From the Assignment Model to Combinatorial Auctions

IPAM Workshop, UCLA

May 7, 2008

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Overview

- LP formulations of the (package) assignment model
- Sealed-bid and ascending-price auctions are different solution methods to the LP formulation
- Focus on the Vickrey-Clarke-Groves (VCG) auction
  - Each buyer demands only one indivisible object
  - A buyer may demand multiple indivisible objects

The combinatorial auction setting

- ◇ An auctioneer with indivisible units of K commodities for sale, $\omega \in Z_+^K$
- ◇ B buyers, indexed $b = 1, 2, ..., B$
- ◇ Buyer b has non-decreasing utility over (z_b, m)

  z_b is a package of the indivisible objects

  m is a divisible good (money)

$$U_b(z, m) = u_b(z) + m$$

- ◇ Each buyer knows his utility function; it is his private information

How to conduct an auction that maximizes the gains from trade?

Incentive compatibility

A buyer *bids truthfully* if his bids on packages are equal to his utilities.

An auction is *dominant strategy incentive compatible* if each buyer maximizes his payoff by bidding truthfully *regardless* of what strategy other buyers follow.

An auction is *ex post incentive compatible* if each buyer maximizes his payoff by bidding truthfully *as long as* other buyers bid truthfully.

Marginal products of buyers[1]

The (maximum) gains from trade are

$$V(N) \equiv \max\{\sum_{\ell=1}^{B} u_\ell(z_\ell) | \sum_{\ell=1}^{B} z_\ell \leq \omega\}$$

where $N = \{s, 1, 2, ..., B\}$. The maximum is attained at an *efficient* assignment $Z^* = (z_1^*, z_2^*, ..., z_B^*)$.

Gains from trade when buyer b is excluded:

$$V(N \backslash b) \equiv \max\{\sum_{\ell \neq b} u_\ell(z_\ell) | \sum_{\ell} z_\ell \leq \omega\}$$

Buyer b's *marginal product* is

$$\text{MP}_b \equiv V(N) - V(N \backslash b)$$

Let $Z^* = (z_1^*, z_2^*, ..., z_B^*)$ be an efficient allocation (at which $V(N)$ is attained). Buyer b's *social opportunity cost* is

$$\text{SOC}_b \equiv V(N \backslash b) - \sum_{\ell \neq b} u_\ell(z_\ell^*)$$

[1] Seller's marginal product $\text{MP}_S = V(N)$. However, seller is not viewed as a strategic player with private information so MP_S will play no role. The seller defines the rules of the auction and steps back.

A sealed-bid VCG auction is efficient and dominant strategy incentive compatible

1. Buyers submit sealed-bids, one for each bundle.
2. Compute the assignment that maximizes the sum of submitted bids.
3. For each buyer, compute the assignment that maximizes sum of bids with the buyer excluded.
4. Each buyer receives the allocation computed in 2, and pays his social opportunity cost (computed under the presumption that bidding is truthful).

In any selling scheme no buyer can hope to extract more than his marginal product. In the VCG auction, buyer b's surplus is

$$u_b(z_b^*) - \mathrm{SOC}_b = u_b(z_b^*) - [V(N \backslash b) - \sum_{\ell \neq b} u_\ell(z_\ell^*)] = \mathrm{MP}_b$$

where $Z^* = (z_b^*)$ is the efficient assignment from step 2., and $V(N \backslash b)$ is from step 3.

Proof that VCG auction is dominant strategy

Let $v_\ell(z_\ell)$, $\ell \neq b$, be the bids of bidders other than b. (It does not matter whether bidders $\ell \neq b$ bid truthfully.)

Bidder b is truthful: submits $u_b(z_b)$ as bids.
Bidder b lies: submits $v_b(z_b)$ as bids.

