Transport via Exit under Non-Gaussian Fluctuations

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Outline: Exit time estimate

Joint work with Dr. *Zhihui Yang*, Western Illinois University

Brownian motion: a Gaussian process

- Independent increments: $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ independent
- Stationary increments with $B_t - B_s \sim N(0, t - s)$
- Continuous sample paths, but nowhere differentiable

Reference:
I. Karatzas and S. E. Shreve,
Brownian Motion and Stochastic Calculus

A sample path for Brownian motion $B_t(\omega)$

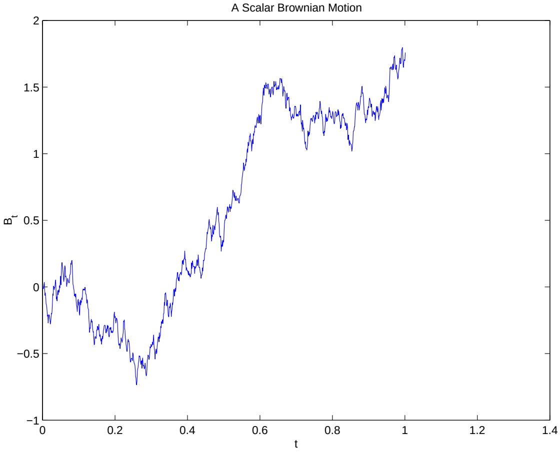


Figure: Continuous path, but nowhere differentable

A math model for Gaussian white noise $\frac{d}{dt}B_t(\omega)$

Generalized time derivative of Brownian motion

Reason: Fourier transform of correlation of $\frac{d}{dt}B_t(\omega)$ is constant

Spectrum is constant: White light (Therefore: *white noise*)

Lévy motion $L_t(\omega)$: a non-Gaussian process

- Independent increments: $L_{t_2} - L_{t_1}$ and $L_{t_3} - L_{t_2}$ independent
- Stationary increments $L_t - L_s \sim$ non-Gaussian distribution (but depends only on $t - s$)
- Stochastically continuous sample paths (continuous in probability):

  $$\mathbb{P}(|L_t - L_s| > \delta) \to 0$$

  when $t \to s$, for all $\delta > 0$.
  Note: There exist a modification of L_t whose paths are continuous from the right and have the left limits at every time.

Reference:
D. Applebaum — **Lévy Processes and Stochastic Calculus**

A sample path for Lévy motion $L_t(\omega)$

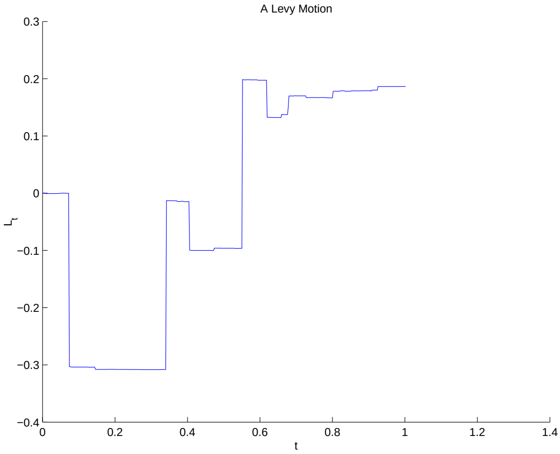


Figure: Jumps!

A math model for non-Gaussian white noise $\frac{d}{dt}L_t(\omega)$

Generalized time derivative of Lévy motion

Reason: Fourier transform of correlation of $\frac{d}{dt}L_t(\omega)$ is constant

Spectrum is constant: White light (Therefore: *white noise*)

Why Lévy motion $L_t(\omega)$: Jumps or flights

- Abrupt change in geophysical processes
- Extreme events (in weather and climate, etc)
- For example:

  1. Abrupt climate change such as Dansgaard-Oeschger events.

  Ditlevsen 1999: Ice record

  2. Diffusion of scalars in some geophysical flows: *Pauses & jumps/flights*

  Shlesinger et al.: *Lévy Flights and Related Topics in Physics* 1995

Random vs. deterministic variables

- Random variable $x(\omega)$: dice
  Ω is sample space (space of multiple chances)
  $\Omega = 1, 2, 3, 4, 5, 6$
- Deterministic variable x: ball
  Ω has a single element (single chance)
  $\Omega = 1$

Random vs. deterministic orbits

- Random orbit: $x(\omega, t)$
  Ω is sample space (space of chances)
  Multiple realizations!
- Deterministic orbit: $x(t)$
  $\Omega = 1$
  Single realization!

