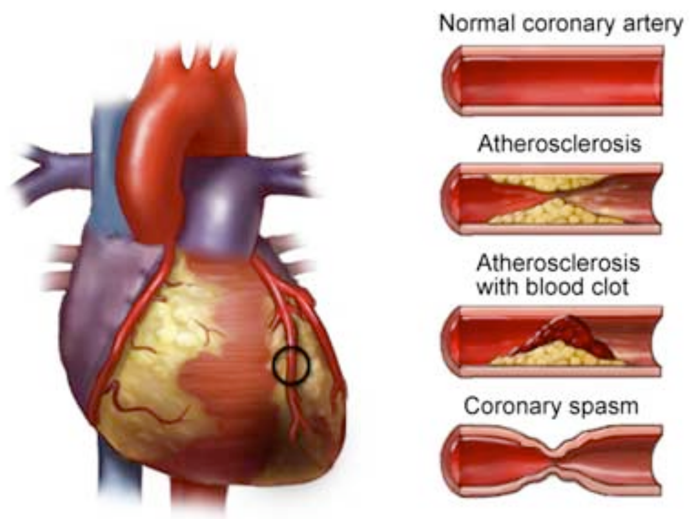
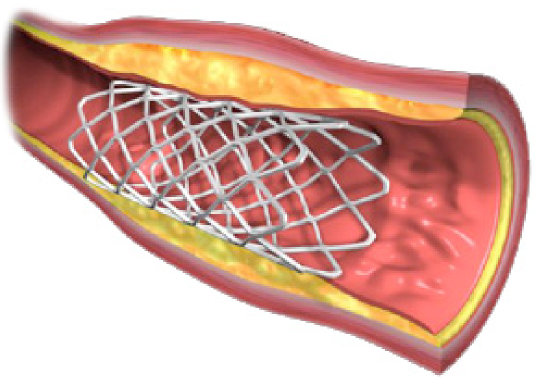
Cardiovascular system modelling ???

- Blood flows and arteries are much more complicated than what will be presented in this talk !

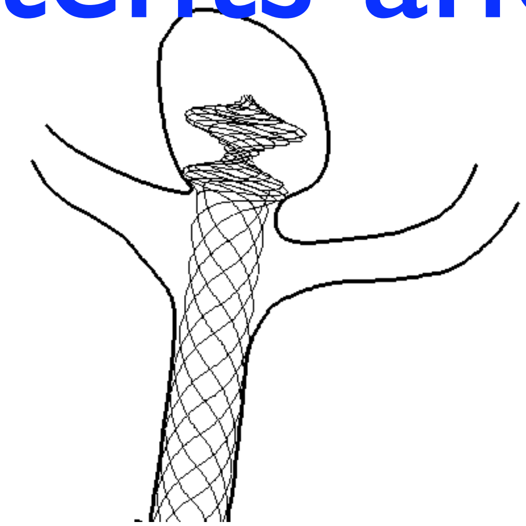


- Nevertheless : even with simplified models, mathematical modelling may help to improve some therapies or medical devices

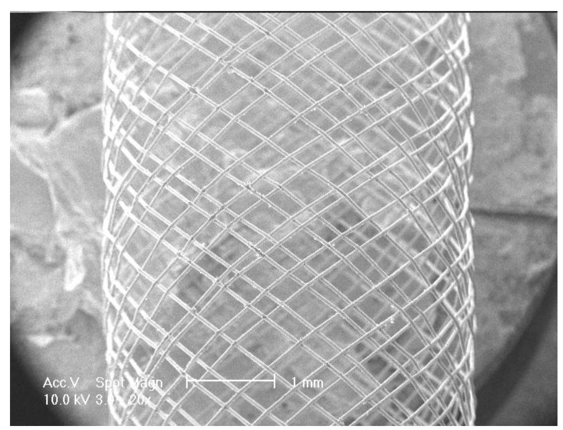


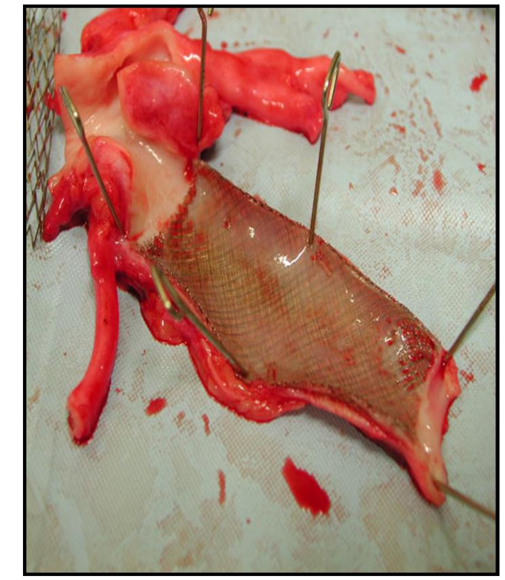
Ex 1: Stents and aneurisms

Coils

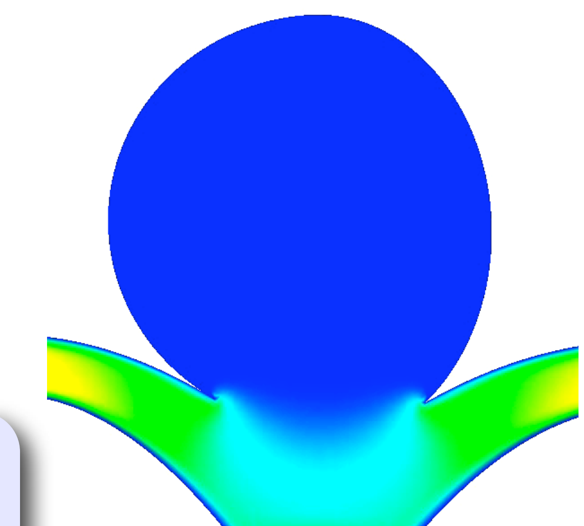


Cardiatis *stent*





In vivo experiments
(Cardiatis /INRA)

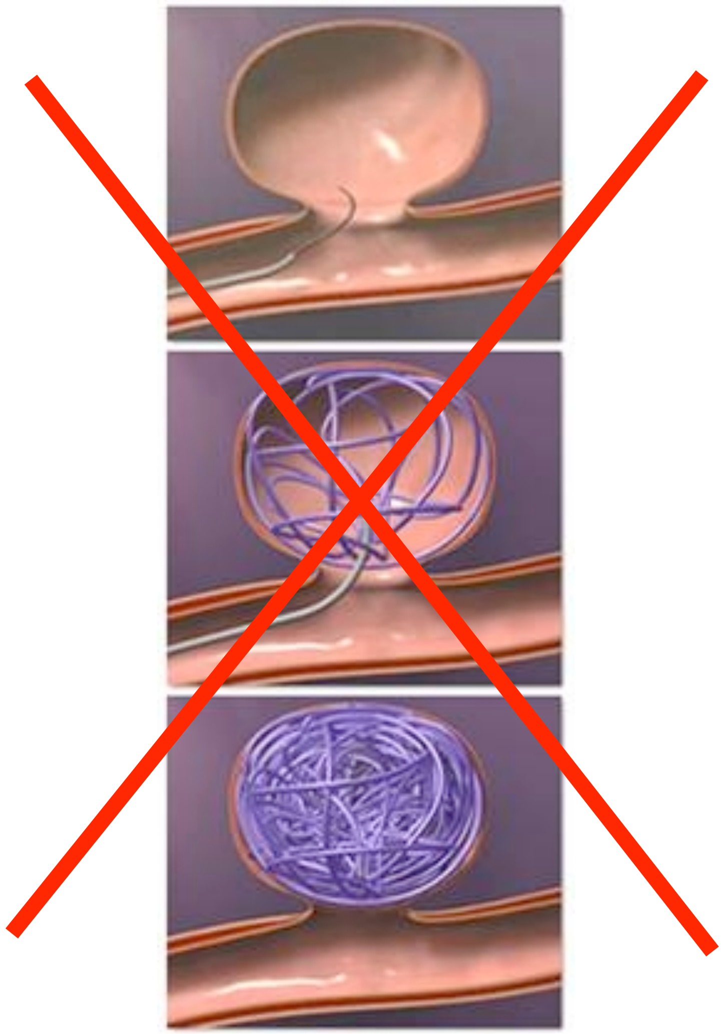


« *In silico* » *experiment*
(Cardiatis /INRIA)

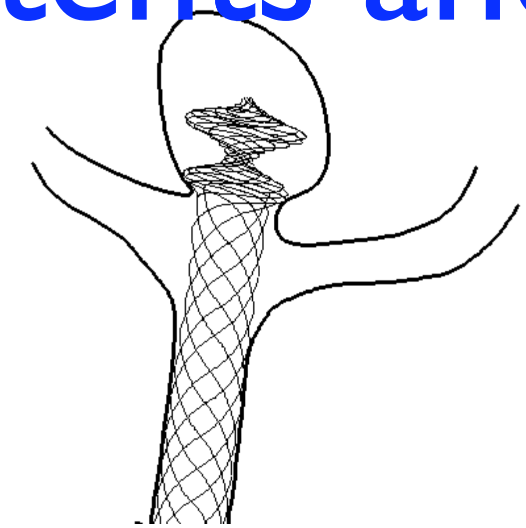
Numerical simulation:

- stent permeability/aneurysm blood flow?
- stent permeability/colateral arteries blood

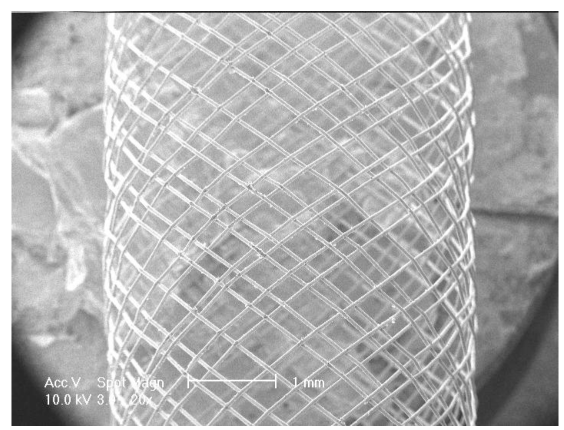
Ex 1: Stents and aneurisms

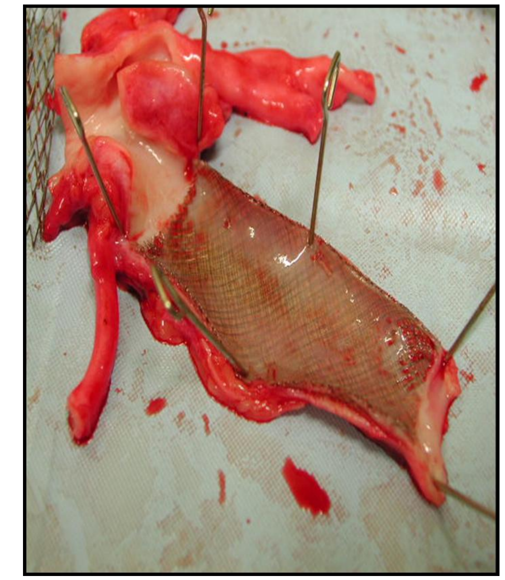


Coils

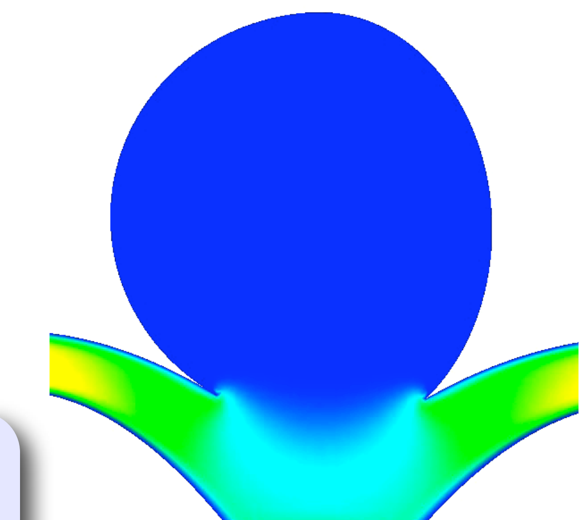


Cardiatis *stent*





In vivo experiments
(Cardiatis /INRA)

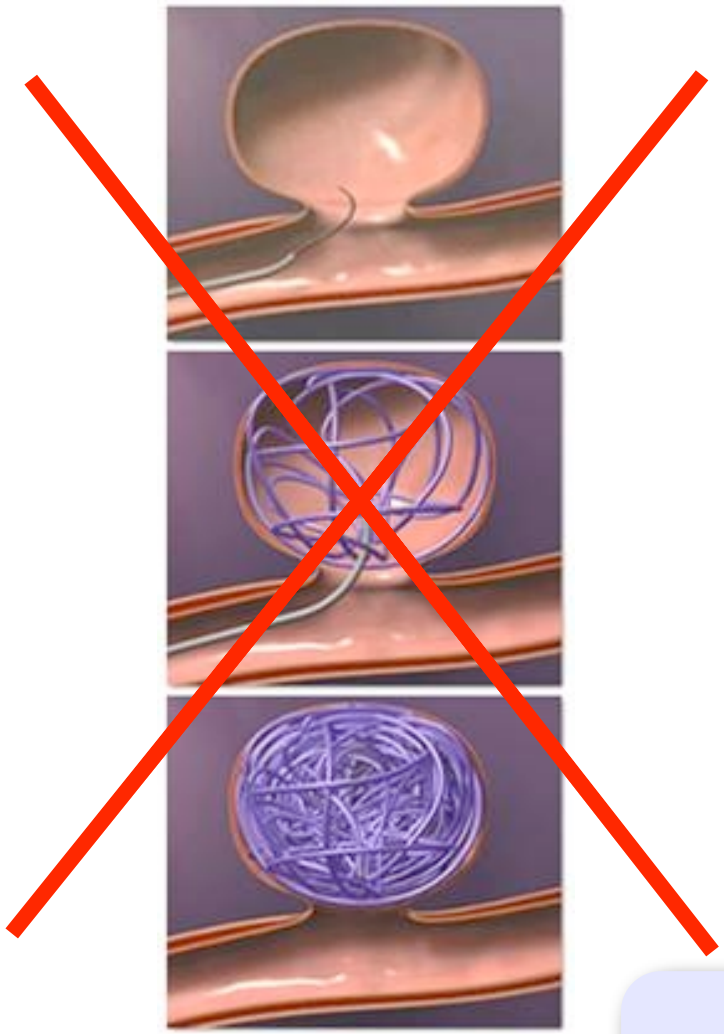


« *In silico* » *experiment*
(Cardiatis /INRIA)
Without stent

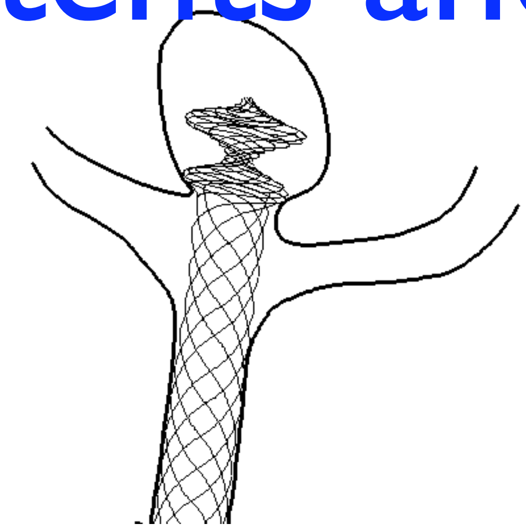
Numerical simulation:

- stent permeability/aneurysm blood flow?
- stent permeability/colateral arteries blood

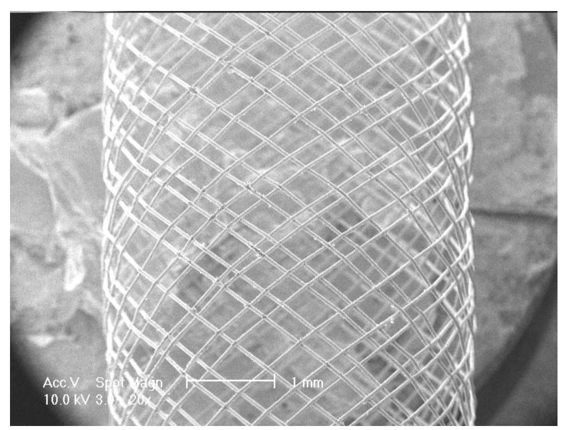
Ex 1: Stents and aneurisms

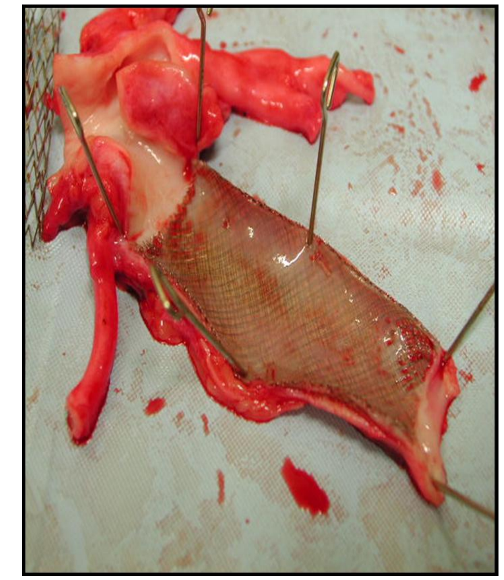


Coils

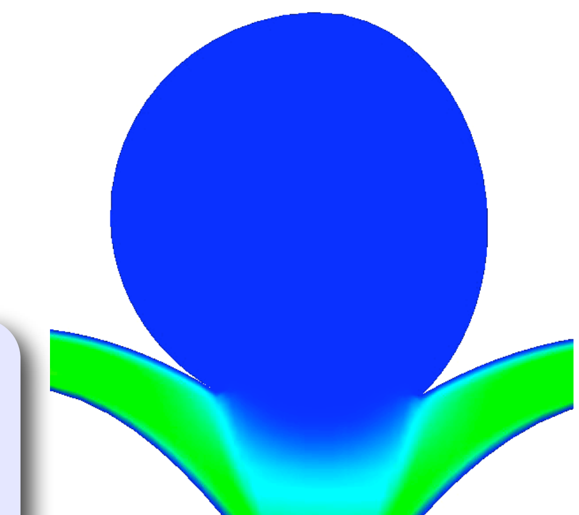


Cardiatis *stent*





In vivo experiments
(Cardiatis /INRA)