The VCG allocations are
Bidder b is truthful: $(z_1^*, z_2^*, ..., z_B^*)$
Bidder b lies: $(z_1', z_2', ..., z_B')$

VCG auction rules imply

$$u_b(z_b^*) + \sum_{\ell \neq b} v_\ell(z_\ell^*) \;\geq\; u_b(z_b') + \sum_{\ell \neq b} v_\ell(z_\ell')$$

To compute VCG payments let

$$\sum_{\ell \neq b} v_\ell(\hat{z}_\ell) \;\geq\; \sum_{\ell \neq b} v_\ell(z_\ell), \qquad \forall (z_\ell)$$

Then,

$$u_b(z_b^*) - \Big[\sum_{\ell \neq b} v_\ell(\hat{z}_\ell) - \sum_{\ell \neq b} v_\ell(z_\ell^*)\Big] \;\geq\; u_b(z_b') - \Big[\sum_{\ell \neq b} v_\ell(\hat{z}_\ell) - \sum_{\ell \neq b} v_\ell(z_\ell')\Big]$$

The assignment model

Koopmans & Beckman (1957), Gale (1960), Shapley & Shubik (1972), Gretsky, Ostroy & Zame (1999)

- ◇ B buyers, indexed b.
- ◇ S sellers (or rather objects), indexed s. Each seller's cost is zero.
- ◇ Each buyer has utility for one object. *Unit demand assumption.*

u_{bs} buyer b's utility for object s, $u_{bs} \geq 0$.

x_{bs} "fraction" of object s allocated to buyer b.

p_s price of s.

π_b $= \max_s\{u_{bs} - p_s, 0\}$ is buyer b's surplus

- ◇ No budget constraint: Each b's endowment of money $> \max_s u_{bs}$

Definitions

- A *feasible assignment*, $(b, s_b)_b$, is an allocation of sellers (objects) to buyers.
- An efficient (feasible) assignment maximizes the sum of utilities of buyers.
- A vector of prices $p = (p_1, p_2, ..., p_S)$ is *Walrasian* if it supports a feasible assignment, $(b, s_b)_b$. That is

$$u_{b\,s_b} - p_{s_b} \geq u_{bs} - p_s, \quad \forall s, \ \forall b.$$

Results

◇ *Walrasian prices exist in the assignment model.*

◇ *The set of assignments supported by a Walrasian price vector is the set of efficient assignments.*

A linear programming formulation of the assignment model

LP^1

$$\max \sum_{b=1}^{n} \sum_{s=1}^{n} u_{bs} x_{bs}$$

s. t.

$$\begin{aligned} \sum_{s} x_{bs} &\leq 1, \quad \forall b \\ \sum_{b} x_{bs} &\leq 1, \quad \forall s \end{aligned}$$

$$x_{bs} \geq 0$$

DLP^1, dual of LP^1

$$\min \sum_{b=1}^{n} \pi_b + \sum_{s=1}^{n} p_s$$

s.t.

$$\begin{aligned} \pi_b + p_s &\geq u_{bs}, \quad \forall b,\ s \\ \pi_b,\ p_s &\geq 0 \end{aligned}$$

Linear programming characterization

1. *All extreme points of LP^1 are integer.*
2. *Any efficient assignment is a solution to LP^1.*
3. *DLP^1 solution set is the set of Walrasian prices and buyer surpluses.*
4. *One corner of the DLP^1 solution set is the smallest Walrasian price vector.*
5. *This corner simultaneously gives each buyer his marginal product.*

6. *Any efficient auction finds a solution to LP^1.*
7. *The sealed-bid VCG auction implements corner of DLP^1 preferred by all buyers.*

Sealed-bid VCG auction in the assignment model

1. Implements the smallest Walrasian price (i.e., price at which Demand = Supply).
   - The smallest Walrasian price exists.
2. At the smallest Walrasian price each bidder gets his *marginal product*.
   - This smallest price is the only market clearing price at which Demand = Supply after any single buyer is removed from the economy.

Ascending-price implementation of VCG auction

1. A dynamic mechanism for discovering the smallest price at which Demand = Supply.
2. A primal-dual algorithm on the LP formulation of the underlying exchange economy.

An ascending-price VCG auction in the assignment model (Demange, Gale, and Sotomayor 1986)

The following auction is ex post incentive compatible.

0. Start with price zero for each object.
1. Buyers report their demand sets at current prices. If there is no overdemanded set, go to Step 3; otherwise go to Step 2.[2]
2. Choose a minimal overdemanded set. Raise prices of all objects in this set until some buyer changes his demand set. Go to Step 1.
3. Assign each buyer an object in his demand set at current prices. Stop.

T – a subset of objects.

$I(T;p)$ – set of buyers whose demand at prices p is in T.

T is *overdemanded* if $|I(T;p)| > |T|$.

[2] An assignment is feasible iff there is no overdemanded set (Hall 1935).