Comparison

- Deterministics: Play a ball (*only one sample*)
- Stochastics: Play a dice (*multiple sample*)
- Random dynamical systems: Play Brownian motion, or Lévy motion, or other noises

Deterministic Dynamical Systems

$$x' = f(x) \quad , x(0) = x_0$$

Solution map: $\varphi(t, x_0)$

“Flow" property: $\varphi(t+s, x_0) = \varphi(t, \varphi(s, x_0))$

What is a random dynamical system?

- Driving system (noise)
- Stochastic flow property (*“Cocycle”*)

Definition of Random Dynamical Systems

Driven flow & Cocycle property

- **Model for noise: Driven flow**

$(\Omega, \mathcal{F}, \mathbf{P})$ — a probability space.
$\{\theta_t\}_{t\in\mathbb{R}}$ be a measurable flow on Ω:

$$\theta : \mathbb{R} \times \Omega \to \Omega$$

such that
(1) θ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable;
(2) $\theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t_1} \circ \theta_{t_2} = \theta_{t_1+t_2}, t_1, t_2 \in \mathbb{R}.$

Driven flow: **P**-preserving measurable flow.

- **Model for evolution:** **Cocycle property (“*stochastic flow*")**

A measurable random dynamical system on the measurable space $(H, \mathcal{B})$ over a driven flow $(\Omega, \mathcal{F}, \mathbf{P}, \theta_t)$ is a mapping

$$\varphi : \mathbb{R}^+ \times \Omega \times H \to H$$

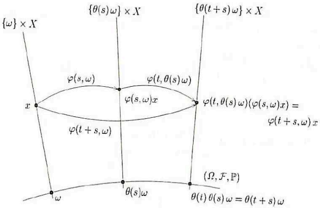
with **Measurability**: φ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F} \otimes \mathcal{B}(\mathcal{H}), \mathcal{F})$-measurable, and

Cocycle property:

$$\varphi(0, \omega, x) = x \in H,$$
$$\varphi(t+s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x)) \quad (1)$$

for t, $s \in \mathbb{R}^+$, $\omega \in \Omega$, and $x \in H$.

Visualizing the cocycle property



Stochastic Differential Equations with Brownian motion

$$dX_t = f(X_t)dt + g(X_t)dB_t, \quad X_0 = x$$

Drift: $f(X_t)$
Diffusion: $g(X_t)$
Solution map: $\varphi(t, \omega, x)$

Does this generate a random dynamical system?

Reference:
B. Oksendal — **Stochastic Differential Equations**

SDEs with Brownian motion

- **Model for noise:** Wiener shift θ_t

Canonical sample space $\Omega := C_0(\mathbb{R}, \mathbb{R}^n)$ (set of continuous functions)

Identifying sample paths: $\omega(t) \equiv B_t(\omega)$

$$\theta_t : \Omega \to \Omega$$

$$\theta_t \omega(s) := \omega(t+s) - \omega(t)$$

- **Model for evolution:** solution map $\varphi(t, \omega, x)$ has cocycle property

SDEs with Brownian Motion: Generate a random dynamical system

Under very general smoothness conditions on the coefficients, a SDE driven by Brownian motion generates a *random dynamical system*.

Reference:
L. Arnold — **Random Dynamical Systems**

SDEs with Lévy motion

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x$$

Drift: $f(X_t)$
Diffusion: $g(X_t)$
Solution map: $\varphi(t, \omega, x)$

Does this generate a random dynamical system?
Answer is incomplete, but very likely
Random dynamical systems driven by non-Gaussian fluctuations

Reference: Kunita 2004

Two approaches for SDEs

Dynamical systems approach:

Invariant sets – Stationary states, periodic orbits, recurrent states, stable manifolds, unstable manifolds, ...

Sample path approach:

Estimating distribution of the solution orbits – Mean, variance, correlation, large deviation, small probability events, ...