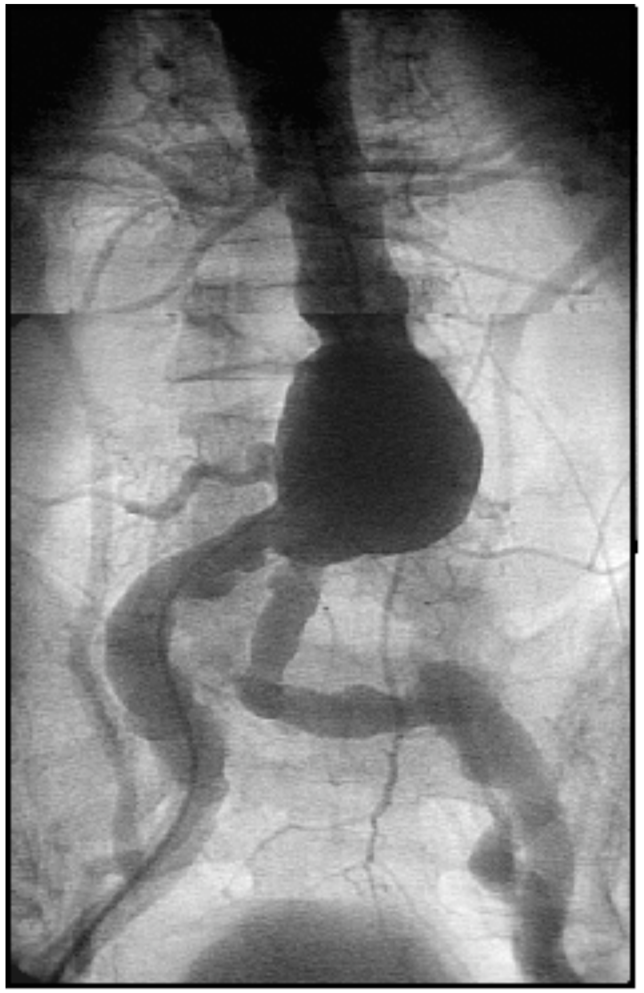
« In silico » experiment
(Cardiatis /INRIA)

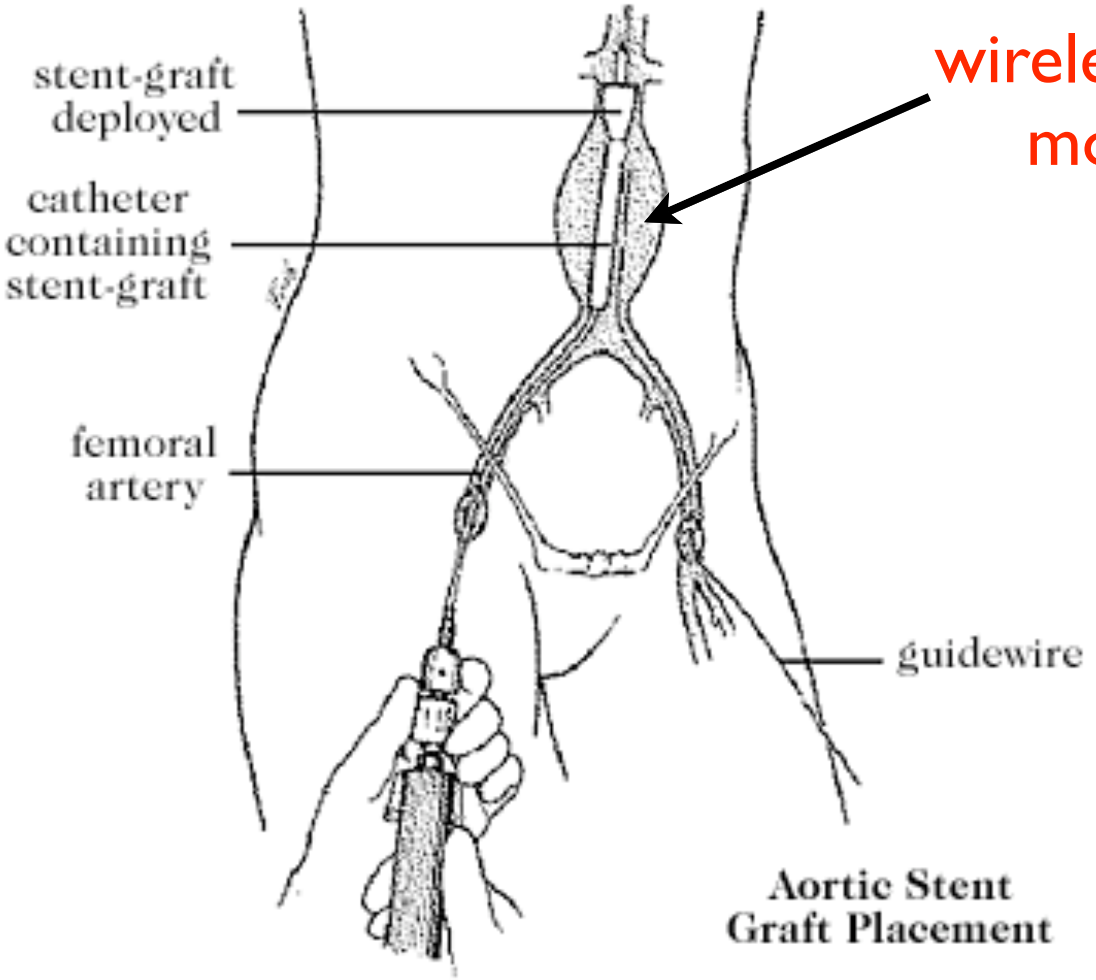
With stent

Numerical simulation:

- stent permeability/aneurysm blood flow?
- stent permeability/colateral blood flow?

Ex 2: Abdominal Aortic Aneurism





Endocom project :

wireless pressure monitoring

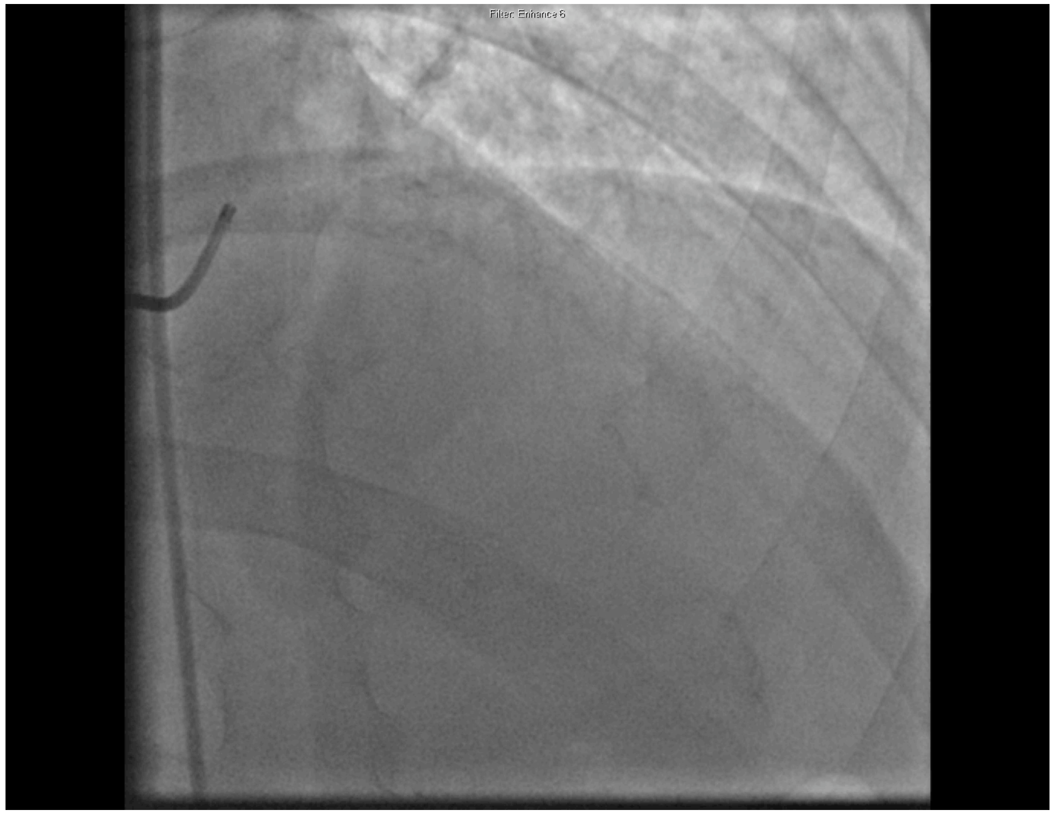
Numerical simulation:

- optimal position of the sensor

Focus of this talk : fluid-structure interaction

- Fluid-structure interaction : interaction with vessel, heart muscle, valves, *etc*.
- Chalenging problems for computational sciences :
  - Efficiency & stability
  - Moving domain: geometrical non-linearities, topological change (contact)
  - Boundary conditions

Fluid-structure interaction in arteries



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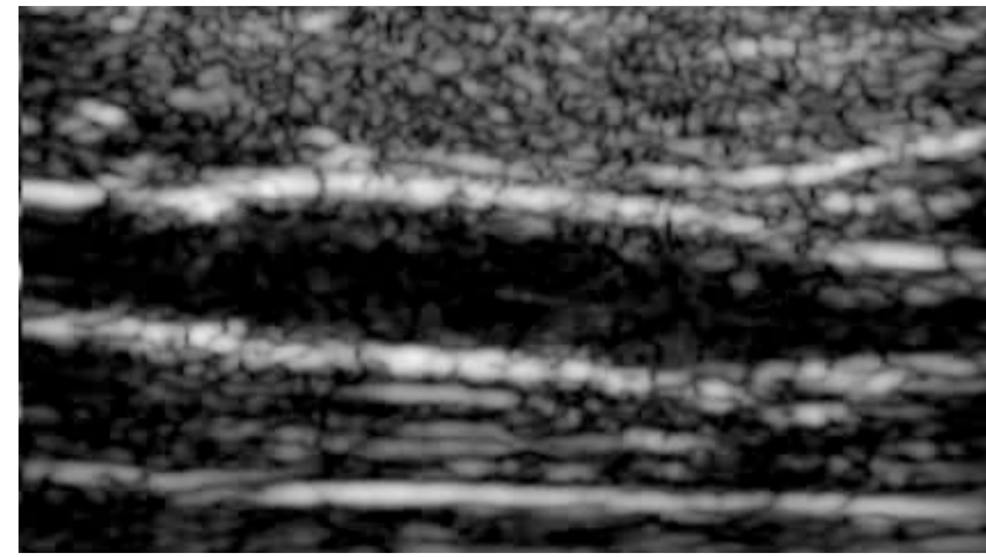
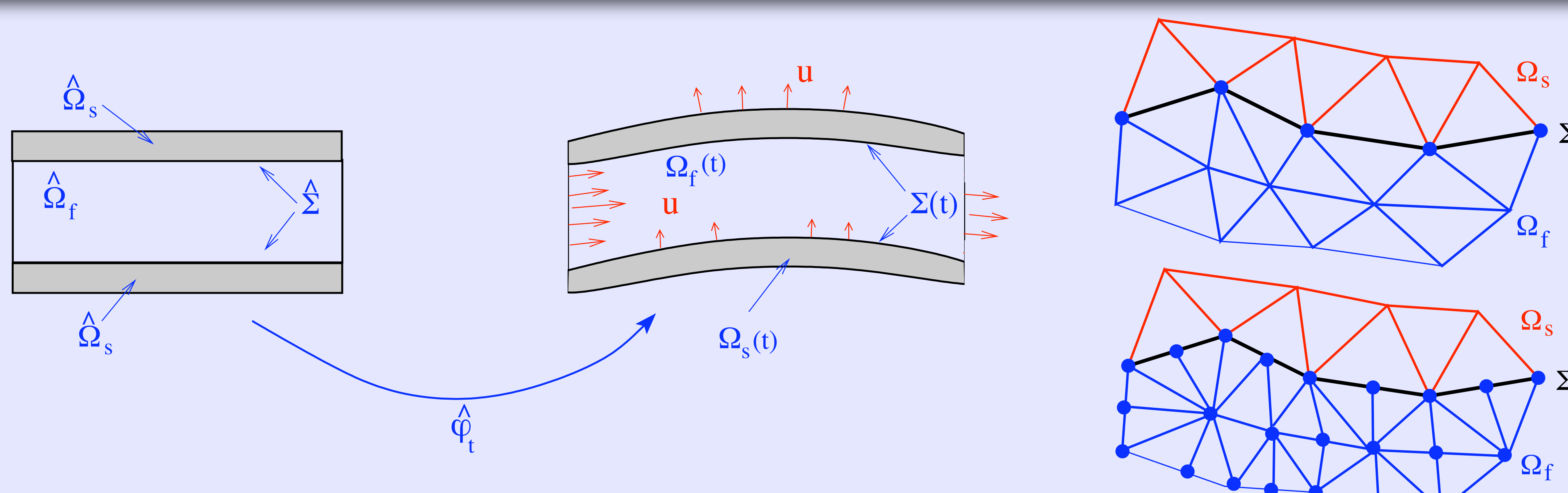
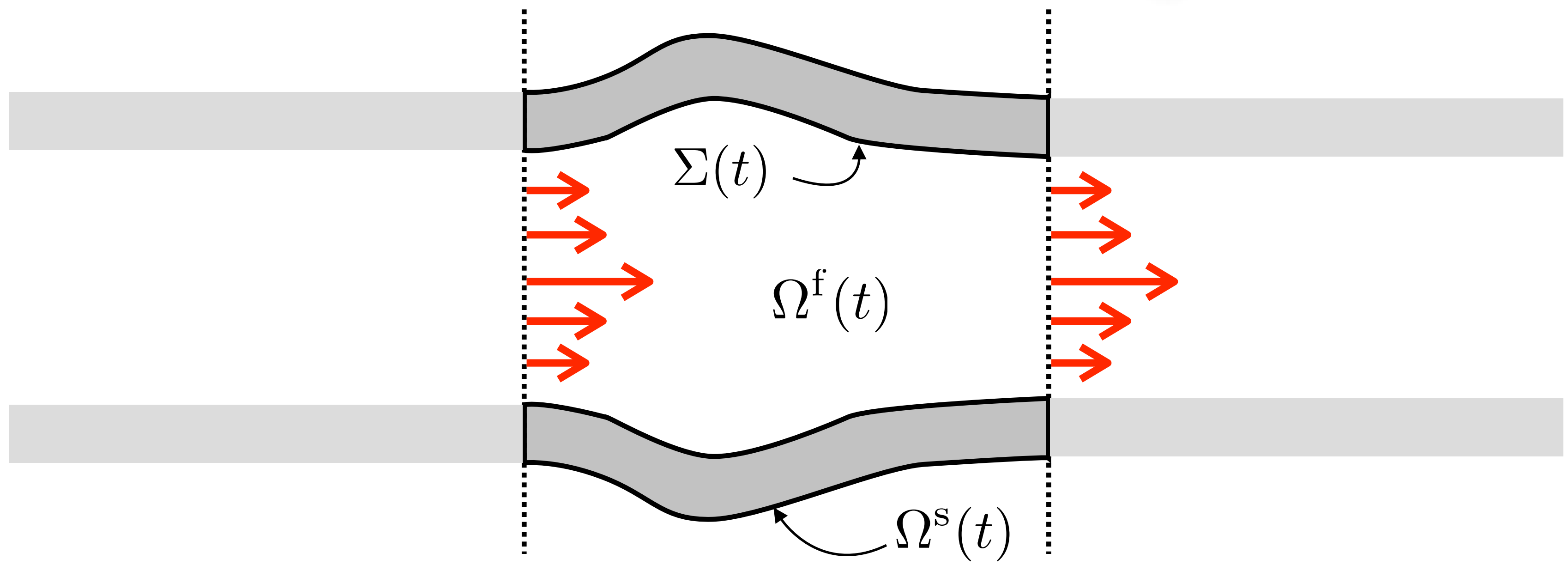


F. Nicoud, H. Vernhet, M. Dauzat

Wall : Arbitrary Lagrangian Eulerian (ALE)



Geometrical description



$\Omega^{\mathrm{f}}(t)$: moving fluid domain (filled with a viscous fluid : blood)

$\Omega^{\mathrm{s}}(t)$: solid current configuration (artery wall)

$\Sigma(t)$: fluid-solid interface, where we enforce

- continuity of velocity
- continuity of stress

The fluid-structure problem:

Determine $\Omega^{\mathrm{f}}(t)$, velocity and stress within the fluid and solid

The coupled problem

- Fluid equations:

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}_{|\widehat{\boldsymbol{x}}} + (\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) - 2\mu\,\mathbf{div}\boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{\nabla}p = \boldsymbol{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} = \boldsymbol{g}, \quad \text{on} \quad \Gamma^{\mathrm{f}}_{\mathrm{N}}$$

- Solid equations:

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\big(\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\big) = \boldsymbol{0}, \quad \text{in} \quad \widehat{\Omega}^{\mathrm{s}}$$

$$\boldsymbol{d} = \boldsymbol{0}, \quad \text{on} \quad \widehat{\Gamma}^{\mathrm{s}}_{\mathrm{D}}$$

$$\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\widehat{\boldsymbol{n}}^{\mathrm{s}} = \boldsymbol{0}, \quad \text{on} \quad \widehat{\Gamma}^{\mathrm{s}}_{\mathrm{N}}$$

- Coupling conditions:

$$\boldsymbol{d}^{\mathrm{f}} = \operatorname{Ext}(\boldsymbol{d}_{|\widehat{\Sigma}}), \quad \boldsymbol{w}(\boldsymbol{d}^{\mathrm{f}}) = \frac{\partial \boldsymbol{d}^{\mathrm{f}}}{\partial t} \quad \text{in} \quad \widehat{\Omega}^{\mathrm{f}}, \quad \Omega^{\mathrm{f}}(t) = (I+\boldsymbol{d}^{\mathrm{f}})(\widehat{\Omega}^{\mathrm{f}}), \quad \text{(geometry)}$$

$$\boldsymbol{u} = \boldsymbol{w}(\boldsymbol{d}^{\mathrm{f}}), \quad \text{on} \quad \Sigma(t), \quad \text{(velocity)}$$

$$\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f}})\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{F}(\boldsymbol{d}^{\mathrm{f}})^{-\mathrm{T}}\widehat{\boldsymbol{n}}, \quad \text{on} \quad \widehat{\Sigma}, \quad \text{(stress)}$$

