Stochastic Representations and Navier-Stokes equations

IPAM Workshop: Transport Systems

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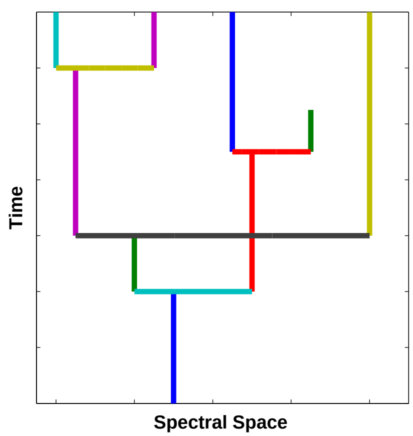
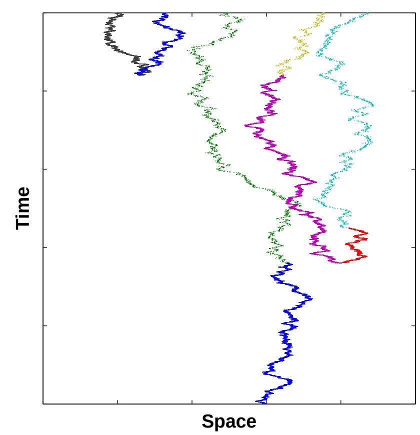
GOAL: Illustrate connection between

deterministic PDE's

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g \qquad \nabla \cdot u = 0 \qquad u_0(x) = u(x, 0)$$

and stochastic processes





Portray solutions of some partial differential equations as expectations of stochastic processes with the aim of garnering insight into properties of solutions. Particular example examined: the Navier-Stokes equations in 3-d. Descriptions will be given of two types of representations. First: Fourier-transformed solutions as given by LeJan and Sznitman (1997). Second: physical space representations. Both give existence and uniqueness of solutions for all time for 'small' initial data and on short time intervals for 'large' initial data.

OUTLINE

Background and illustration

- **two simple examples**

Navier-Stokes: an extra twist

- **Fourier space**

- **Physical space**

Comments

ANTECEDENTS:

Skorohod (1964)

Ikeda, Watanabe, Dawson and others, 1968 and on

Savits (1969)

McKean (1975)

LeJan and Sznitman (1997)

Bhattacharya et al (2003)

OTHER DIRECTIONS:

Constantin (2001), Constantin and Iyer (2008), Iyer (2006)

Albeverio and Belopolskaya (2003), (2005), (2007)

Busnello, Flandoli, Romito (2005)

Fontana (2006)

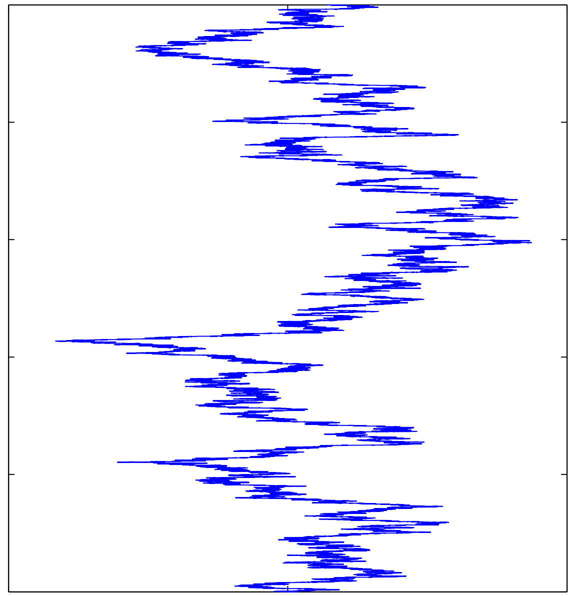
Birnir (2007)

Brzezniak and Neklyudov (2008)