Transport via Exit

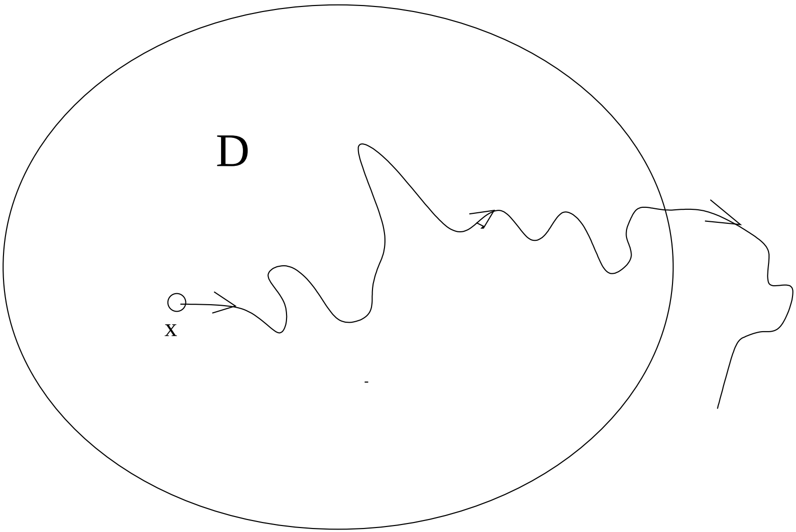


Figure:

Exit time

$$dX_t = f(X_t)dt + g(X_t)dB_t, \quad X_0 \text{ given} \tag{2}$$

where f is an $n-$dimensional vector function, g is an $n \times m$ matrix function, and $B_t(\omega)$ is an $m-$dimensional Brownian motion.

The generator is **local:**

$$Av = (\nabla v)^T f(x) + \frac{1}{2} Tr[g(x)g^T(x)D^2(v)], \tag{3}$$

where D^2 is the Hessain differential matrix and Tr denotes the trace.

Exit time from a domain D: $\sigma_x(\omega) := \inf\{t : \ X_t \in \partial D\}$

How to compute mean exit time?

Mean exit time $u(x) := \mathbb{E}\sigma_x(\omega)$:

$$\begin{aligned} Au &= -1, \\ u|_{\partial D} &= 0. \end{aligned}$$

Refs: *Naeh, Klosek, Matkowsky and Schuss 1990*

Transport via Exit

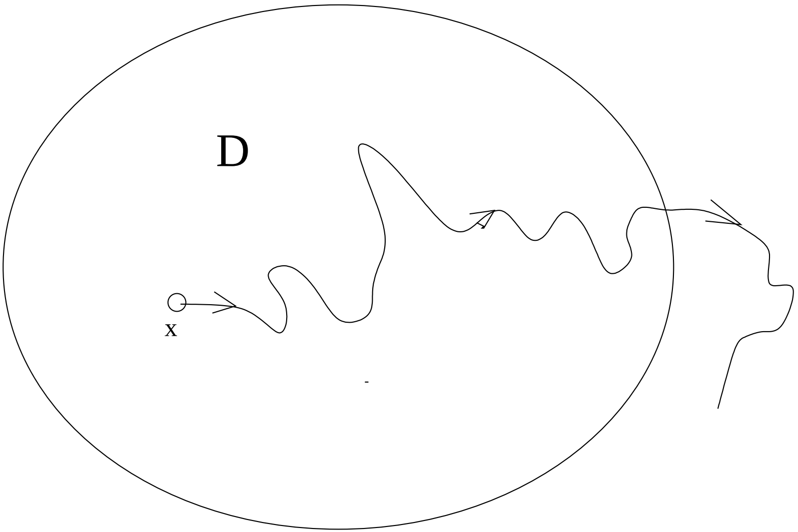


Figure:

How to compute exit probability?