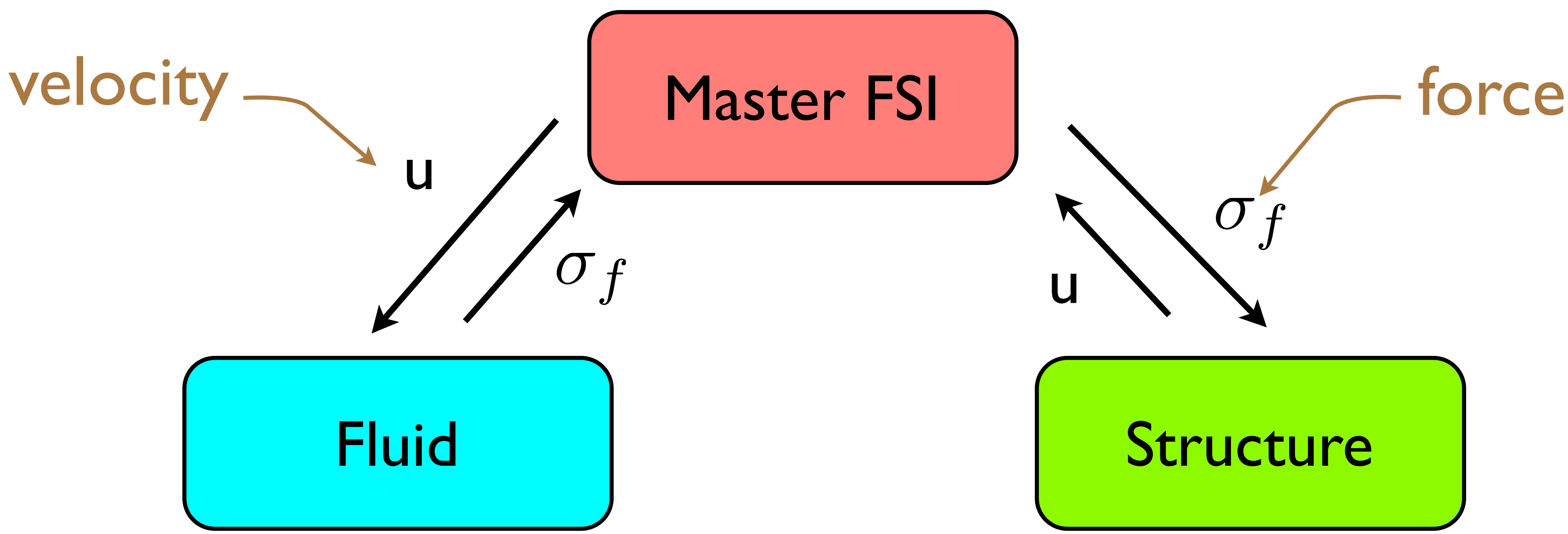
Remark: alternatives

- Interesting simplified approaches have been proposed:
  - Figueroa, Vignon-Clementel, Jansen, Hughes, Taylor, 2006
  - Nobile, Vergara, 2008

- In those cases, the FSI cost is almost the fluid cost.
- In the sequel, we only consider the cases where a “real” structure problem has to be solved. Useful
  - if the stress within the wall is required

Implementation issues

- Use independent solvers for fluid and structure :
  - Advantage: re-usability of state-of-the-art algorithms
  - Difficulties: possible troubles with the coupling algorithms



- **Strong** coupling: sub-iterations at each time step
- **Weak** coupling : 1 or 2 iterations per time step

Example: Dirichlet-Neumann

- Fluid sub-problem: $(\boldsymbol{d}^{\mathrm{f},n+1}, \boldsymbol{u}^{n+1}, p^{n+1}) = \mathcal{F}\big(\boldsymbol{d}^{n+1}_{|\widehat{\Sigma}}\big)$
- Solid sub-problem: $\boldsymbol{d}^{n+1}_{|\widehat{\Sigma}} = \mathcal{S}\big(\boldsymbol{d}^{\mathrm{f},n+1}, \boldsymbol{u}^{n+1}, p^{n+1}\big)$

Fixed-point iterations with acceleration

(1) Initiallization:

$$\boldsymbol{\lambda}^0 = \boldsymbol{d}^n_{|\widehat{\Sigma}}$$

(2) Until convergence ($k \geq 0$):

(a) Solve fluid and solid:

$$\widetilde{\boldsymbol{\lambda}}^{k+1} = (\mathcal{S} \circ \mathcal{F})\big(\boldsymbol{\lambda}^k\big)$$

(b)Relaxation:

$$\boldsymbol{\lambda}^{k+1} = \omega_k \widetilde{\boldsymbol{\lambda}}^{k+1} + (1-\omega_k)\boldsymbol{\lambda}^k, \quad \omega_k \in (0,1]$$

- **Aitken acceleration(*Wall, Ramm,2001*)**:

$$\omega_k = \frac{(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) \cdot (\boldsymbol{\lambda}^k - \widetilde{\boldsymbol{\lambda}}^k - \boldsymbol{\lambda}^{k-1} + \widetilde{\boldsymbol{\lambda}}^{k-1})}{\big|\boldsymbol{\lambda}^k - \widetilde{\boldsymbol{\lambda}}^k - \boldsymbol{\lambda}^{k-1} + \widetilde{\boldsymbol{\lambda}}^{k-1}\big|^2}$$

Domain decomposition

Option 1 : decompose first then linearize

- Dirichlet-Neumann
  - Fixed-point : *Le Tallec-Mouro (1999) Wall-Ramm (2001), ...*
  - Newton : *Fernandez-Moubachir (2003), ...*
  - *Inexact* Newton: *Matthies-Steindorf (2003), JFG-Vidrascu (2003), Mischler-van Brummelen-de Borst (2005), ...*
- Neumann - Neumann
  - *Deparis-Discacciati-Quarteroni (2005), ...*
- Robin - Neumann
  - *Badia-Nobile-Vergara (2007)*

Option 2 : linearize first then decompose

- *Fernandez, JFG, Gloria, Vidrascu (2007)*

Common feature:

*These algorithms are strongly coupled, for **stability**.*

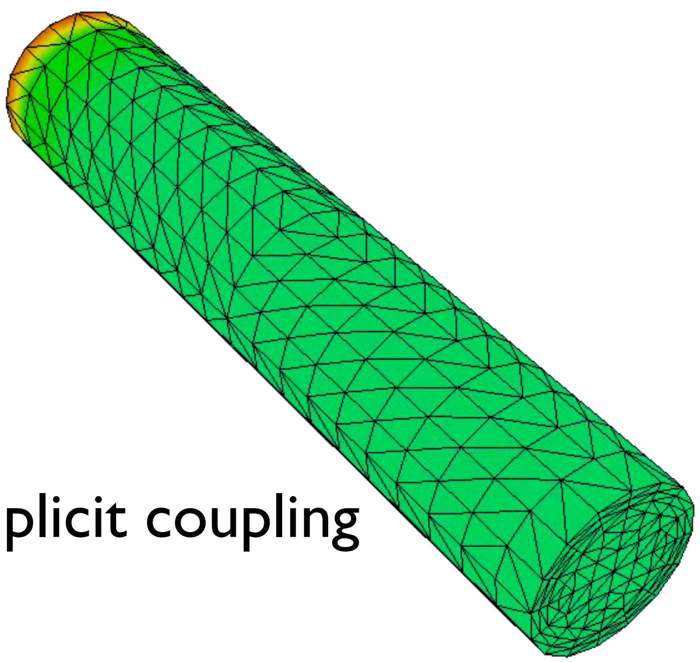
Number of sub-iterations : between 10 and 100 !

Explicit coupling: some observations

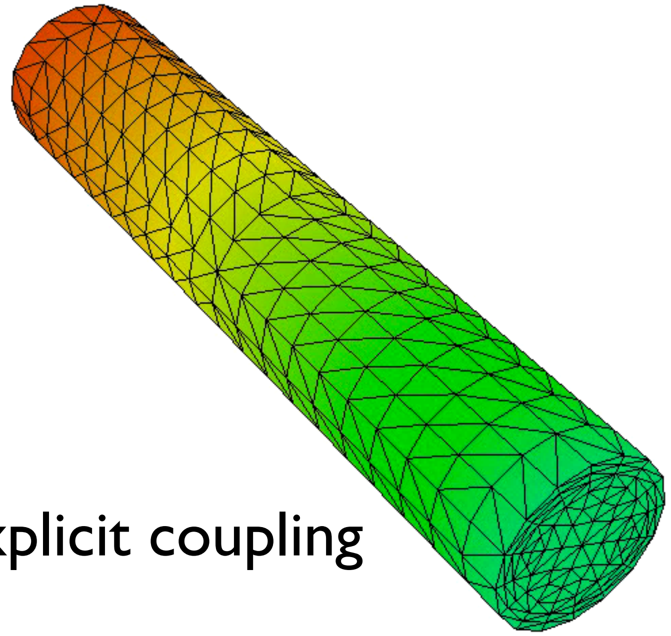
- An explicit algorithm is *a priori* very efficient...

$$\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}$$

- **... but unstable !**



implicit coupling



explicit coupling

- Energy estimate with an artificial interface power term:

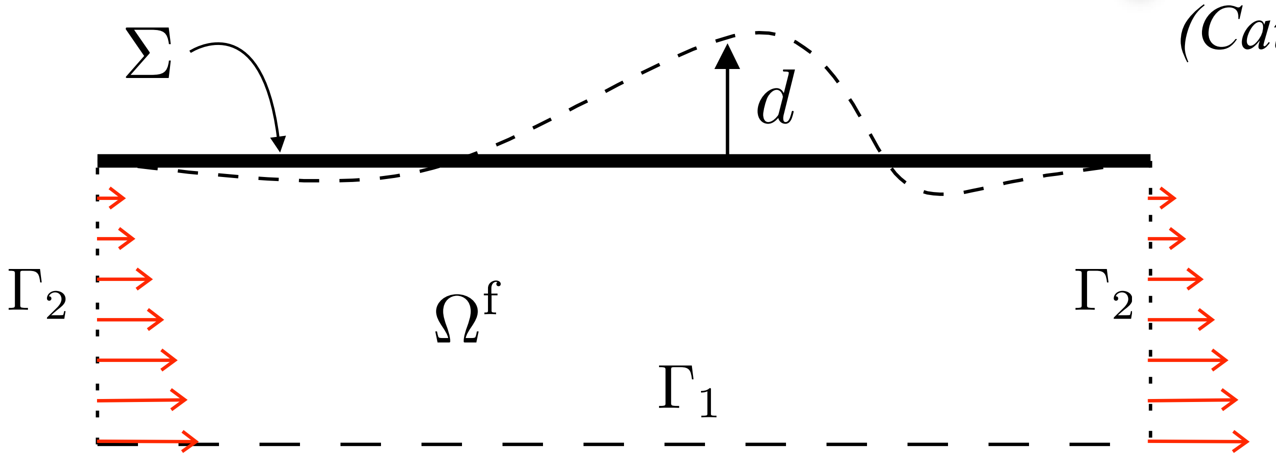
$$\int_{\Sigma^{n+1}} \boldsymbol{\sigma}(\boldsymbol{u}^{n+1}, p^{n+1})\boldsymbol{n} \cdot \left(\boldsymbol{u}^{n+1} - \frac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n}}{\delta t}\right)$$

- Explicit coupling is stable and widely used in aeroelasticity !

> ▸ What is the **source of instabilities of those schemes in blood flows** ?

A 2D simplified model

(Causin, JFG, Nobile, 2004)



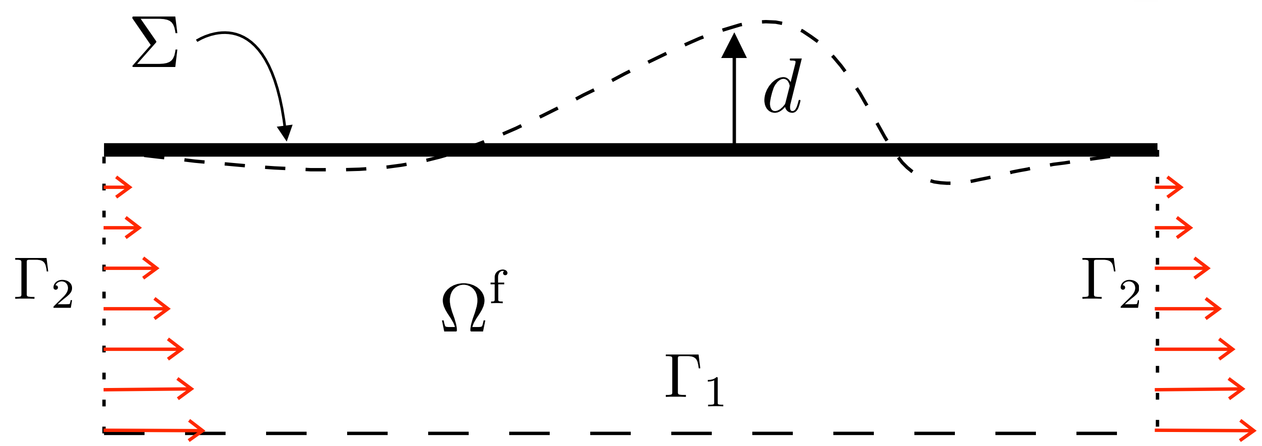
- Solid: string model (small displacements)

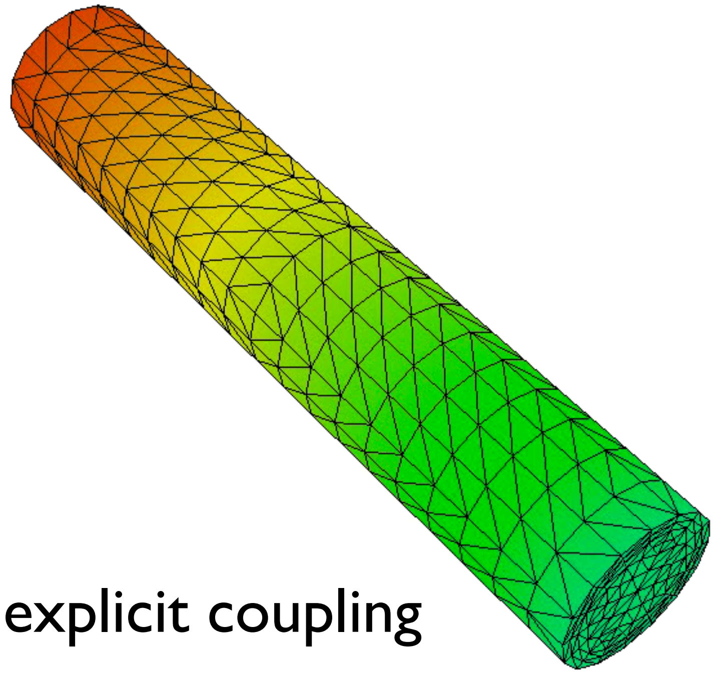
$$\rho^{\mathrm{s}} \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

with

- d: vertical displacement
- ε: vessel thickness
- L: linear operator (for instance $L\eta = a\eta - b\frac{\partial^2 \eta}{\partial x^2}$)

A 2D simplified model





explicit coupling

- Solid: string model (infinitesimal displacements)

$$\rho^{\mathrm{s}}\varepsilon\ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

- Fluid: fixed fluid domain, no viscous/convective terms

$$\begin{cases} \rho^{\mathrm{f}}\dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u} = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u}\cdot\boldsymbol{n} = \dot{d}, & \text{on} \quad \Sigma \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \text{on} \quad \Gamma_1 \\ p = 0, & \text{on} \quad \Gamma_2 \end{cases} \quad \underset{\text{div}}{\Longrightarrow} \quad \begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}}\dfrac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{n} = -\rho^{\mathrm{f}}\ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Interest of this model :

- **Physics**: reproduces propagation phenomena
- **Numerics**: explicit coupling unstable

The added-mass operator

Fluid: $$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \varepsilon \ddot{d} + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

Theorem (Steklov-Poincaré operator)

The operator $\mathcal{M}_{\mathrm{A}} : H^{-\frac{1}{2}}(\Sigma) \rightarrow H^{\frac{1}{2}}(\Sigma)$ defined as: for each $g \in H^{-\frac{1}{2}}(\Sigma)$ we set $\mathcal{M}_{\mathrm{A}}(g) \stackrel{\text{def}}{=} q_{|\Gamma^{\mathrm{w}}}$, where $q \in H^1(\Omega^{\mathrm{f}})$ solves

$$\begin{cases} -\Delta q = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = g, & \text{on} \quad \Sigma \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ q = 0, & \text{on} \quad \Gamma_2 \end{cases}$$

is a linear, compact, positive and self-adjoint operator in $L^2(\Sigma)$.

✓ From this definition, we have $\boxed{p_{|\Sigma} = \mathcal{M}_{\mathrm{A}}(-\rho^{\mathrm{f}} \ddot{d}) = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}}$

The added-mass effect

$$\text{Fluid:} \quad \begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

$$\text{Solid:} \quad \rho^{\mathrm{s}} \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma, \tag{1}$$

$$\boxed{p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}}$$

$$\left(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}}\right) \ddot{d} + Ld = 0, \quad \text{in} \quad \Sigma \tag{2}$$

Remarks:

- This equation looks like a structure equation, except for the extra “mass” term
- The **fluid-structure coupling** can be condensed into an **extra mass** action on the structure (hence the terminology “added-mass effect”)

Question :

What kind of time integration scheme of (2) arises from the explicit coupling of (1)?

Explicit coupling and added-mass

Fluid: $$\left\{\begin{aligned} \rho^{\mathrm{f}}\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\delta t}+\nabla p^{n+1}&=0\\ \operatorname{div}\boldsymbol{u}^{n+1}&=0\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}&=\frac{d^{n}-d^{n-1}}{\delta t}\end{aligned}\right. \quad\underset{\text{div}}{\Longrightarrow}\quad \left\{\begin{aligned} -\Delta p^{n+1}&=0\\ \frac{\partial p^{n+1}}{\partial\boldsymbol{n}}&=-\rho^{\mathrm{f}}\frac{d^{n}-2d^{n-1}+d^{n-2}}{\delta t^{2}}\end{aligned}\right.$$

Solid: $$\rho^{\mathrm{s}}\varepsilon\frac{d^{n+1}-2d^{n}+d^{n-1}}{\delta t^{2}}+Ld^{n+1}=p^{n+1}_{|\Sigma}$$

$$p^{n+1}_{|\Sigma}=-\rho^{\mathrm{f}}\mathcal{M}_{\mathrm{A}}\frac{d^{n}-2d^{n-1}+d^{n-2}}{\delta t^{2}}$$

Condensed FSI problem:

$$\underbrace{\rho^{\mathrm{s}}\varepsilon\frac{d^{n+1}-2d^{n}+d^{n-1}}{\delta t^{2}}}_{\textbf{implicit}}+\rho^{\mathrm{f}}\mathcal{M}_{\mathrm{A}}\underbrace{\frac{d^{n}-2d^{n-1}+d^{n-2}}{\delta t^{2}}}_{\textbf{explicit}}+L\eta^{n+1}=0$$

Observation:

Weak coupling leads to an explicit discretization of the added-mass

An unconditional instability result

Proposition *(Causin-JFG-Nobile 04)*

Let $\lambda_{\max}$ be the largest eigenvalue of $\mathcal{M}_{\mathrm{A}}$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^{\mathrm{f}}\lambda_{\max}}{\rho^{\mathrm{s}}\varepsilon} \geq 1. \tag{1}$$

Remarks:

- The instability condition does not depend on the time step
- The instability condition confirms two numerical observations:
  - Instabilities might occur when the structure is **light**, **thin** and **slender**
  - In aeroelasticity $\rho^{\mathrm{f}} \ll \rho^{\mathrm{s}}$, hence weak (*i.e.* explicit) coupling is stable

Implicit / Explicit coupling: summary

- Implicit coupling stable but too expensive
- Explicit coupling cheap but unstable
- Other time schemes have been considered by *Förster-Wall-Ramm 07* with analogous conclusions
- Geometrical non-linearities (moving domains), convective and viscous effects do not seem to affect the stability of a coupling algorithm. However, they are implicitly treated in fully implicit schemes (very expensive!)