Buyers' utilities

	A	B	ϕ
u_1	4	**7**	0
u_2	**8**	7	0
u_3	6	4	0

Steps in the auction

	p_A	p_B	D_1	D_2	D_3	OD
$-$	0	0	B	A	A	$\{A\}$
$0 < \theta < 1$	θ	0	B	A	A	$\{A\}$
$-$	1	0	B	$A,\ B$	A	$\{A,\ B\}$
$0 < \theta < 5$	$1 + \theta$	θ	B	$A,\ B$	A	$\{A,\ B\}$
$-$	6	5	$\mathbf{B}$	$\mathbf{A},\ B$	$\phi,\ A$	$\emptyset$

The auction as a primal-dual algorithm

- Fix a DLP^1 feasible solution, $(\pi_b^0,\ p_s^0)$

- b's demand set at prices (p_s^0):

  $$D_b(p^0) = \{\, s \mid \pi_b^0 = u_{bs} - p_s^0 \geq u_{bs'} - p_{s'}^0, \forall s' \}$$

- $x_{bs} \geq 0$ satisfies complementary slackness w.r.t. $(\pi_b^0,\ p_s^0)$:

  $$\text{If } x_{bs} > 0 \quad \text{then} \quad s \in D_b(p^0).$$

- Solutions which satisfy complementary slackness w.r.t. $(\pi_b^0,\ p_s^0)$ (but are not necessarily LP^1 feasible) are in the feasible region of RP^1 below.

RP^1

$$\min \sum_{b=1}^{N} \sum_{s=1}^{N} c_s^0 w_s$$

s. t.

$$\begin{aligned} \sum_{s \in D_b(p^0)} x_{bs} &= 1, \quad \forall b \\ \sum_{\{b | s \in D_b(p^0)\}} x_{bs} - w_s &= 1, \quad \forall s \in \cup_b D_b(p^0) \\ x_{bs},\ w_s &\geq 0, \end{aligned}$$

where

$$c_s^0 = \begin{cases} 1, & \text{if } s \text{ in minimal overdemanded set at prices } p^0 \\ 0, & \text{otherwise.} \end{cases}$$

DRP^1

$$\max \sum_b \mu_b + \sum_s \nu_s$$

s. t.

$$\begin{aligned} \mu_b + \nu_s &\leq 0, \quad \forall b,\ s \in D_b(p^0) \\ \nu_s &\leq c_s, \quad \forall s. \end{aligned}$$

New variables in DRP^1 and RP^1:

- w_s is amount of excess demand for object s.
- μ_b is rate of decrease of b's surplus.
- ν_s is rate of increase of price of object s.

The ascending-price auction

0. Start with price zero for each object.
1. Buyers report their demand sets at current prices. If there is no overdemanded set, go to Step 3; otherwise go to Step 2.
2. Choose a minimal overdemanded set. Raise prices of all objects in this set until some buyer changes his demand set. Go to Step 1.
3. Assign each buyer an object in his demand set at current prices. Stop.

The primal-dual algorithm

0. Let $i = 0$ and let the initial DLP^1 feasible solution be $p^0 = 0$ (with the implied π_b's).
1. Obtain LP^1 "solution" satisfying complementary slackness w.r.t. prices p^i. If solution is LP^1 feasible go to Step 3; otherwise go to Step 2.
2. Compute minimal overdemanded set and use it to determine objective fn. coefficients of RP^1. Increase p^i in the direction indicated by solution to DRP^1, until further increases render DRP^1 infeasible; call this price p^{i+1}. Increment $i \leftarrow i + 1$. Go to Step 1.
3. Assign each buyer an object in his demand set at current prices. Stop.

The Package Assignment Model

(Bikhchandani and Ostroy 2002)

- B buyers, $b = 1, 2, ..., B$.
- 1 seller, indexed s.
- k=1,2,...,K distinct indivisible commodities.
- $\omega \in \mathcal{Z}_+^K$, endowment of seller s.
- $N = \{s, 1, 2, ..., B\}$ is the set of agents.
- Agent's utility functions

$$\begin{aligned} U_b(z, m) &= u_b(z) + m, \quad \forall b. \\ U_s(y, m) &= m, \quad 0 \le y \le \omega. \end{aligned}$$

- No budget constraint:
  Each b's endowment of money $> u_b(\omega)$
- All agents are price takers.