For a solution orbit started at $x \in D$, which part of the boundary does it escape through?
$p(x)$: Exit probability from D through a part of the boundary $\Gamma \subset \partial D$

$$\begin{aligned} Ap &= 0, \\ p|_{\Gamma} &= 1, \\ p|_{\partial D - \Gamma} &= 0. \end{aligned}$$

Refs: *Schuss; Brannan-Duan-Ervin; Evans*

Transport via Exit

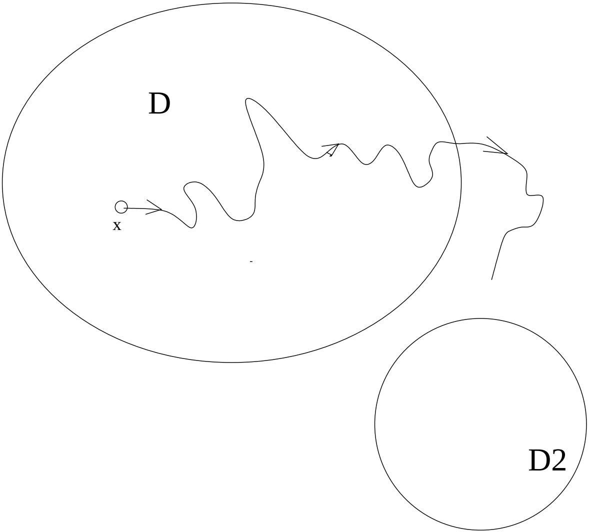


Figure:

2D Example

$$\begin{aligned} \dot{x} &= y + \varepsilon \dot{B}_t^1 \\ \dot{y} &= -x - y^3 + \varepsilon \dot{B}_t^2 \end{aligned}$$

Phase portrait for unperturbed system: $\dot{x} = y, \quad \dot{y} = -x - y^3$

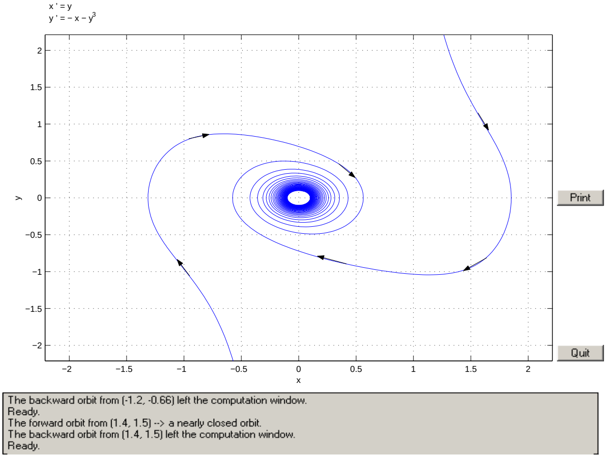


Figure:

Mean exit time from a domain: $\varepsilon = 0.6$

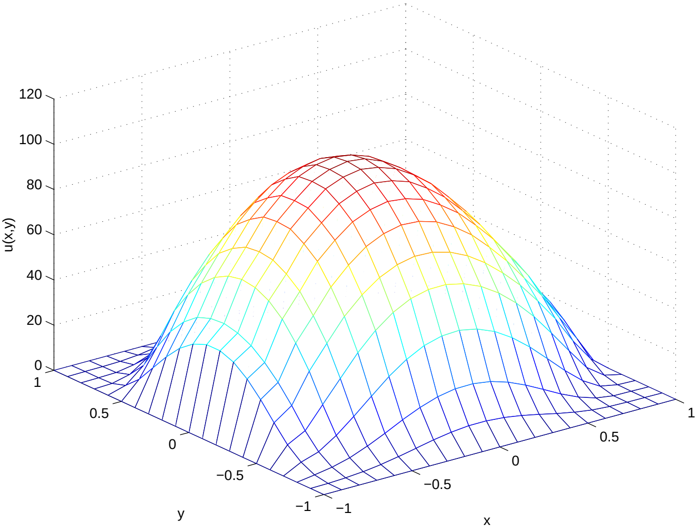


Figure: $u(x, y)$: Mean exit time for orbit staring at (x, y)