Three ideas: *(Fernandez, JFG, Grandmont, 2006)*

- Treat implicitly the added-mass effect (incompressibility, pressure stress)
- Treat explicitly the fluid domain motion, convective and viscous effects
- Perform this using a projection scheme (Chorin-Teman) within the fluid

The Chorin-Teman projection scheme

Main feature:
Incompressibility and viscous/convective effects are decoupled

- Incompressible Navier-Stokes equations:

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)-2\mu\,\mathbf{div}\,\boldsymbol{\epsilon}(\boldsymbol{u})+\boldsymbol{\nabla}p=\mathbf{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}$$
$$\operatorname{div}\boldsymbol{u}=0, \quad \text{in} \quad \Omega^{\mathrm{f}}$$

- **Viscous** step:

$$\begin{cases} \rho^{\mathrm{f}}\left(\dfrac{\widetilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^{n}}{\delta t}+\widetilde{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{u}}^{n+1}\right)-2\mu\operatorname{div}\boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1})=0, & \text{in} \quad \Omega \\ \widetilde{\boldsymbol{u}}^{n+1}=0, & \text{on} \quad \partial\Omega \end{cases}$$

- **Projection** step:

$$\begin{cases} \rho^{\mathrm{f}}\dfrac{\boldsymbol{u}^{n+1}-\widetilde{\boldsymbol{u}}^{n+1}}{\delta t}+\boldsymbol{\nabla}p^{n+1}=0, & \text{in} \quad \Omega \\ \operatorname{div}\boldsymbol{u}^{n+1}=0, & \text{in} \quad \Omega \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}=0, & \text{on} \quad \partial\Omega \end{cases} \quad \xRightarrow[\text{div}]{} \quad \begin{cases} -\Delta p^{n+1}=-\dfrac{\rho^{\mathrm{f}}}{\delta t}\operatorname{div}\widetilde{\boldsymbol{u}}^{n+1}, & \text{in} \quad \Omega \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}}=0, & \text{on} \quad \partial\Omega \end{cases}$$

Semi-implicit coupling: explicit part

- Viscous sub-step:

$$\boldsymbol{d}^{\mathrm{f},n+1} = \mathrm{Ext}(\boldsymbol{d}^n_{|\widehat{\Sigma}}), \quad \boldsymbol{w}^{n+1} = \frac{\boldsymbol{d}^{\mathrm{f},n+1} - \boldsymbol{d}^n}{\delta t}, \quad \Omega^{\mathrm{f},n+1} = (I + \boldsymbol{d}^{\mathrm{f},n+1})(\widehat{\Omega}^{\mathrm{f}}),$$

$$\rho^{\mathrm{f}} \left(\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^n}{\delta t} + (\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{w}^{n+1}) \cdot \boldsymbol{\nabla} \widetilde{\boldsymbol{u}}^{n+1} \right) - 2\mu \,\mathbf{div}\, \boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1}) = 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}$$

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{w}^{n+1}, \quad \text{on} \quad \Sigma^{n+1}$$

Observation:

- Fluid domain, viscous and convective effects explicitly treated

Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

$$\left\{\begin{aligned} \rho^{\mathrm{f}}\frac{\boldsymbol{u}^{n+1}-\widetilde{\boldsymbol{u}}^{n+1}}{\delta t}+\boldsymbol{\nabla} p^{n+1} &= 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}\\ \mathrm{div}\boldsymbol{u}^{n+1} &= 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = \frac{\boldsymbol{d}^{n+1}-\boldsymbol{d}^{n}}{\delta t}\cdot\boldsymbol{n}&, \quad \text{on} \quad \Sigma^{n+1} \end{aligned}\right. \underset{\mathrm{div}}{\Longrightarrow} \left\{\begin{aligned} -\Delta p^{n+1} &= -\frac{\rho^{\mathrm{f}}}{\delta t}\mathrm{div}\widetilde{\boldsymbol{u}}^{n+1}, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}\\ \frac{\partial p^{n+1}}{\partial \boldsymbol{n}} &= -\rho^{\mathrm{f}}\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^{n}+\boldsymbol{d}^{n-1}}{\delta t^2}, \quad \text{on} \quad \Sigma^{n+1} \end{aligned}\right.$$

- Solid equation:

$$\left\{\begin{aligned} \rho^{\mathrm{s}}\frac{\boldsymbol{d}^{n+1}-2\boldsymbol{d}^{n}+\boldsymbol{d}^{n-1}}{\delta t^2}-\mathbf{div}\left(\boldsymbol{F}(\boldsymbol{d}^{n+1})\boldsymbol{S}(\boldsymbol{d}^{n+1})\right) &= \mathbf{0}, \quad \text{in} \quad \widehat{\Omega}^{\mathrm{s}}\\ \boldsymbol{F}(\boldsymbol{d}^{n+1})\boldsymbol{S}(\boldsymbol{d}^{n+1})\widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f},n+1})\boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{n+1},p^{n+1})\boldsymbol{F}(\boldsymbol{d}^{\mathrm{f},n+1})^{-\mathrm{T}}\widehat{\boldsymbol{n}}&, \quad \text{on} \quad \widehat{\Sigma} \end{aligned}\right.$$

Observations:

- Projection sub-step in a fixed fluid domain (fixed matrix)
- Implicit part solved with cheaper (inner) iterations

A stability result (linear case)

Theorem: *(Fernandez-Gerbeau-Grandmont 07)*

Assume the interface matching operator to be L^2-stable. Then, under condition

$$\rho^{\mathrm{s}} \geq C\left(\rho^{\mathrm{f}}\frac{h}{H^{\alpha}} + 2\frac{\mu\delta t}{hH^{\alpha}}\right), \quad \text{with} \quad \alpha \stackrel{\mathrm{def}}{=} \begin{cases} 0, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} = \Sigma, \\ 1, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} \neq \Sigma, \end{cases}$$

the following discrete energy inequality holds:

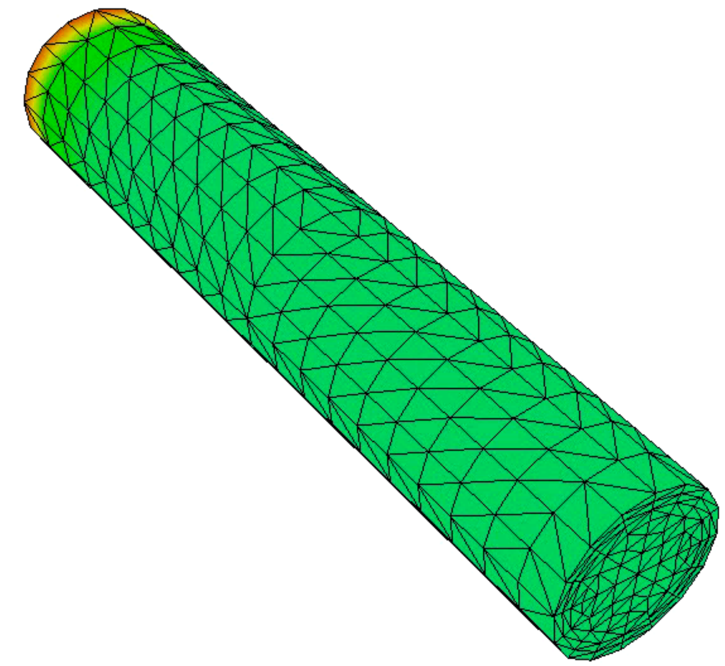
$$\frac{1}{\delta t}\left[\frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n+1}\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n}\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n+1}-\boldsymbol{d}_H^{n}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n}-\boldsymbol{d}_H^{n-1}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}}\right]$$
$$+ \frac{1}{2\delta t}\left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1},\boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^{n},\boldsymbol{d}_H^{n})\right] + \mu\|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|^2_{0,\Omega^{\mathrm{f}}} \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionnally stable in the energy norm.

Remark: this semi-implicit algorithm has been extended by *Badia, Quarteroni, Quaini* to other projection schemes.

3D Navier-Sokes / Non-linear Shell coupling

- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4} s$



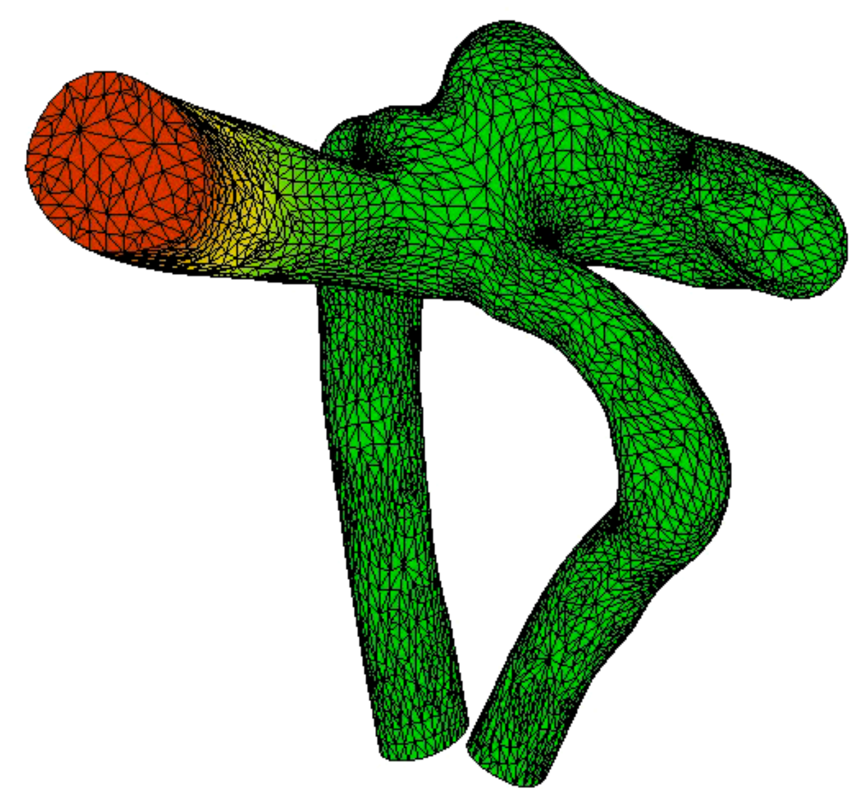
COUPLING	ALGORITHM	CPU time
Implicit	FP-Aitken	24.86
	quasi-Newton	6.05
	Newton	4.77
Semi-Implicit	Newton	1

2001

2003

2006

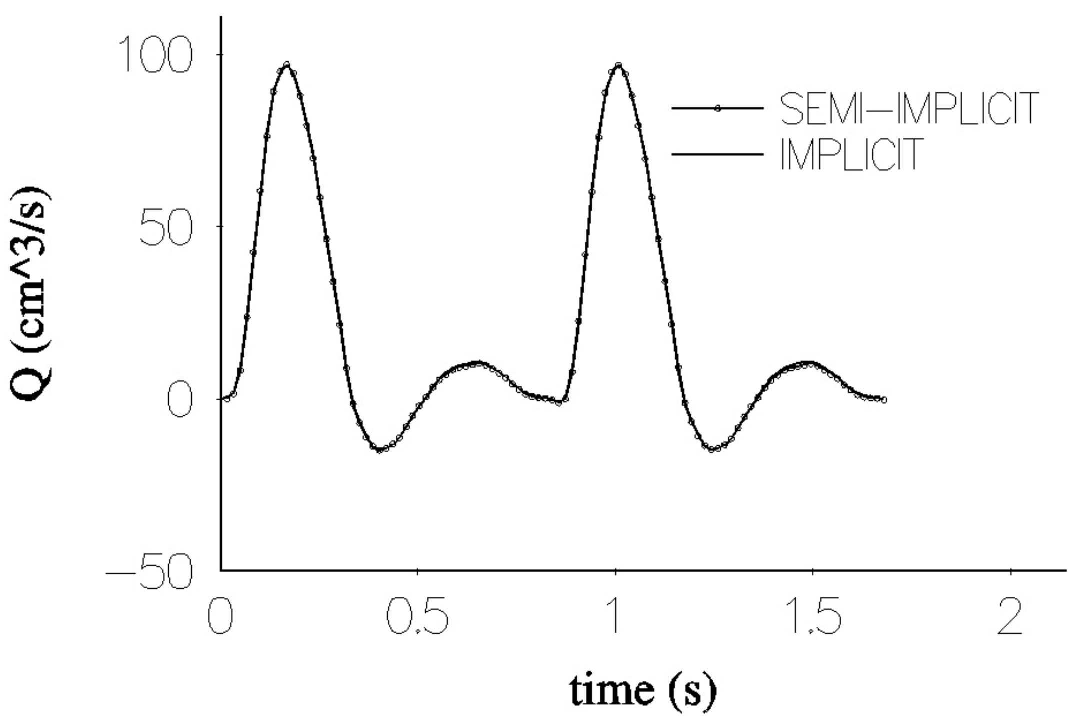
- Cerebral aneurysm: 20 time steps of length $\delta t = \times 10^{-3} s$



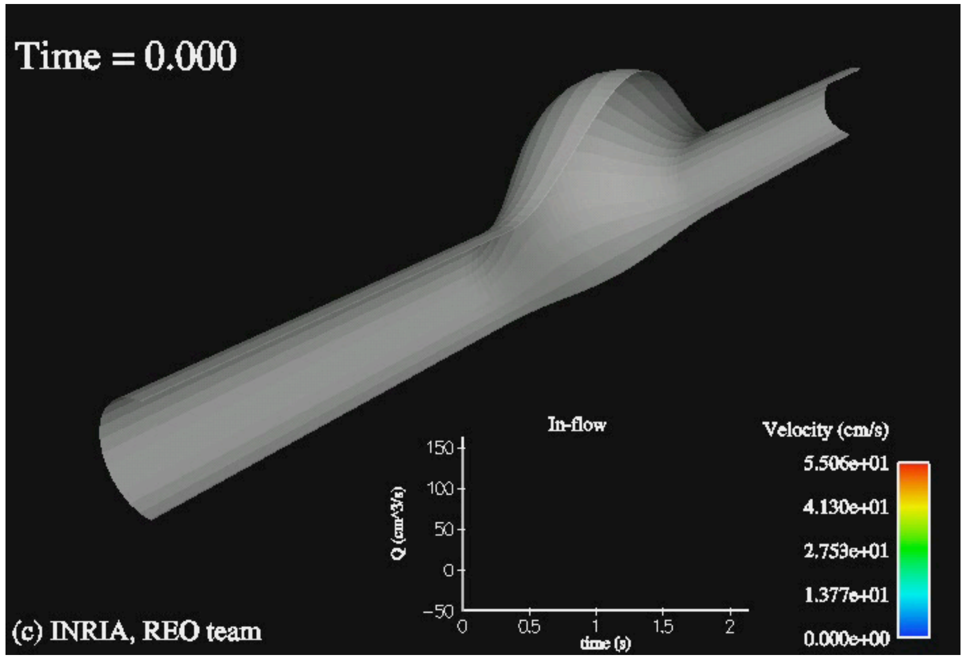
COUPLING	CPU time
Implicit	4.70
Semi-Implicit	1

3D Navier-Sokes / Non-linear Shell coupling

- Abdominal aortic aneurysm (in-vitro model): 2 cardiac cycles, 1000 times steps
  - $\delta t = 1.68 \times 10^{-3} s$
  - Fluid: 26950 Hexahedra ($\mathbb{Q}_1/\mathbb{Q}_1$ FE)
  - Solid: 2240 Quadrilaterals (MITC4 FE)
  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 6 10^6\, dynes/cm^2$, $\nu = 0.3$