An *efficient assignment* maximizes the sum of buyers' utilities:

$$Z^* \equiv \arg\max_{Z=(z_b)} \{\sum_{b=1}^{B} u_b(z_b) | \sum_b z_b \leq \omega\}.$$

A *pricing function* is $\langle p_b(z) \rangle$, $\forall z \leq \omega$, $\forall b$.

The price paid or received for a package may be *non-linear* in the objects in the package and may different for different buyers, i.e., *non-anonymous*.

The revenue received by the seller for feasible assignment $Z = (z_b)$ is

$$P(Z) \equiv \sum_{b=1}^{B} p_b(z_b).$$

A *pricing equilibrium* is a $[\, Z^* = (z_b^*),\ \langle p_b^*(\cdot) \rangle \,]$ such that $p_b^*(\cdot) \geq 0$, and

- Z^* is a feasible assignment,
- buyers maximize utilities: for all b,

$$u_b(z_b^*) - p_b^*(z_b^*) \geq u_b(z) - p_b^*(z), \qquad \forall z$$

- seller maximizes profits: for all feasible assignments Z'

$$P^*(Z^*) \geq P^*(Z').$$

To rationalize non-linear, non-anonymous pricing, impose the following trade restrictions:

B1. Each buyer may buy at most one package from the seller

B2. Buyers may not resell packages to each other after buying from the seller

A Linear Programming Formulation

LP^2

$$\max \sum_{b=1}^{B} \sum_{z} x_b(z) u_b(z)$$

s.t.

$$
\begin{aligned}
\sum_{z} x_b(z) &= 1, \qquad \forall b && (1)\\
\sum_{Z} x_s(Z) &= 1 && (2)\\
x_b(z) - \sum_{Z \in G^b(z)} x_s(Z) &= 0, \qquad \forall z, \forall b && (3)
\end{aligned}
$$

$$x_b(\cdot),\ x_s(\cdot) \geq 0$$

$x_b(z)$ is fraction of package z consumed by b
$x_s(Z)$ is fraction of assignment Z sold
$G^b(z)$ is the set of assignments in which b gets package z.

(1) Sum of fractions of packages bought is = 1 for each buyer.

(2) Sum of fractions of assignments sold is = 1.

(3) Demand = Supply.

$\pi_b = \max_z\{u_b(z) - p_b(z)\}$ is buyer b's surplus

$\pi_s = \max_Z\{P(Z)\}$ is seller's revenue (profit)

DLP^2

$$\min_{\pi_s,(\pi_b)} \ \pi_s + \sum_{b=1}^{B} \pi_b$$

s.t.

$$\begin{aligned} \pi_b - [u_b(z) - p_b(z)] \ &\geq \ 0, \qquad \forall z\, \forall b \\ \pi_s - P(Z) \ &\geq \ 0, \qquad \forall Z = (z_b) \end{aligned}$$

Complementary slackness

Utility max.	$x_b(z)[\pi_b - (u_b(z) - p_b(z))]$	$=$	0
Profit max.	$x_s(Z)[\pi_s - P(Z)]$	$=$	0

If LP^2 has an integer-valued optimal solution then solutions to DLP^2 are pricing equilibria.

The set of efficient assignments are (integer) optimal solutions to LP^2.

1. All extreme points of feasible region of LP^2 are integer valued.

$[\, Z^* = (z_b^*), \langle p_b^*(\cdot)\rangle \,]$ *is a pricing equilibrium if and only if* Z^* *is an optimal solution to LP^2 and* $\langle p_b^*(\cdot)\rangle$ *is an optimal solution to DLP^2.*

2. Pricing equilibrium exists.
3. Pricing equilibrium outcomes are efficient.

However, pricing equilibria may be manipulable. (That is, an auction with payments determined by a pricing equilibrium may not be incentive compatible.)

Existence of non-manipulable prices

Buyers are *substitutes* if for any subset of buyers T

$$\mathrm{MP}_T \geq \sum_{b \in T} \mathrm{MP}_b$$

Existence of non-manipulable prices

MP_b is an upper-bound on pricing equilibrium payoffs of buyer b

Define $\langle \underline{p}_b(\cdot)\rangle$ as a *marginal product (mp) pricing equilibrium* if $\pi^*(\underline{p}_b) = \mathrm{MP}_b$, $\forall b$.

Equivalently, an mp pricing equilibrium satisfies $\underline{p}_b(z_b^*) =$ social opportunity cost of z_b^*, $\forall b$.