Exit probability through a part of boundary: $\varepsilon = 0.6$

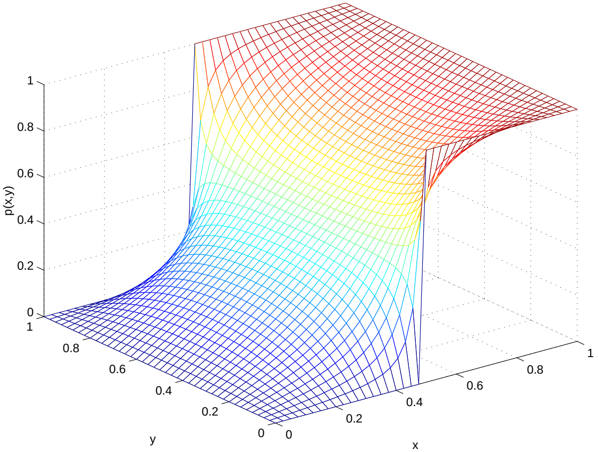


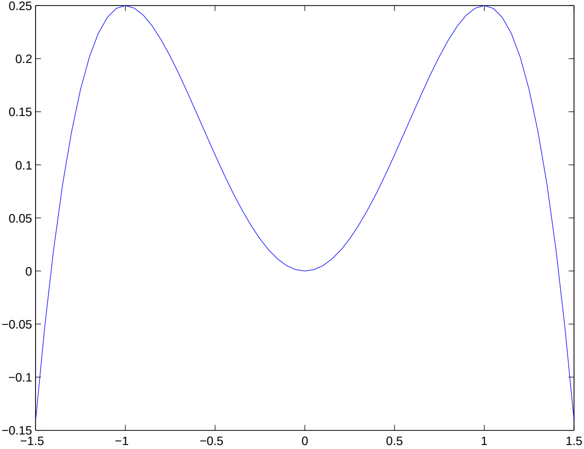
Figure: $p(x, y)$: Likelihood for orbit staring at (x, y) to escape through $\Gamma \subset \partial D$

1D Example: Exit time vs. noise intensity

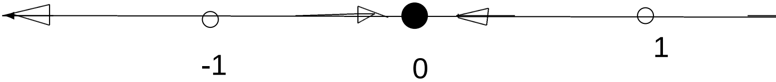
$$\dot{x} = -x + x^3 + \varepsilon \dot{B}_t$$

Potential function $U(x) = \frac{1}{2}x^2 - \frac{1}{4}x^4$:
$-x + x^3 = -U'(x)$

Potential function $U(x)$ for $\dot{x} = -x + x^3 = -U'(x)$



Phase portrait for unperturbed system $\dot{x} = -x + x^3$



Mean exit time from an interval: $\varepsilon = 0.5$

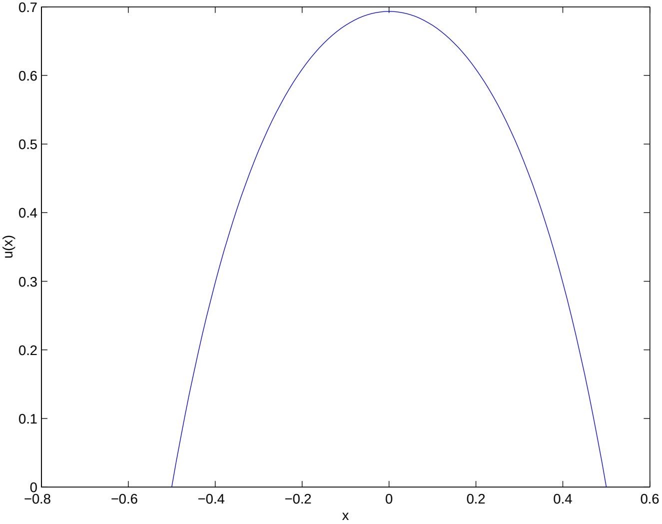


Figure:

Mean exit time from an interval: $\varepsilon = 0.3$

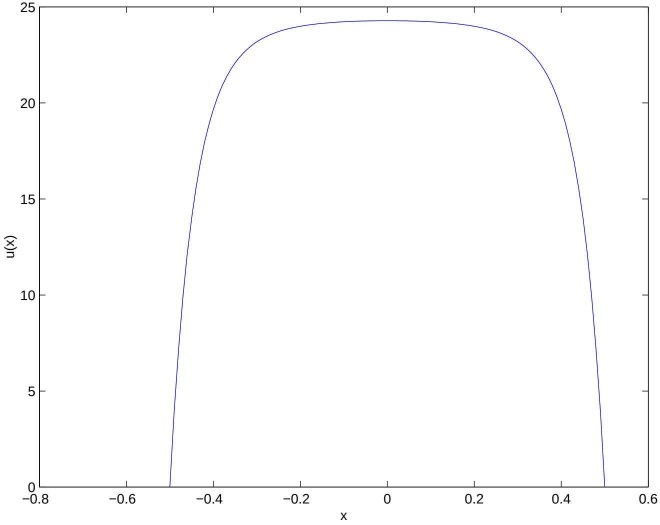


Figure:

Mean exit time from an interval: $\varepsilon = 0.08$

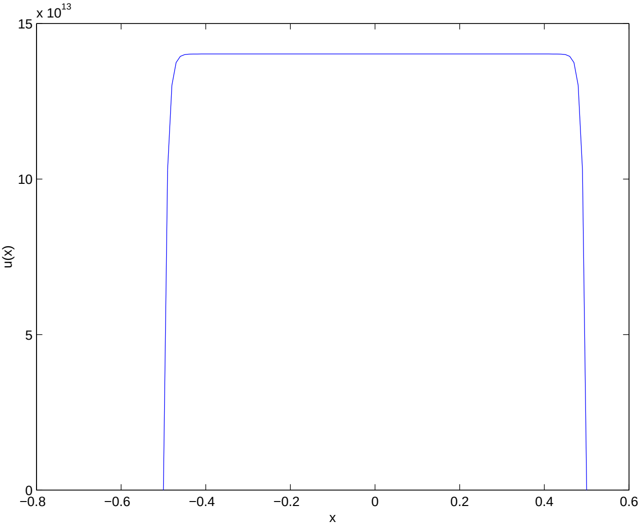


Figure:

Theorem: Freidlin-Wentzell

Theorem

$$\dot{x} = -U'(x) + \varepsilon \dot{B}_t,$$

Mean exit time from a domain of an unperturbed stable equilibrium:

$$E_x \sigma(\varepsilon) \sim O(e^{\frac{1}{\varepsilon^2}})$$

1D SDE with non-Gaussian Lévy noise

$$dX_t^{\varepsilon} = -U'(X_t^{\varepsilon})dt + \varepsilon dL_t, \quad X_0 = x$$

Potential function $U(\cdot)$ has a minimum at 0
Unperturbed system has a stable equilibrium at 0
Lévy process $L_t(\omega)$: Stationary and independent increments, with jumps. The jumps are described by Lévy jump measure $\nu(du)$ on $\mathbb{R}$:

$$\int_{\mathbb{R}\setminus\{0\}} (u^2 \wedge 1)\ \nu(du) < \infty,$$

or equivalently,

$$\int_{\mathbb{R}\setminus\{0\}} \frac{u^2}{1+u^2}\ \nu(du) < \infty.$$

Lévy motion $L_t(\omega)$: a non-Gaussian process

- Independent increments: $L_{t_2} - L_{t_1}$ and $L_{t_3} - L_{t_2}$ independent
- Stationary increments $L_t - L_s \sim$ non-Gaussian distribution (but depends only on $t - s$)
- Sample paths are right continuous with left limits at every time ("**Càdlàg**").

Lévy-Khintchine formula:

It is known that any Lévy process is completely determined by the Lévy-Khintchine formula. This says that for any one-dimensional Lévy process L_t, there exists a $a \in R$, $d > 0$ and a measure ν such that

$$Ee^{i\lambda L_t} = \exp\{ia\lambda t - dt\frac{\lambda^2}{2} + t\int_{\mathbb{R}\setminus\{0\}}(e^{i\lambda u} - 1 - i\lambda u\, I\{|u| < 1\})\nu(du)\},$$

where $I(S)$ is the indicator function of the set S, i.e., it takes value 1 on this set and takes zero value otherwise.