Out-flow

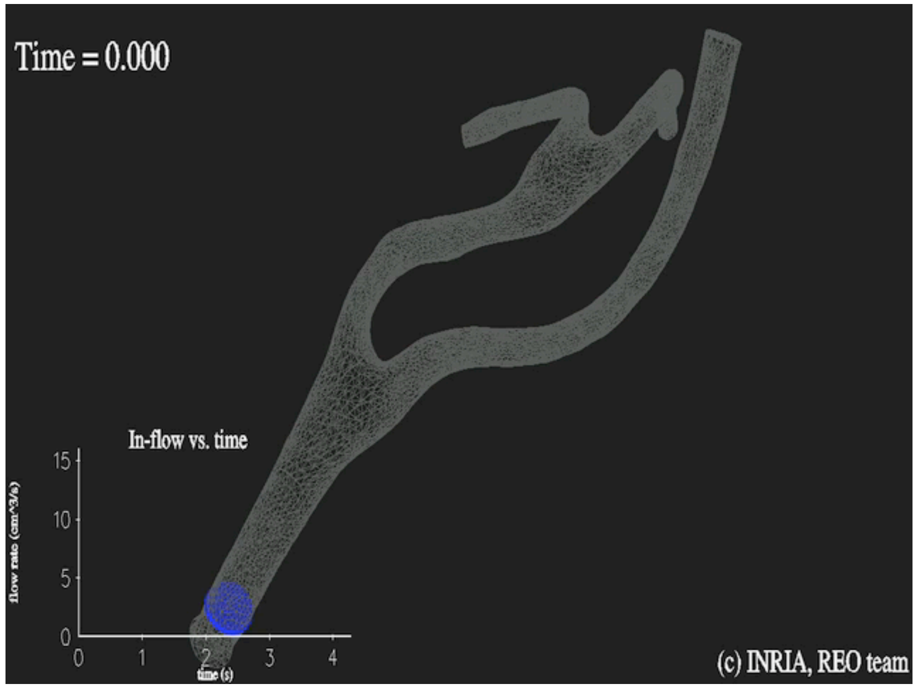


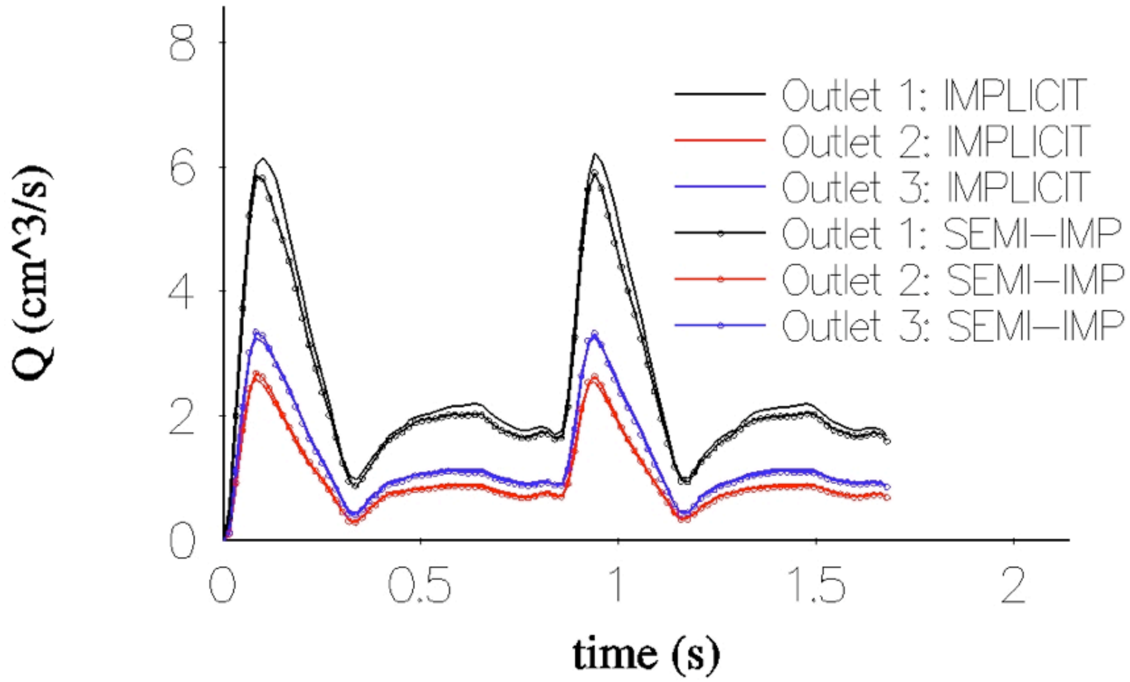
COUPLING	CPU time
Implicit	**9.3**
Semi-Implicit	**1.0**

Dimensionless CPU time

3D Navier-Sokes / Non-linear Shell coupling

- Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps
  - $\delta t = 1.68 \times 10^{-3} s$
  - Fluid: 70047 Tetrahedra ($\mathbb{P}_1/\mathbb{P}_1$ FE)
  - Solid: 8103 Quadrilaterals (MITC4 FE)
  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 6 \times 10^6\, dynes/cm^2$, $\nu = 0.3$.





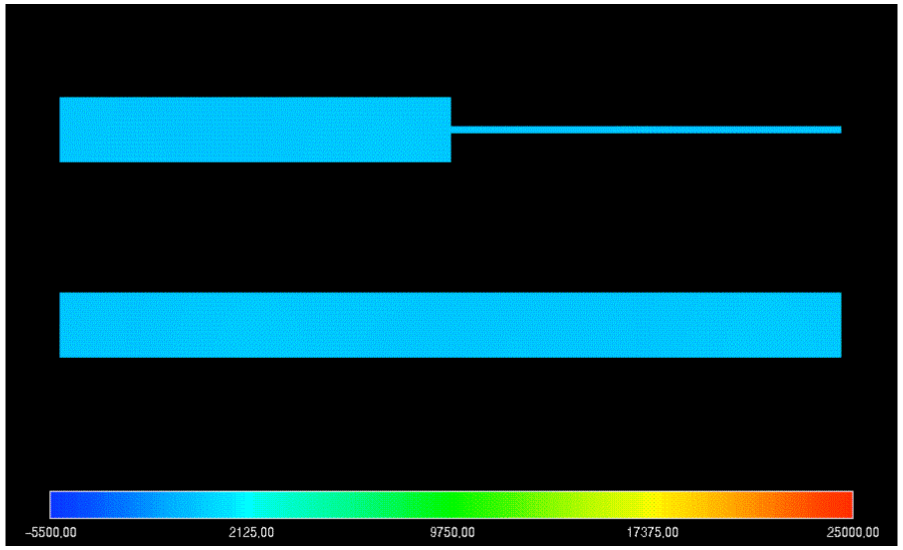
Out-flow

COUPLING	CPU time
Implicit	**6.7**
Semi-Implicit	**1.0**

Dimensionless CPU time

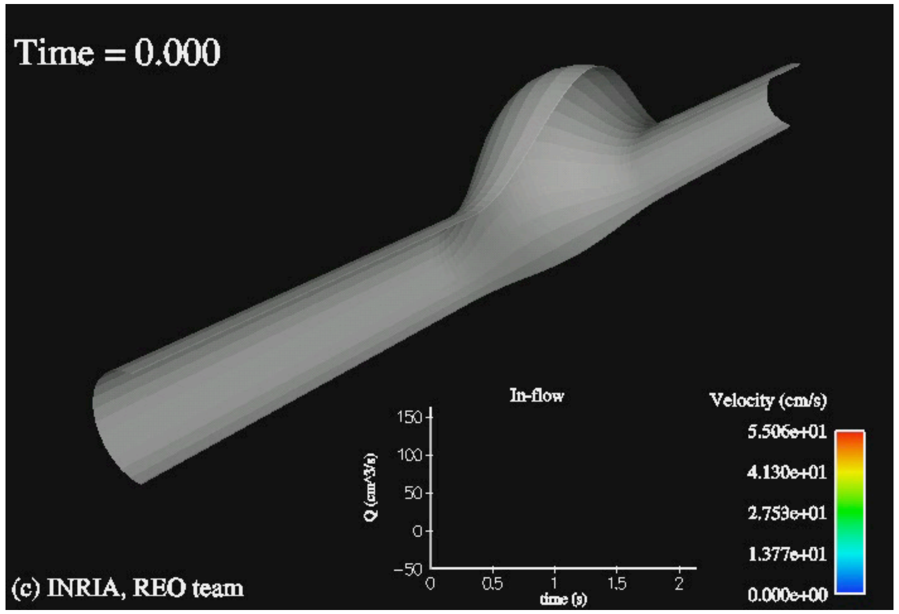
Remarks on boundary conditions

- Spurious reflexion of pressure wave: 3D-1D coupling



Formaggia, JFG, Quarteroni, Nobile, 2001
Formaggia, Moura, Nobile, 2007
Formaggia, Veneziani, Vergara, 2006

- But pressure wave reflexion is not the only issue



The best non-reflecting outlet boundary condition cannot prevent the global (non physiological) bending !

Remarks on boundary conditions

- Surrounding tissues play a key role

- Typical b.c. on the external part of the vessel :

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = p_0$$

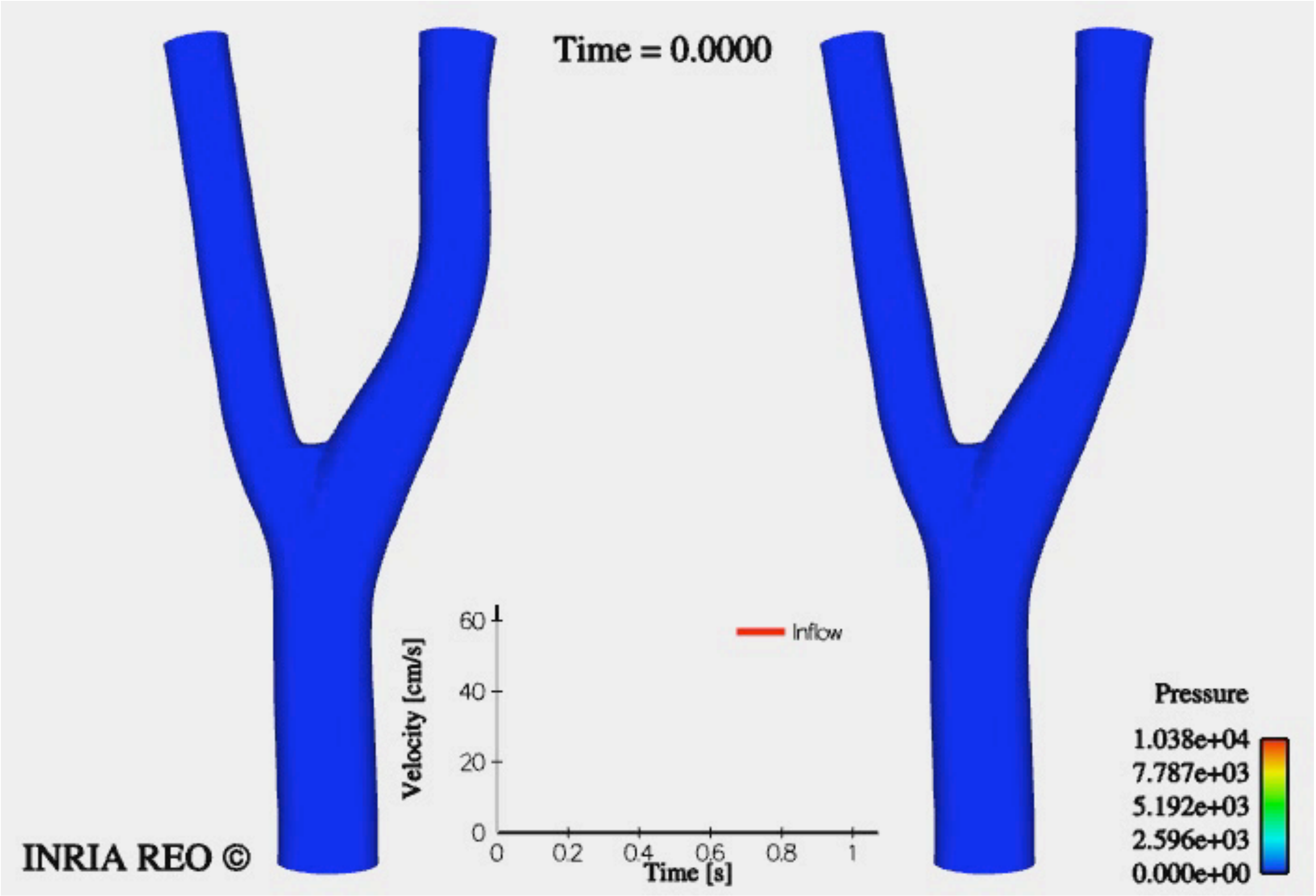
- A simple and affordable way to model the surrounding tissues :

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s \boldsymbol{d} - c_s \frac{\partial \boldsymbol{d}}{\partial t}$$

Remarks on boundary conditions

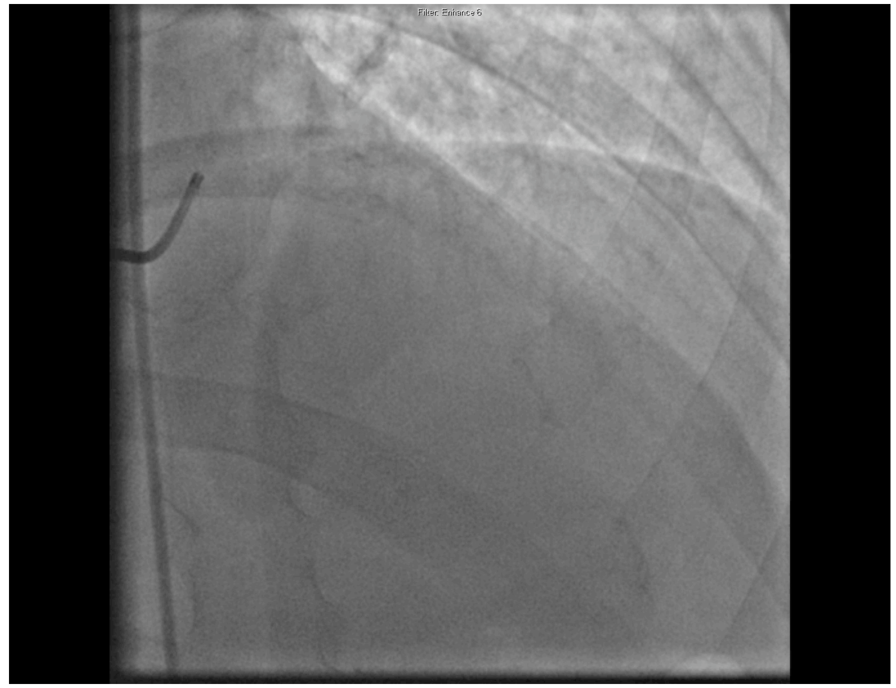
$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = -k_s\boldsymbol{d} - c_s\frac{\partial \boldsymbol{d}}{\partial t}$$

$$\boldsymbol{F}(\boldsymbol{d})\ \boldsymbol{S}(\boldsymbol{d})\ \widehat{\boldsymbol{n}} = p_0$$

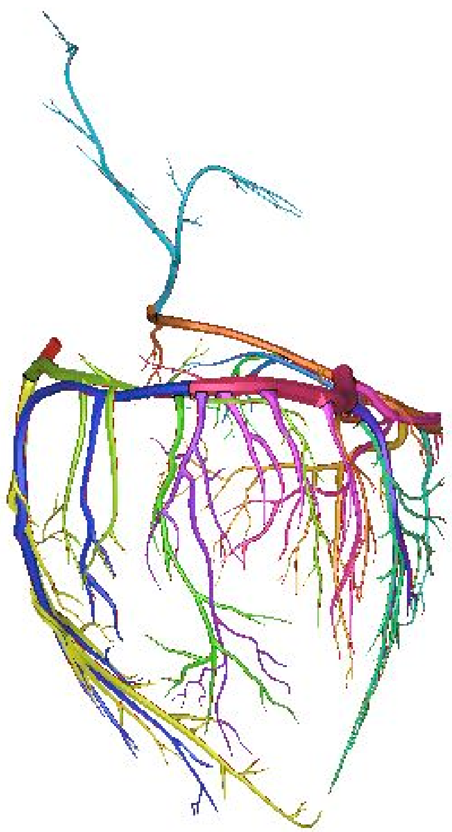


M. Astorino

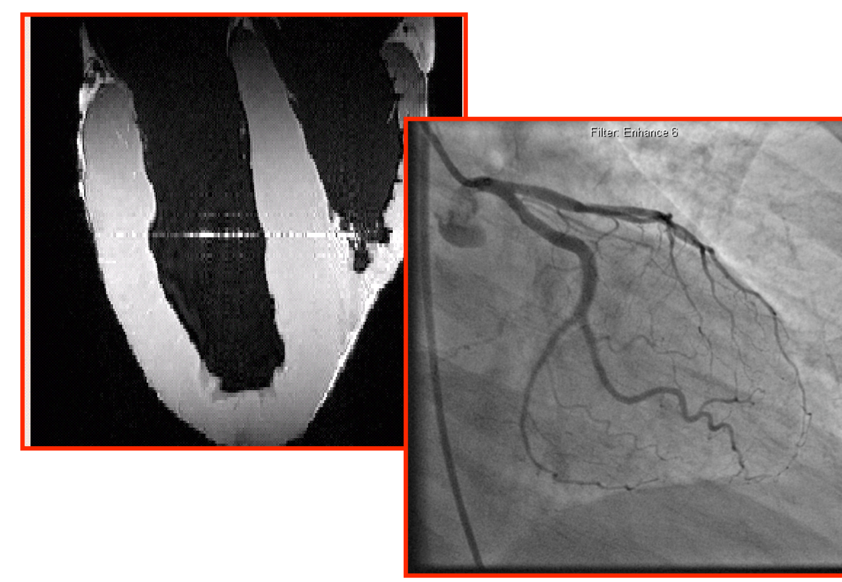
Myocardium perfusion



© Pediatric Cardiology - Stanford University



3D flows

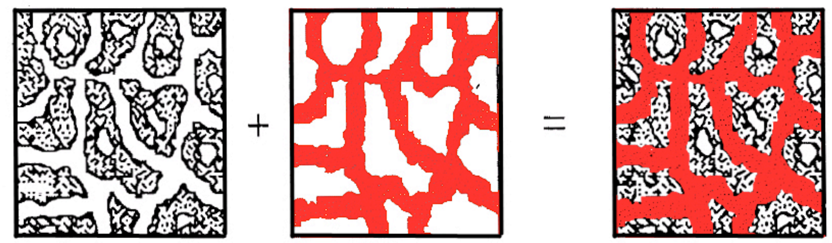


Poroelastic flows

Two levels of modelling

I. Vignon-Clementel & G. Rossi

Poroelastic model

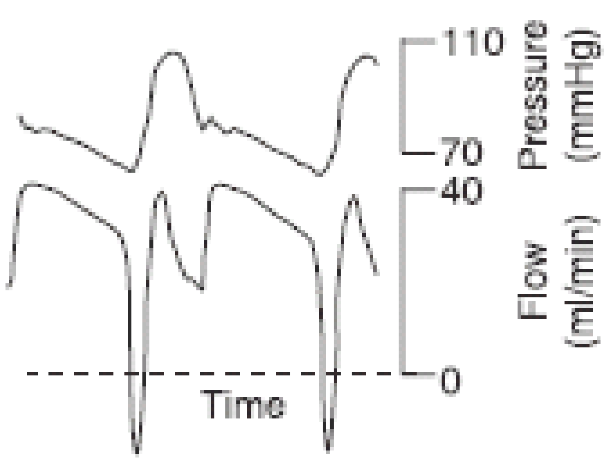


$$\mathrm{div}(\boldsymbol{\sigma}_s - p\mathrm{Id}) = 0$$

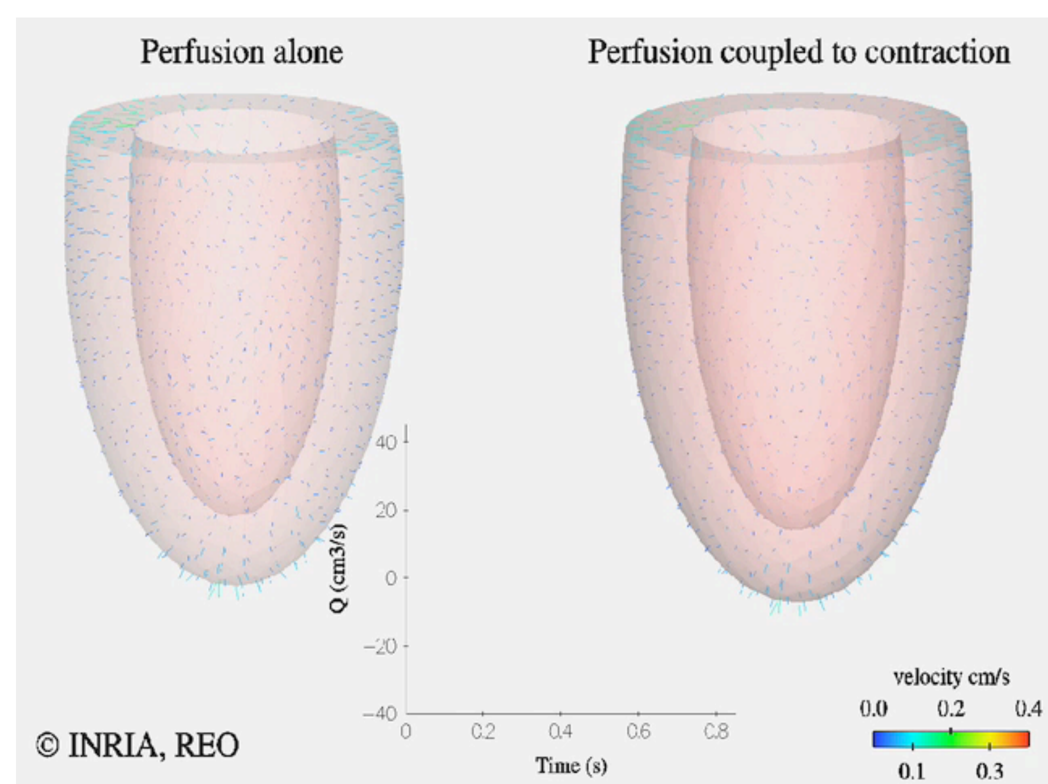
$$\boldsymbol{w} + K(J,\phi)\boldsymbol{\nabla} p = 0$$
$$\mathrm{div}\,\boldsymbol{w} + \mathrm{div}\,\boldsymbol{u}_s = f(x,p)$$