> *An mp pricing equilibrium exists if and only if buyers are substitutes.*
>
> $\langle \underline{p}_b(\cdot)\rangle$ *is an mp pricing equilibrium for* $\mathcal{E}$ *if and only if for all* b, $\langle \underline{p}_b(\cdot)\rangle_{b'\neq b}$ *is a pricing equilibrium for* $\mathcal{E}_{-b}$.

An mp pricing equilibrium (and only an mp pricing equilibrium) is non-manipulable.

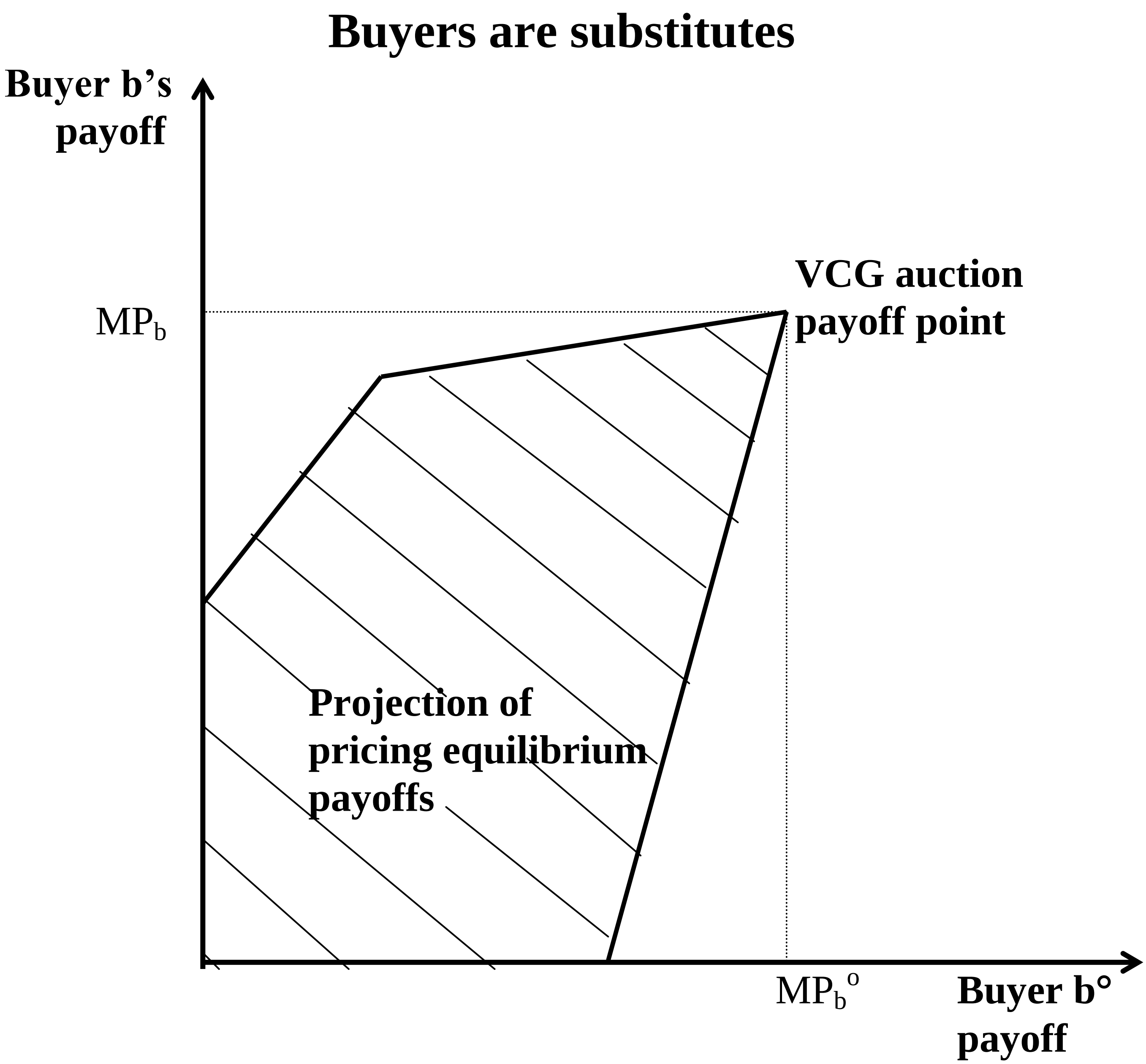

Figure 1a: Projection of pricing equilibrium payoffs into the space of payoffs of any two buyers, when (all) buyers are substitutes