Intuition for Lévy jump measure $\nu(u)$

$\nu(A)$: Number of jumps of “size" A, for $A \subset \mathbb{R}$
$\nu(\mathbb{R} - \{0\})$: Intensity of jumps (how often it jumps)

For example:
$\nu(\mathbb{R} - \{0\}) = \infty \Rightarrow$ countable jumps on any finite time interval
$\nu(\mathbb{R} - \{0\}) < \infty \Rightarrow$ countable jumps on the whole time axis

α-stable symmetric Lévy Noise

Lévy jump measure: $\nu(du) = \frac{1}{|u|^{1+\alpha}}(du)$, with $0 < \alpha < 2$
$\alpha = 2$: Brownian motion B_t

Heavy tail for $0 < \alpha < 2$:

$$\mathbb{P}(|L_t| > u) \sim \frac{1}{u^\alpha}$$

Light tail for $\alpha = 2$:

$$\mathbb{P}(|B_t| > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi}u}$$

α-stable symmetric Lévy Noise

Lévy jump measure: $\nu(du) = \frac{1}{|u|^{1+\alpha}}(du)$, with $0 < \alpha \leq 2$
The generator is nonlocal:
$A = -(-\Delta)^{\frac{\alpha}{2}}$
$\alpha = 2$: Brownian motion B_t, generator $A = \Delta$

Fractional Fokker-Planck equation: Schertzer et al 2001

Question: Exit time in terms of the nonlocal generator?

Theorem: Exit time under α-stable Lévy Noise

Theorem

$$\dot{x} = -U'(x) + \varepsilon \dot{L}_t,$$

Mean exit time from a domain containing an unperturbed stable equilibrium:

$$E_x \sigma(\varepsilon) \sim \frac{1}{\varepsilon^\alpha}$$

Imkeller and Pavlyukevich 2006

Another family of symmetric Lévy Noise

Lévy jump measure: $\nu(du) = f(\ln|u|)\,\frac{du}{|u|^{1+\alpha}}$, with $0 < \alpha < 2$ and $f > 0$ measurable

Theorem: Mean exit time with a family of symmetric Lévy noises

Lemma

$$\dot{x} = -U'(x) + \varepsilon \dot{L}_t,$$

with Lévy jump measure $\nu(du) = f(\ln|u|)\,\frac{du}{|u|^{1+\alpha}}$.
Main assumption:

$$\frac{f(\ln|u/\varepsilon|)}{\tilde{f}(\varepsilon)}\,\frac{du}{|u|^{1+\alpha}} \rightharpoonup \nu^*(du).$$

Mean exit time from $[-b, a]$ *containing an unperturbed stable equilibrium:*

$$E_x\,\sigma(\varepsilon) \sim \frac{1}{\nu^*(\mathbb{R}\setminus[-b,a])}\,\frac{1}{\tilde{f}(\varepsilon)}.$$

Ideas of Proof:

1. Estimating $\int_{\mathbb{R}\setminus[-K,K]} \nu(d(\frac{u}{\varepsilon}))$
2. Weak convergence of measures $\frac{f(\ln|u/\varepsilon|)}{\tilde{f}(\varepsilon)} \frac{du}{|u|^{1+\alpha}}$
3. Applying a result of Godovanchuk 1981

A specific example of symmetric Lévy Noise

Lévy jump measure: $\nu(du) = \frac{1}{|\ln|u||+1} \frac{du}{|u|^{1+\alpha}}$, with $0 < \alpha < 2$

Theorem: Mean exit time with a specific Lévy noise

Theorem

$$\dot{x} = -U'(x) + \varepsilon \dot{L}_t,$$

with Lévy jump measure $\nu(du) = \frac{1}{|\ln|u||+1} \frac{du}{|u|^{1+\alpha}}$.
Mean exit time from a domain containing an unperturbed stable equilibrium:

$$E_x\sigma(\varepsilon) \sim O(\frac{|\ln \varepsilon|}{\varepsilon^{\alpha}})$$

Yang and Duan: 2008

Mean exit time comparison

Mean exit times are in the order:

$$O(\frac{1}{\varepsilon^{\alpha}}) < O(\frac{|\ln \varepsilon|}{\varepsilon^{\alpha}}) < \exp(\frac{C}{\varepsilon^{2}}).$$

or symbolically:

Polynomial $\prec$ Combined polyn. and natural logarithm $\prec$ Exponential

Brownian noise $\prec$ Our noise $\prec$ α-stable symmetric Lévy Noise

Conclusions

- Non-Gaussian Lévy noise
- Systems driven by non-Gaussian Lévy noise
- Mean exit time estimate: exponential, polynomial, combined polynomial & natural logarithm