Perfusion velocity : $\boldsymbol{w} = \phi(\boldsymbol{u}_f - \boldsymbol{u}_s)$

Venous compartment: $f(x,p) = b(x)(p - p_v)$

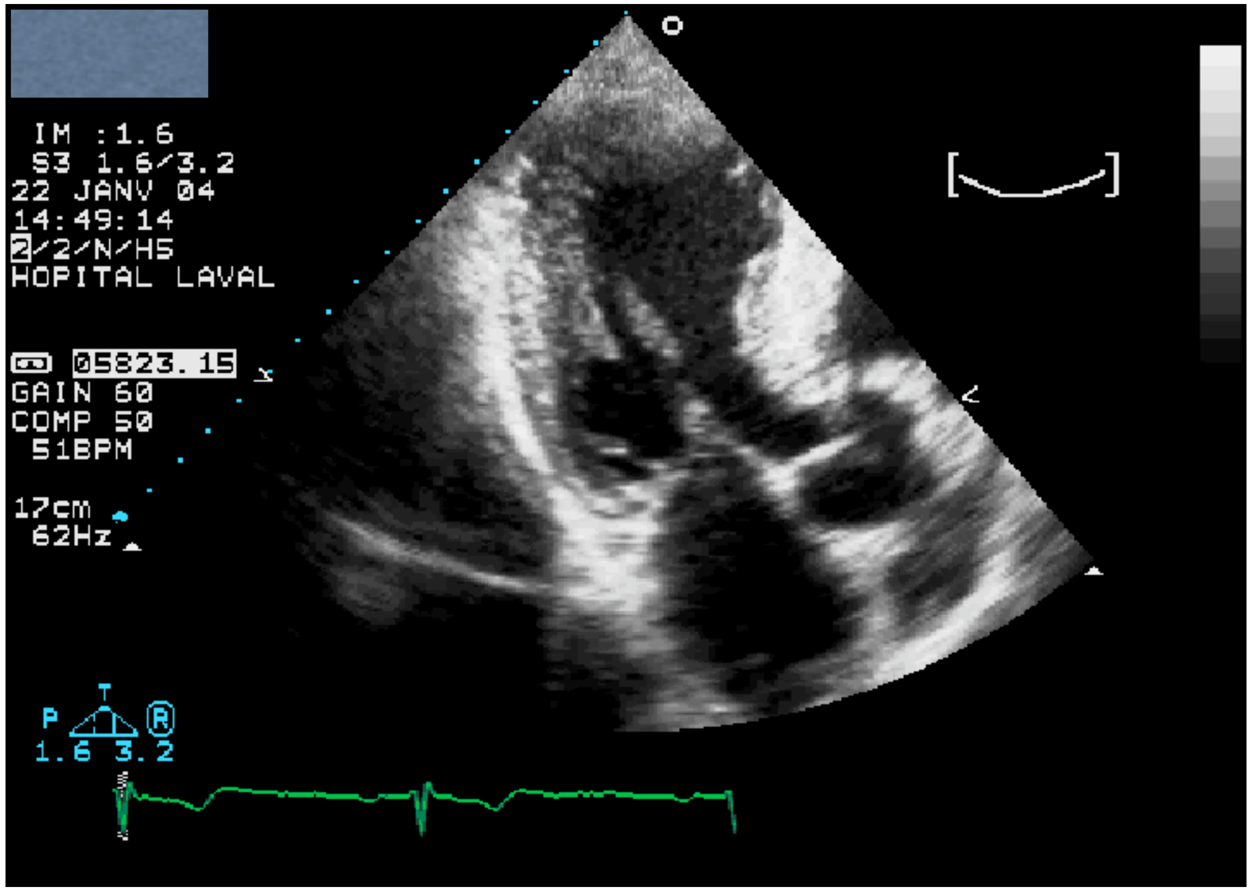


Adapted from Greg & Green

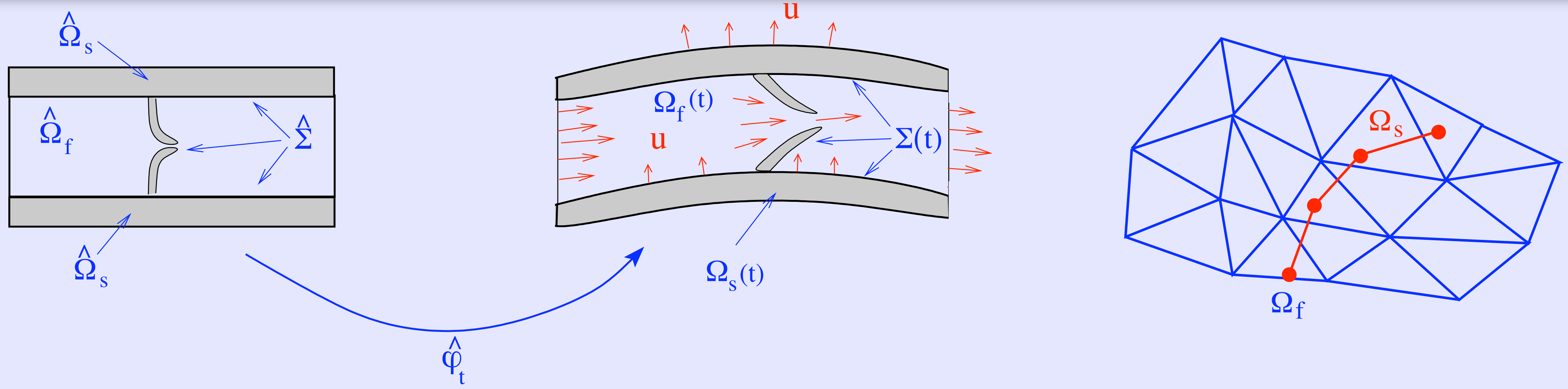


I. Vignon-Clementel & G. Rossi

FSI with valves



Valve : "Fictitious Domains" (FD)



"Fictitious Domain" for valves

- Basic idea: impose the kinematic constraint in a weak form

$$\langle \boldsymbol{\mu}, \boldsymbol{u}_f \rangle_\Sigma = \langle \boldsymbol{\mu}, \boldsymbol{u}_s \rangle_\Sigma, \forall \boldsymbol{\mu} \in \Lambda$$

- A saddle point problem has to be solved in the fluid

$$\begin{cases} a_f(\boldsymbol{u}_f, \boldsymbol{v}_f) + \langle \lambda, \boldsymbol{v}_f \rangle_\Sigma & = & \displaystyle\int_{\Omega_f(t)} \boldsymbol{f}_f \cdot \boldsymbol{v}_f, & \forall \boldsymbol{v}_f \in X_f \\ \langle \mu, \boldsymbol{u}_f \rangle_\Sigma - \langle \mu, \boldsymbol{u}_s \rangle_\Sigma & = & 0 & \forall \mu \in \Lambda \\ \hat{a}_s(\hat{\boldsymbol{u}}_s, \hat{\boldsymbol{v}}_s) - \langle \lambda, \boldsymbol{v}_s \rangle_\Sigma & = & \displaystyle\int_{\hat{\Omega}_s} \hat{\boldsymbol{f}}_s \cdot \hat{\boldsymbol{v}}_s & \forall \hat{\boldsymbol{v}}_s \in \hat{X}_s \end{cases}$$

- Lagrange multiplier space:

$$\Lambda_h = \{\boldsymbol{\mu}_h \text{ measure on } \Sigma,\ \boldsymbol{\mu}_h = \sum_{i=1}^{N_\Sigma} \boldsymbol{\mu}_i \delta(\mathbf{x}_i^{n+1}), \boldsymbol{\mu}_i \in \mathbb{R}^n\}$$

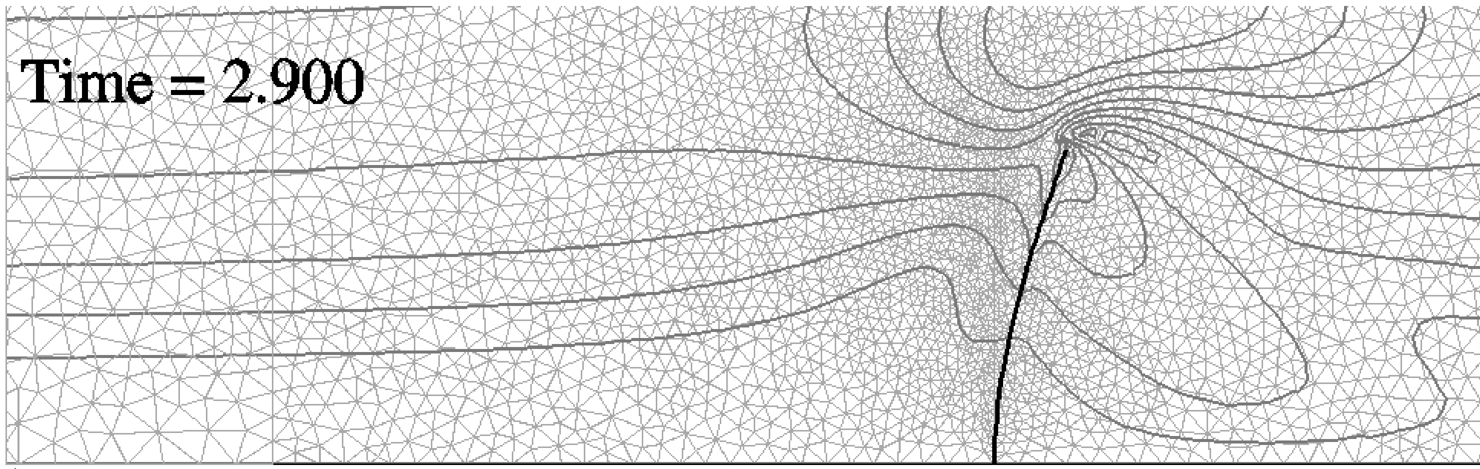
- Other Lagrange multiplier spaces are possible

Baaijens, 2001, de Hart et al. 2003

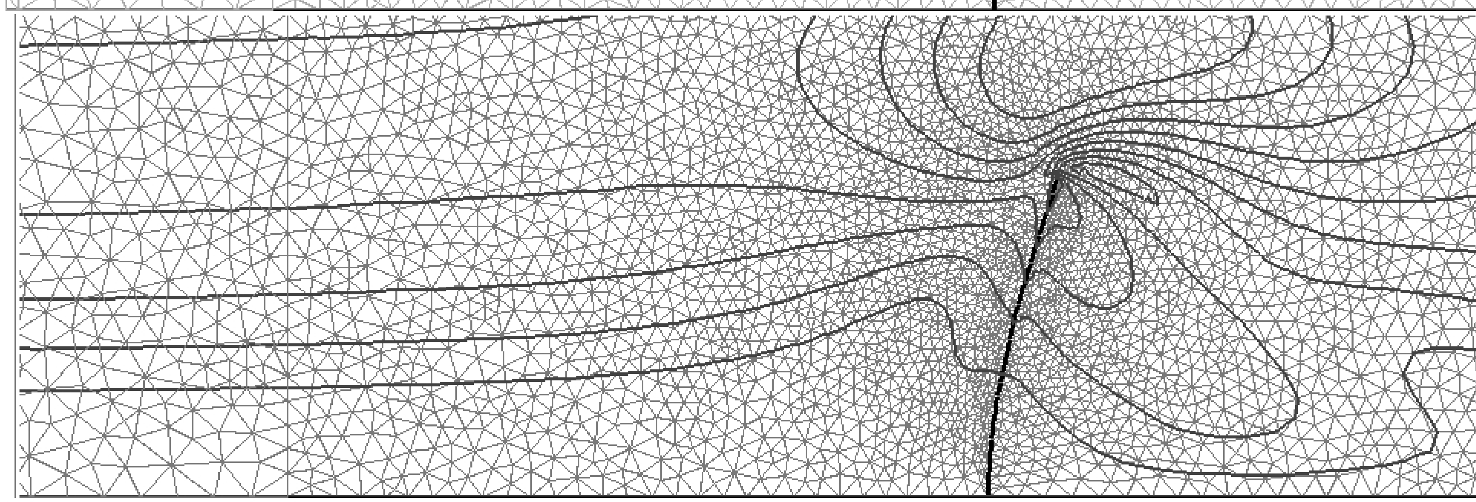
ALE / FD

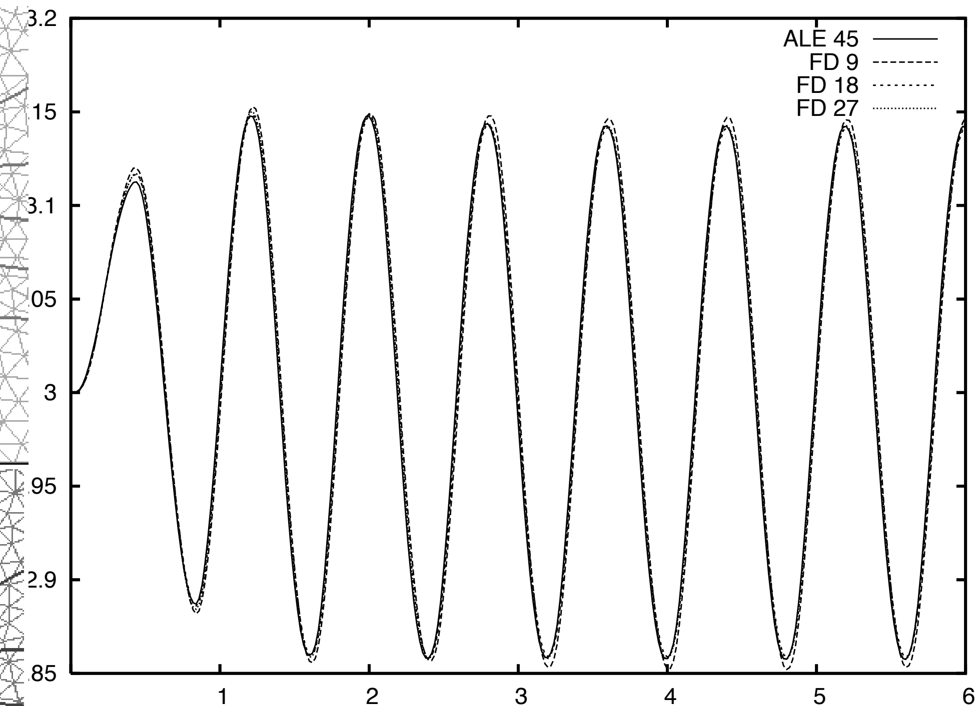
Comparison:

FD :



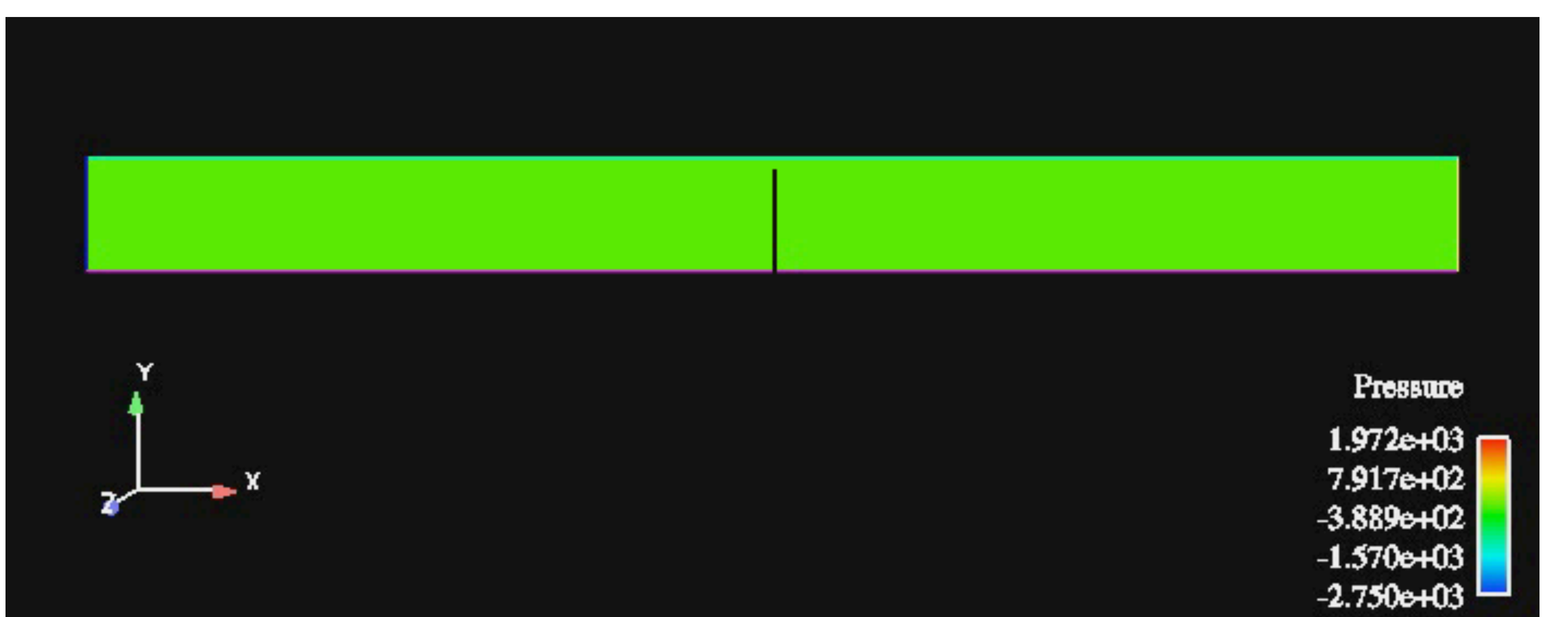
ALE :