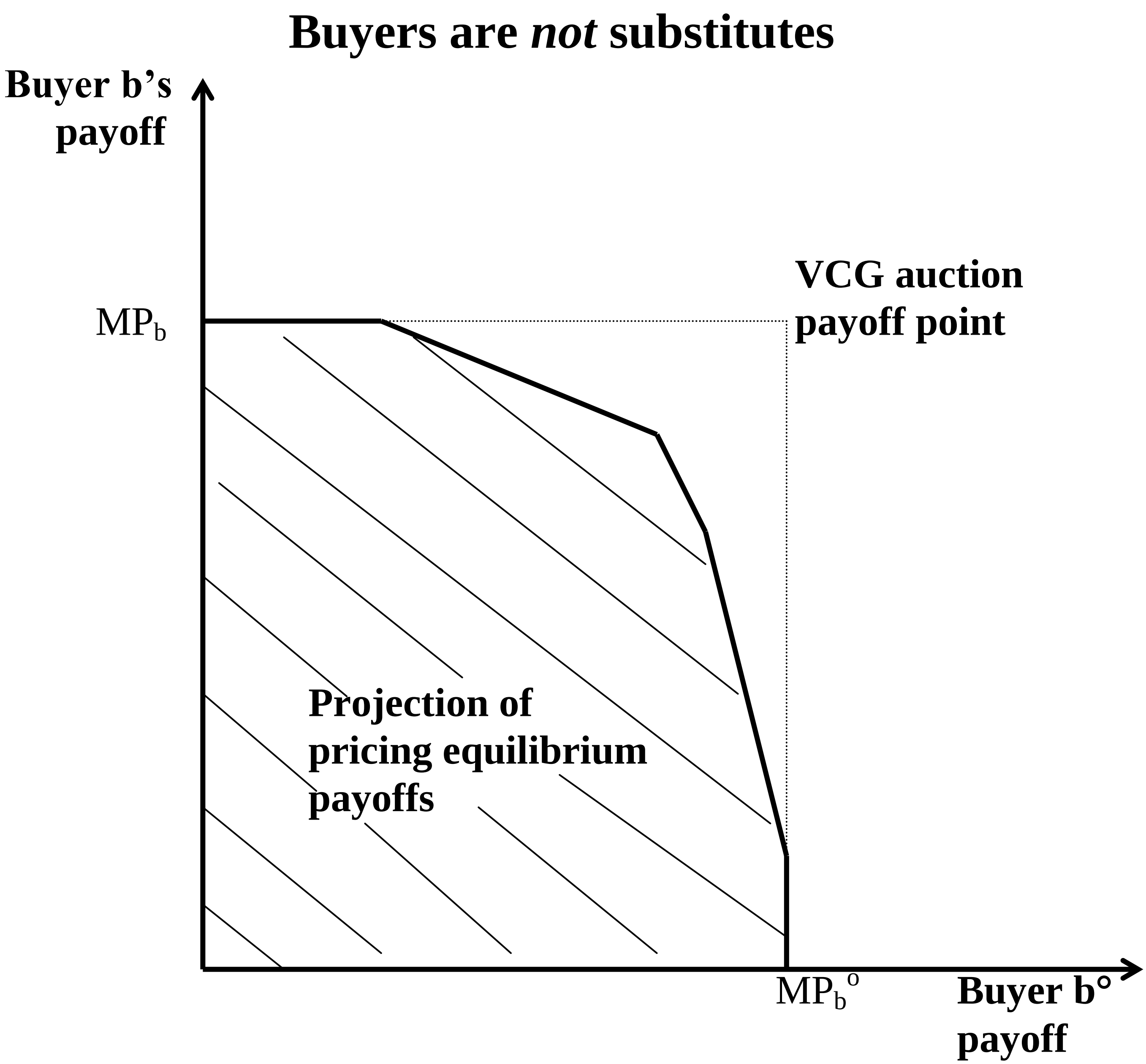

Figure 1b: Projection of pricing equilibrium payoffs into the space of payoffs of two buyers, when buyers are not substitutes

Gross substitutes condition on utilities.