Valve displacement

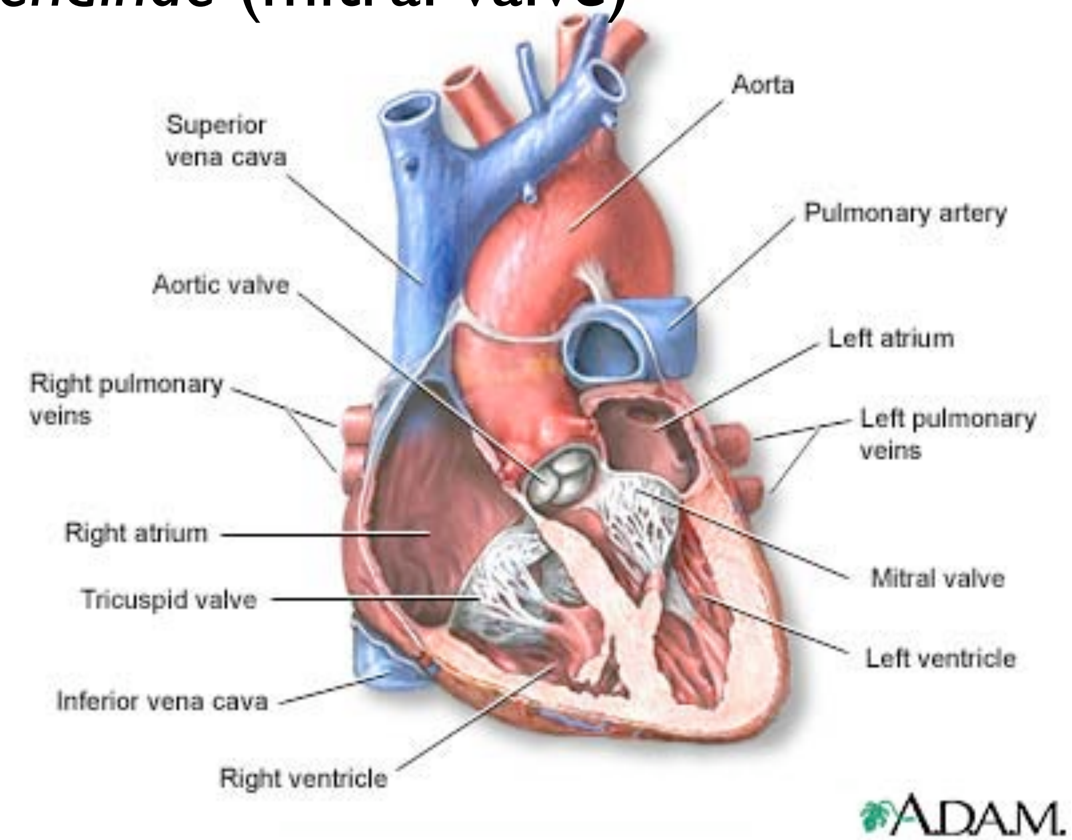
Mix ALE + FD:



N. Diniz dos Santos

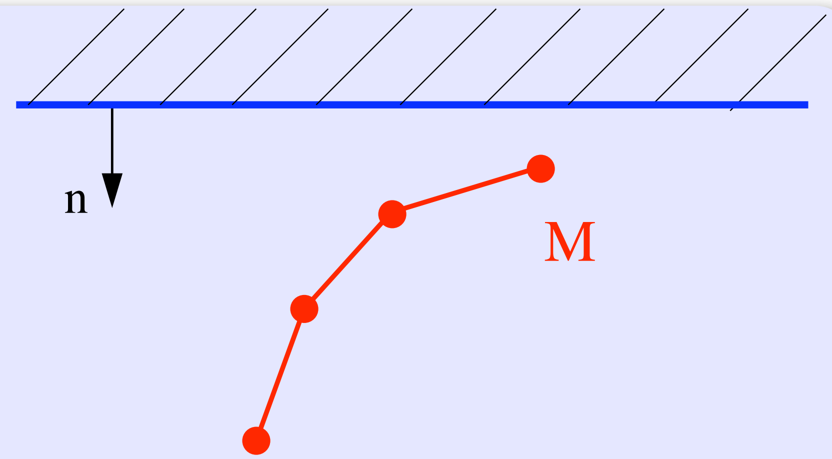
FSI and kinematic constraints

- Valves are submitted to various kinematic constraints:
  - contact between leaflets
  - *chordae tendinae* (mitral valve)



- We propose a framework to deal with these constraints in partitionned FSI algorithms.

Solid-wall contact



M a solid with energy J

$\mathcal{T}_h$ a mesh of M

$$X_h = \{\varphi_h \in C^0(M;\mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h\}$$

Mininization with convex constraint : $\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)$

With $\mathcal{U}_h = \{\varphi_h \in X_h, F_{x_i}(\varphi_h) \leq 0, \forall x_i \in M\}$

$$F_{x_i}(\boldsymbol{\varphi}_h) = \varepsilon - \boldsymbol{n} \cdot \boldsymbol{\varphi}_h(x_i) - c$$

van Loon, Anderson, van de Vosse, 2006

Dual approach

$$G(\mu) = \inf_{\boldsymbol{\varphi}\in X_h} \left[J(\boldsymbol{\varphi}) + \sum_{i=1}^{N_\Sigma} \mu_i F_{x_i}(\boldsymbol{\varphi}) \right]$$

$$G(\lambda_c) = \max_{\mu_i \geq 0} G(\mu)$$

λ_c : contact pressure

- The constraint is now simple to enforce

Gradient method with projection

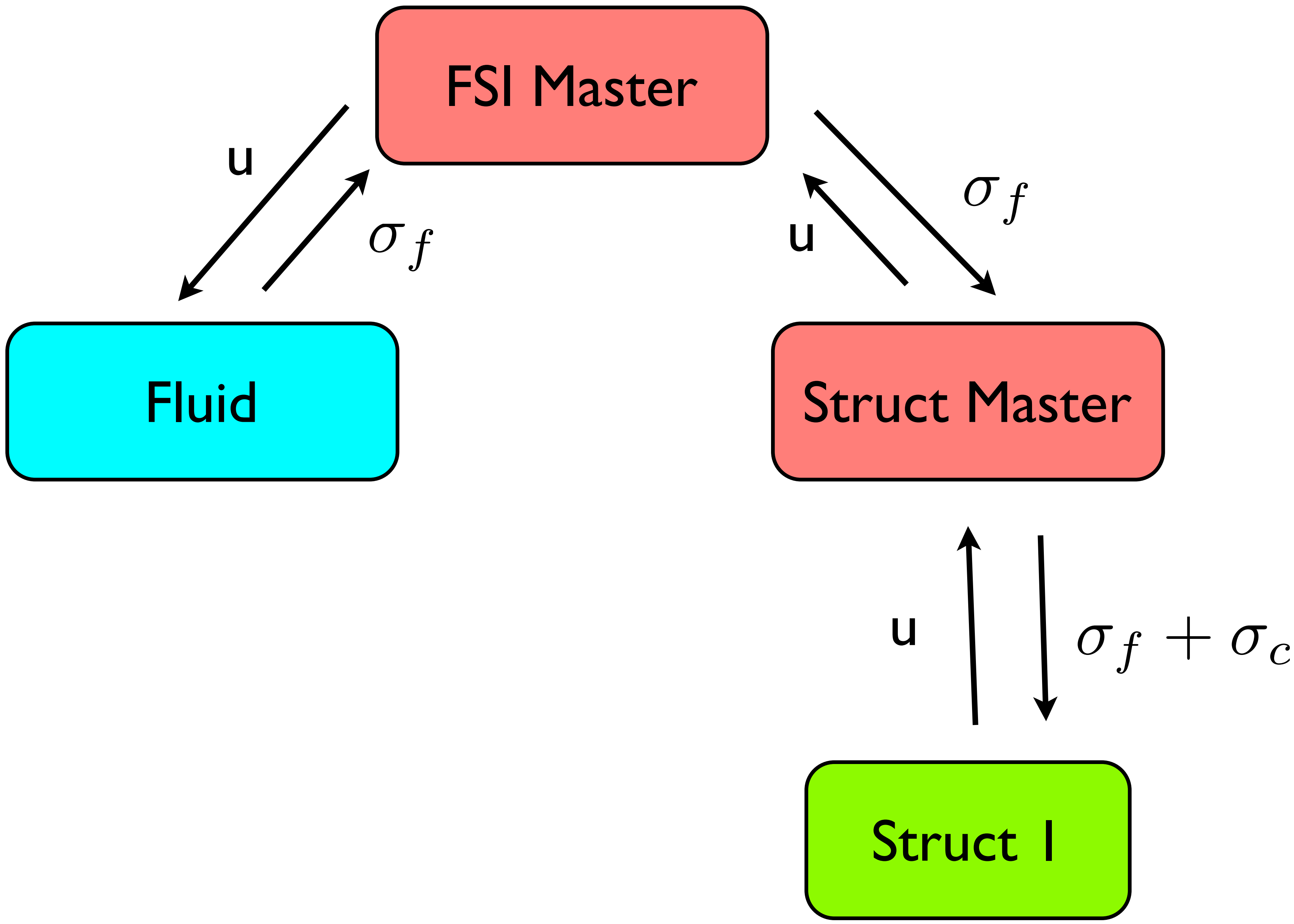
I $\langle J'(\boldsymbol{\varphi}^k), \boldsymbol{\xi} \rangle = -\sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \langle F'_{x_i}(\boldsymbol{\varphi}^k), \xi \rangle = \sum_{i=1}^{N_\Sigma} \lambda_{c,i}^k \boldsymbol{n} \cdot \boldsymbol{\xi}(x_i)$

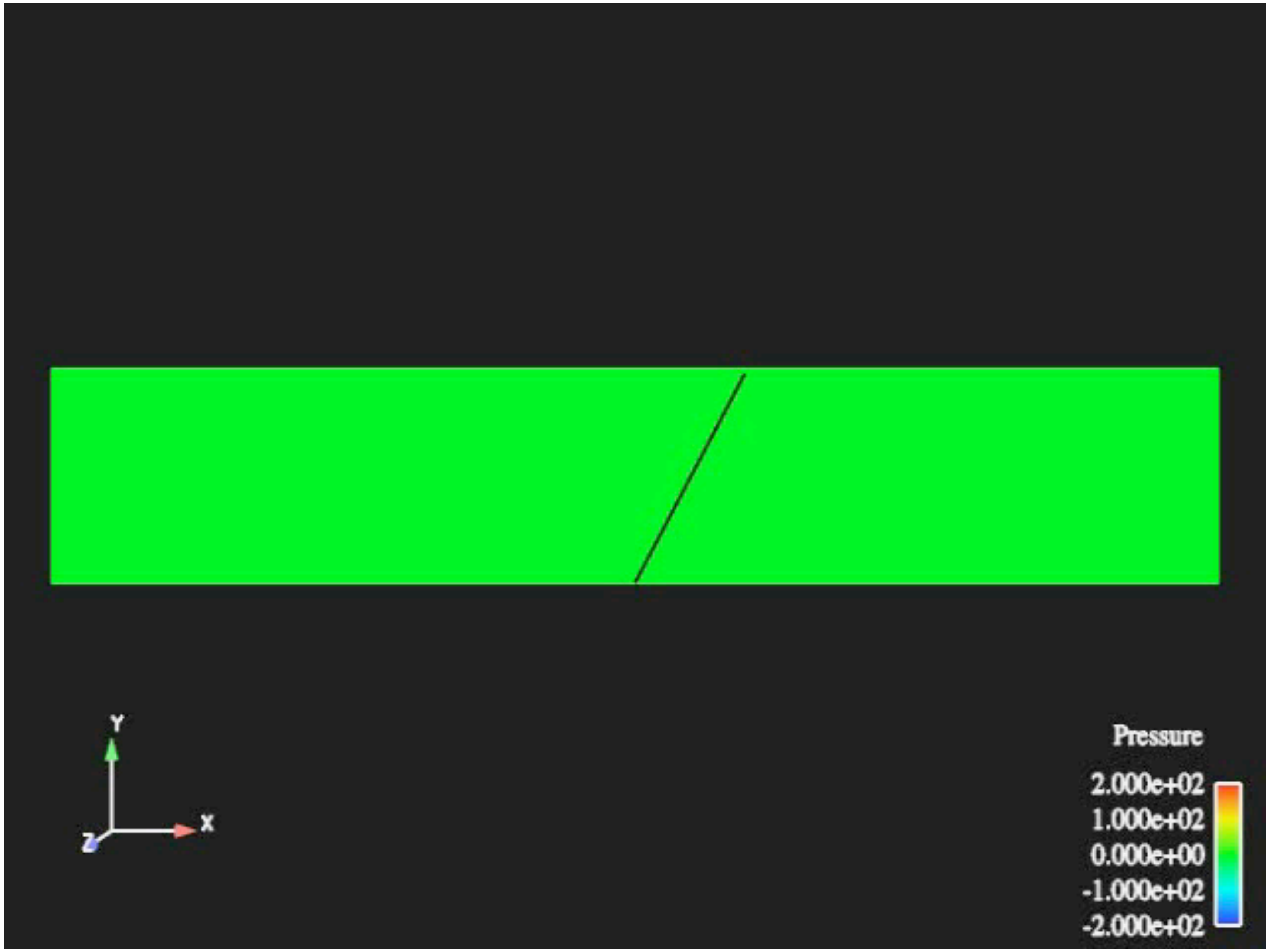
2) $\lambda_{c,i}^{k+1} = \mathbf{P}_{\mathbb{R}+} \left(\lambda_{c,i}^k + \alpha^k \boldsymbol{\nabla} G(\lambda_c^k)_i \right) = \mathbf{P}_{\mathbb{R}+} \left(\lambda_{c,i}^k + \alpha^k F_{x_i}(\boldsymbol{\varphi}^k) \right)$

3) Iterate on k

- The contact force is added to the hydrodynamics force

Implementation





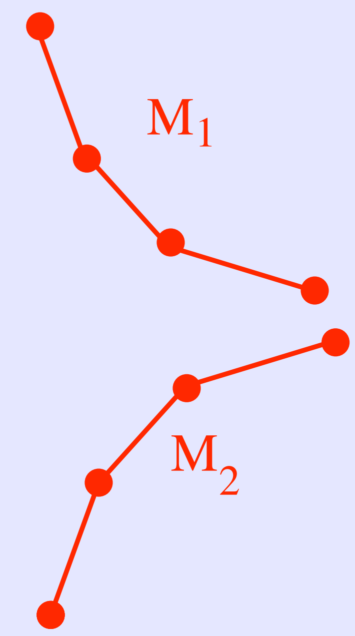
N. Diniz dos Santos

Solid-solid contact

$M = (M_1, M_2, \dots)$ a family of solid with energy J

$\mathcal{T}_h$ a mesh of M

$X_h = \{\varphi_h \in C^0(M;\mathbb{R}^3), \varphi_h|_T \in P_1, \forall T \in \mathcal{T}_h\}$



Mininization with *non convex* constraints :

$$\inf_{\varphi_h \in \mathcal{U}_h} J(\varphi_h)$$

with

$$\mathcal{U}_h = \{\boldsymbol{\varphi}_h \in X_h, dist(\boldsymbol{\varphi}_h(T_1), \boldsymbol{\varphi}_h(T_2)) \geq \varepsilon, \forall T_1, T_2 \in \mathcal{T}_h\}$$

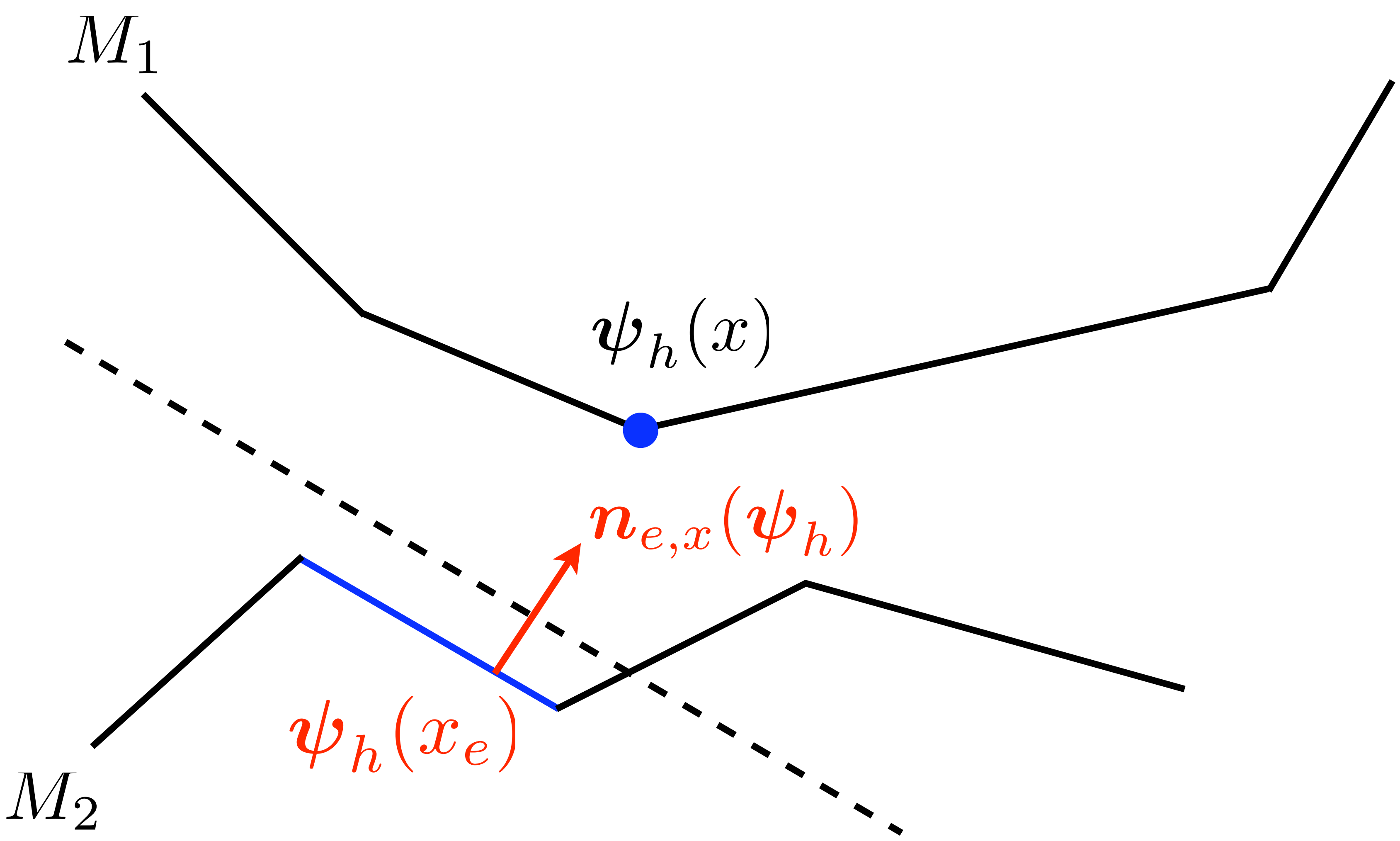
Solid-solid contact

Optimization algorithm (O. Pantz, 2007) :