The demand sets at two linear price vectors $\underline{P} = (\underline{p}_1, \underline{p}_2, ..., \underline{p}_K)$, $\overline{P} = (\overline{p}_1, \overline{p}_2, ..., \overline{p}_K)$:

$$\begin{aligned} D_b(\underline{P}) &\equiv \arg\max_z \{u_b(z) - \underline{P} \cdot z\} \\ D_b(\overline{P}) &\equiv \arg\max_z \{u_b(z) - \overline{P} \cdot z\} \end{aligned}$$

Let

$$S(\underline{P}, \overline{P}) \equiv \{k \in K | \, \underline{p}_k = \overline{p}_k \}$$

For any z, define z^S

$$z_k^S \equiv \begin{cases} z_k & \text{if } k \in S \\ 0, & \text{otherwise.} \end{cases}$$

The utility function $u_b(\cdot)$ satisfies *gross substitutes* if $\underline{P} \leq \overline{P}$ then for any $\underline{z} \in D_b(\underline{P})$ there exists $\overline{z} \in D_b(\overline{P})$ such that $\underline{z}^S \leq \overline{z}$.

If buyers' utilities are gross substitutes then buyers are substitutes.

Gross substitute functions satisfy discrete concavity (Murota 2003)

Ascending-price auctions as primal-dual algorithms

If buyers are substitutes, then ascending-price auctions that implement the sealed-bid VCG auction outcome exist.

- Formulate an "appropriate" LP which yields efficient assignments.
- One solution to dual of LP is a non-manipulable price vector.
- Run a primal-dual algorithm which finds the non-manipulable price vector.

In order to get an ex post incentive compatible auction, primal-dual algorithm must end at an mp pricing equilibrium.

Ascending-price VCG auctions

- Homogeneous objects
  Decreasing marginal utility (Ausubel 1997)

- Heterogeneous objects
  Gross substitutes (de Vries, Schummer, and Vohra 2007)

In all of the above settings

◇ The buyers are substitutes condition is satisfied

◇ The nice properties of auctions under unit demand assumption extend

Ascending-price but not VCG auctions

- Heterogeneous objects
  Gross substitutes: (Kelso and Crawford 1982), (Gul and Stacchetti 2000)
  Without gross substitutes assumption: (Parkes and Ungar 2002), Ausubel and Milgrom 2002)

These auctions are not ex post incentive compatible.

If bidders bid truthfully then these auctions yield an efficient assignment. But no reason to expect truthful bidding.

Efficient, ascending-price auctions in private value models with quasilinear utility

MODEL/PAPER	BUYERS ARE SUBSTITUTES?	IMPLEMENT MP PRICING EQUILIBRIUM?	INCENTIVE COMPATIBLE?	ASCENDING-PRICE VICKREY AUCTION?
Assignment Model (CK, DGS)	Yes	Yes (smallest Walrasian equilibrium)	Ex post incentive compatible (EPIC)	Yes
Homogenous objects, diminishing marginal utility (A1)	Yes	Yes	EPIC	Yes
Heterogeneous objects, gross substitute preferences (KC, GS, M)	Yes	No (Walrasian but not MP pricing eq.)	No	No
Heterogeneous objects, gross substitute preferences (A2, AM, dVSV)	Yes	Yes	EPIC	Yes
Minimum Spanning Tree (BdVSV)	Yes	Yes	EPIC	Yes
Heterogeneous objects, no restriction on preferences (PU, AM, dVSV)	No	No (MP pricing eq. need not exist)	No	No

Abbreviations:

A1: Ausubel (1997)
AM: Ausubel and Milgrom (2002)
CK: Crawford and Knoer (1981)
DVSV: de Vries, Schummer, and Vohra (2003)
GS: Gul and Stacchetti (2000)
PU: Parkes & Ungar (2000)
A2: Ausubel (2000)
BdVSV Bikhchandani, de Vries, Schummer, and Vohra (2002)
DGS: Demange, Gale, and Sotomayor (1986)
KC: Kelso and Crawford (1982)
M: Milgrom (2001)

Summary

When buyers are substitutes, an ascending-price VCG auction is:

1. A dynamic mechanism for discovering an mp pricing equilibrium.

   A. An mp pricing equilibrium exists.

   B. An mp pricing equilibrium is the only pricing equilibrium at which Demand = Supply after any single buyer is removed from the economy.

2. Gives each bidder his marginal product. That is, it implements the VCG auction.

3. A primal-dual algorithm on an appropriate LP formulation of the underlying exchange economy.

Limitations

Very little is known about combinatorial auctions when

- Budget constraints
- Information externalities
- Consumption externalities