- replace a problem with nonconvex constraints with a sequence of problems with convex constraints.
- Solve $$J(\varphi_h^{k+1}) = \inf_{\psi_h \in T(\varphi_h^k)} J(\psi_h)$$

  where $T(\varphi_h^k)$ is a convex neighborhood of φ_h^k
- Iterate on k until convergence

Solid-solid contact



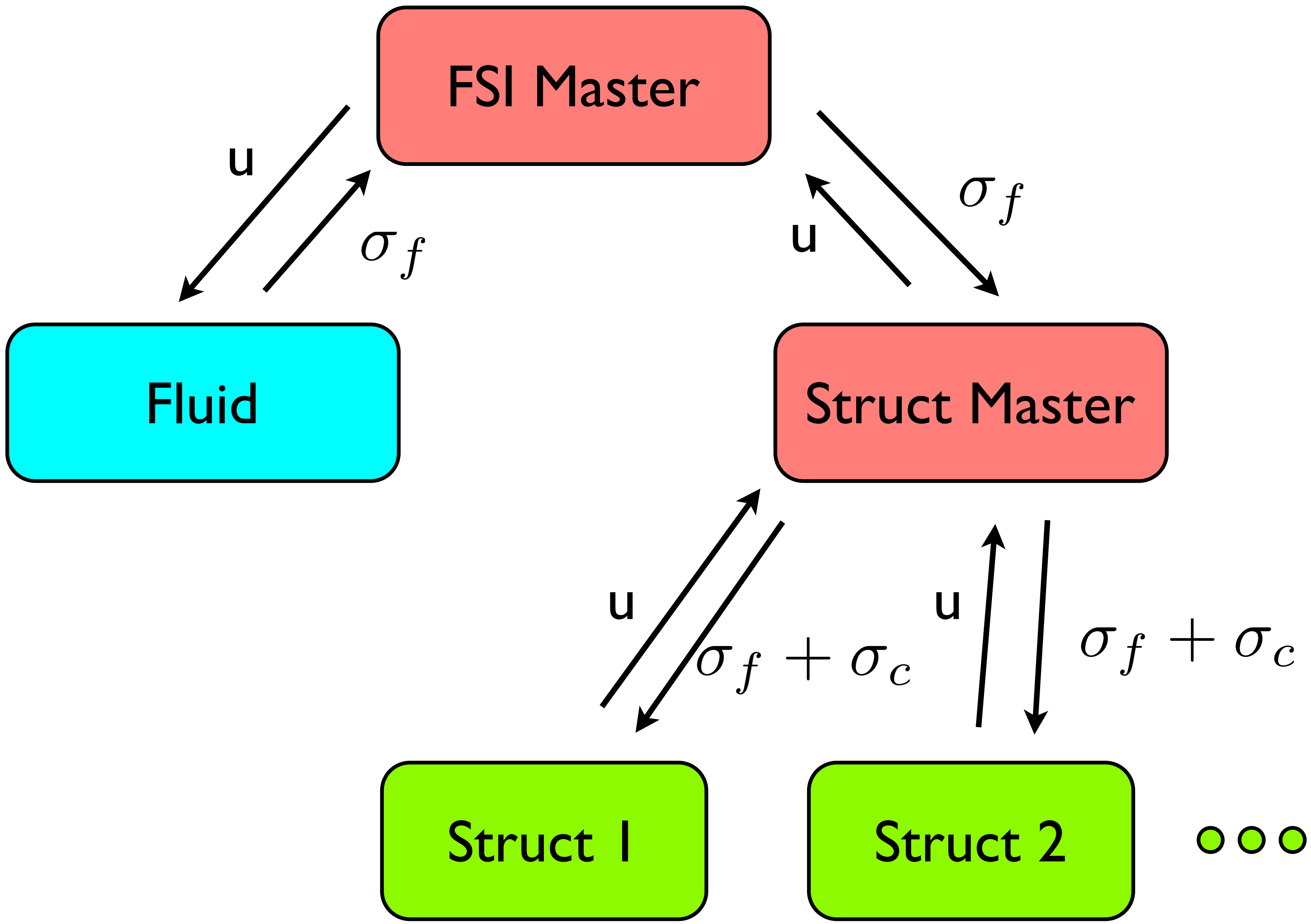
$$T(\boldsymbol{\psi}_h) = \Big\{ \boldsymbol{\varphi}_h \in X_h,\ \min_{x_e \in e} \boldsymbol{n}_{e,x}(\boldsymbol{\psi}_h) \cdot (\boldsymbol{\varphi}_h(x_e) - \boldsymbol{\varphi}_h(x)) \geq \varepsilon$$
$$\text{for all edges } e \text{ and all node } x \notin e \Big\}$$

- At convergence, φ_h does not satisfy *a priori* the optimality conditions of the original problem
- ... but the error is O(h) (*O. Pantz, 2007*)
- Same kinds of constraint as for the solid-wall case:

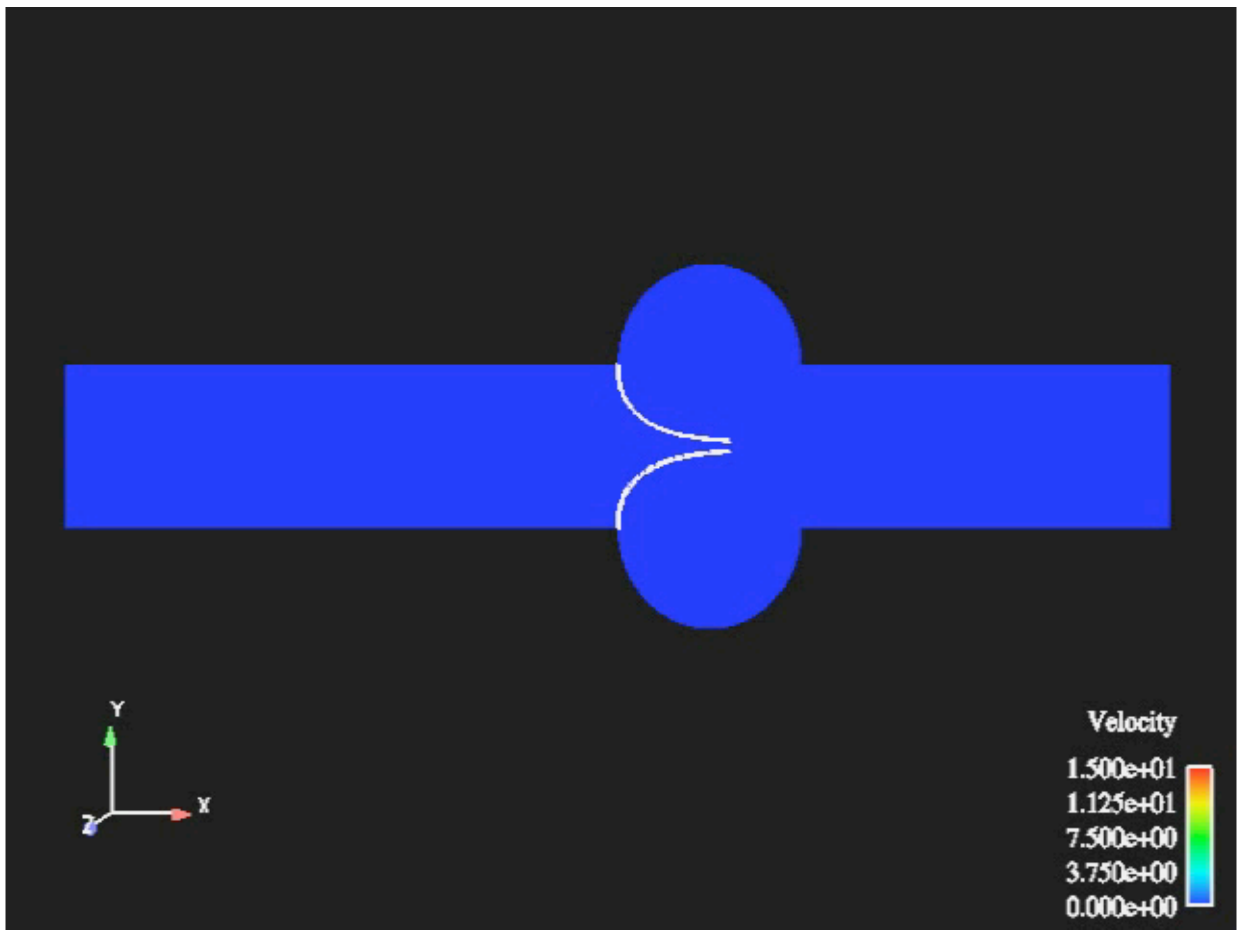
$$T(\boldsymbol{\psi}_h) = \Big\{\boldsymbol{\varphi}_h \in X_h, F^0_{e,x}(\boldsymbol{\varphi}_h) \leq 0, F^1_{e,x}(\boldsymbol{\varphi}_h) \leq 0, \\ \text{for all edges } e \text{ and all node } x \notin e\Big\}$$

$$F^j_{e,x}(\boldsymbol{\varphi}_h) = \varepsilon - \boldsymbol{n}_{e,x}(\boldsymbol{\psi}_h) \cdot (\boldsymbol{\varphi}_h(e_j) - \boldsymbol{\varphi}_h(x))$$

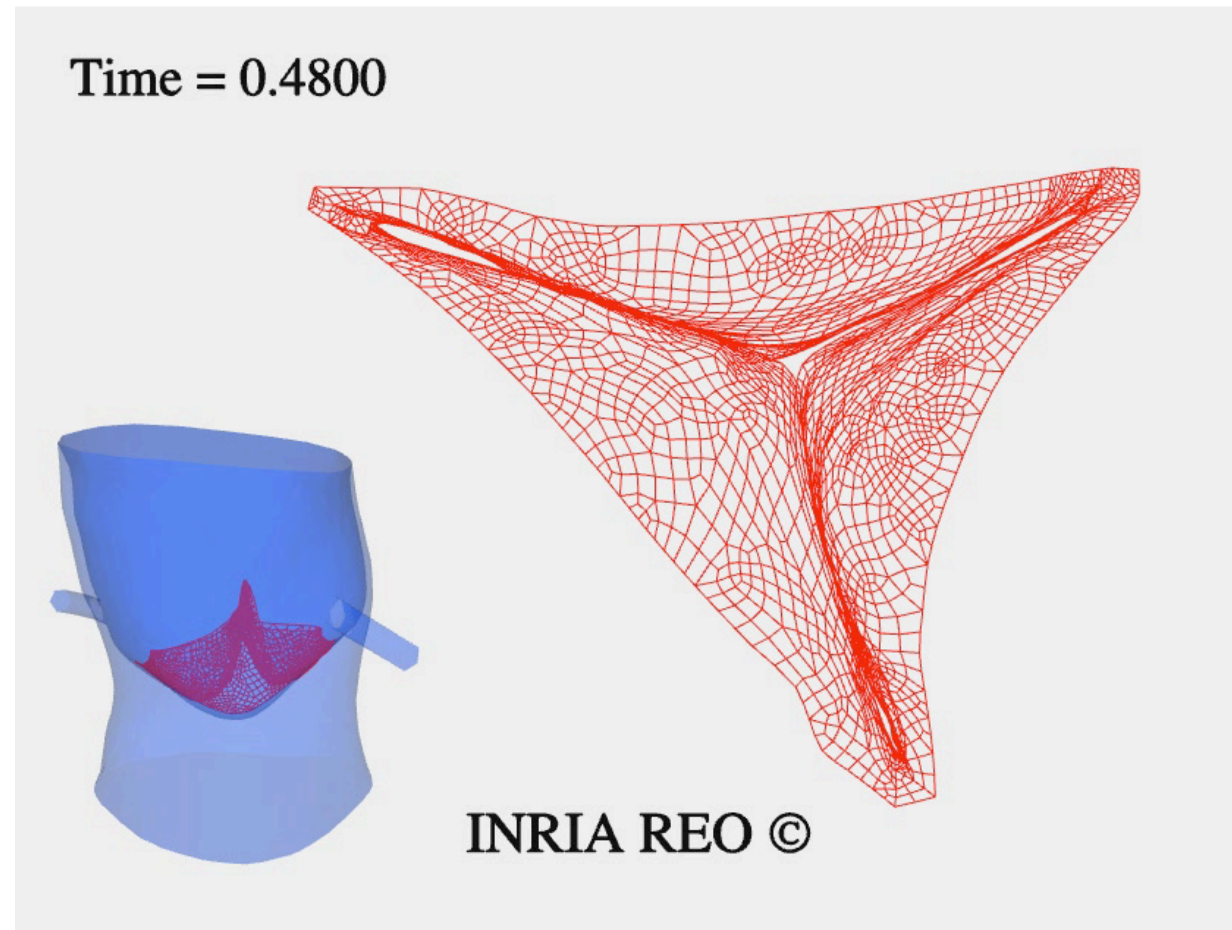
Implementation



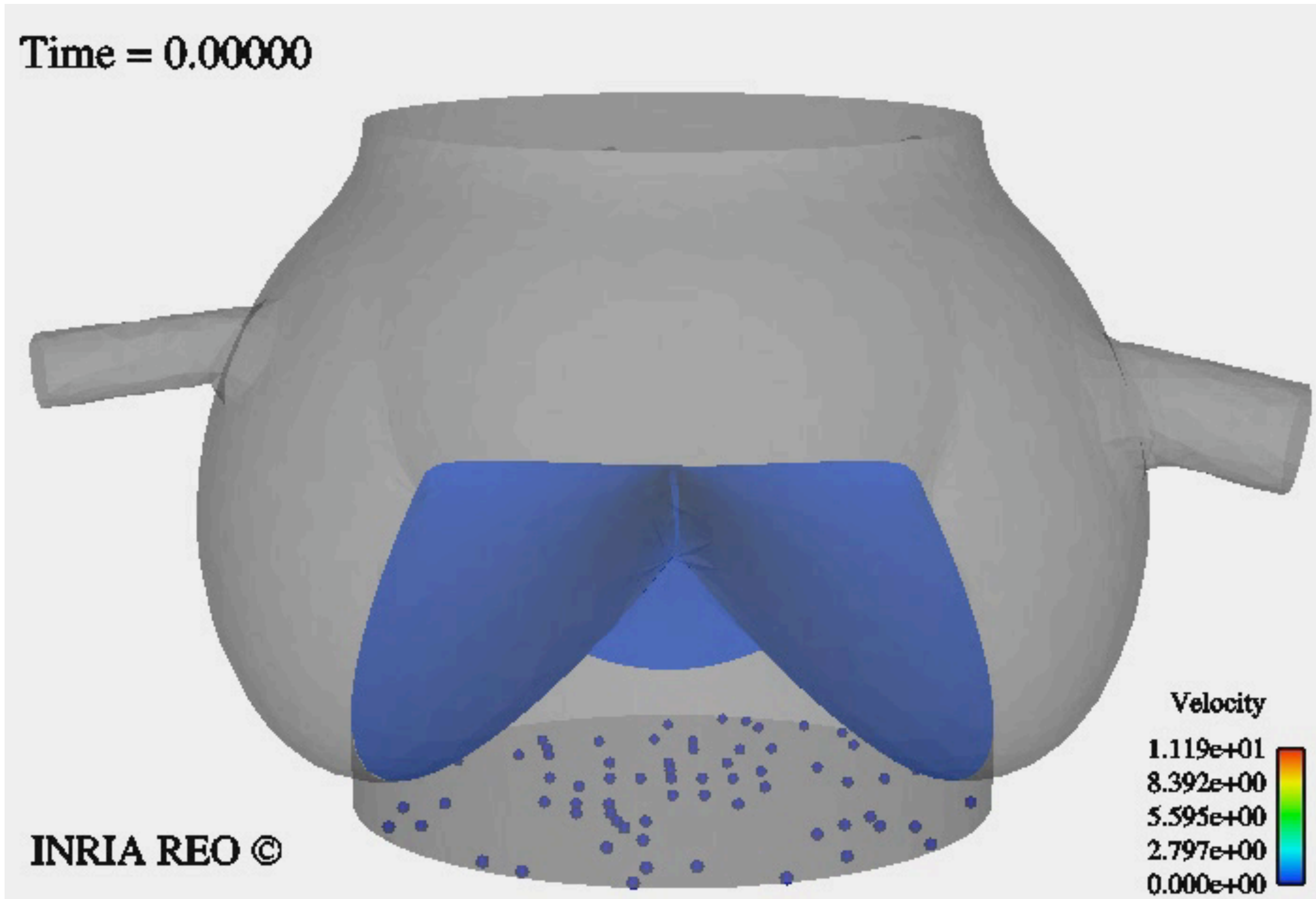
FSI & Solid-solid contact



Diniz dos Santos, JFG, 2007



Astorino, JFG, Pantz, Traoré



Astorino, JFG, Pantz, Traoré

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