EXAMPLE : The heat equation and Brownian motion

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$$

$$u(x,0) = f(x)$$



Brownian motion W

$$u(x,t) = \int f(x-y)K(y,t)dy \qquad K(y,t) = (2\pi t)^{-d/2}\exp^{-|y|^2/2t}$$

$$u(x,t) = E_x f(W(t))$$

Fourier transform:

$$\widehat{u}(\xi,t) = (2\pi)^{-d/2} \int_{R^d} e^{-i\xi\cdot x} u(x,t)dx, \quad \xi \in \mathbf{R}^d$$

$$\frac{\partial \widehat{u}}{\partial t} = -\frac{|\xi|^2}{2}\widehat{u}$$

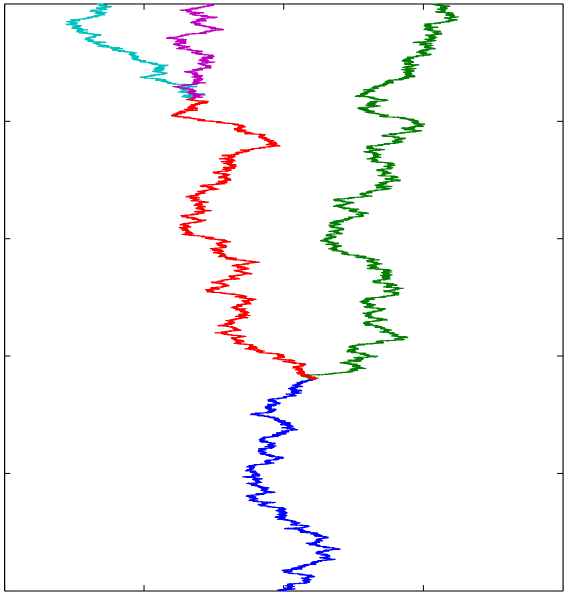
$$\widehat{u}(\xi,t) = \widehat{f}(\xi)e^{-|\xi|^2 t/2} = \widehat{f}(\xi)P(S_\xi > t)$$

$e^{-|\xi|^2 t/2}$ is the Fourier transform of $K(y,t)$

$$u(x,t) = \int f(x-y)K(y,t)dy = E_x f(W(t))$$

KPP reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u \quad \text{with} \quad u(x,0) = f(x)$$



McKean(1975):If $|f(x)| \leq 1$

$$u(x,t) = E_x \prod_{i \leq N(t)} f(W^{(i)}(t))$$

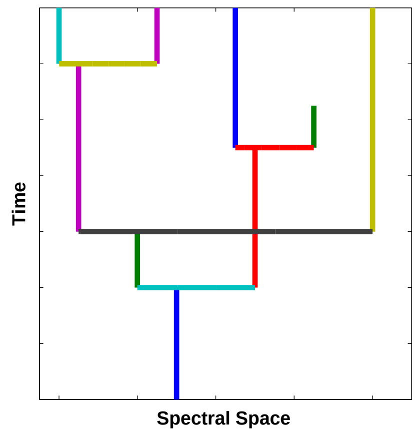
Markov process W: branching Brownian motion

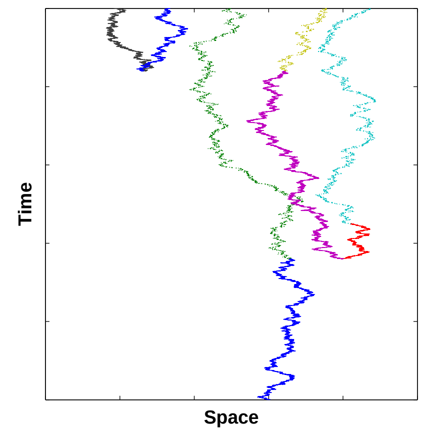
Omnibus idea:

- **Mild formulation**
- **Recognize transition probabilities**

USE IDEA TO: Connect

random processes:





solutions to the Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g \qquad \nabla \cdot u = 0 \qquad u_0(x) = u(x, 0)$$

Navier-Stokes equations in 3-d

- $\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p + g$, **pressure** p **and forcing** g

- **incompressible fluids:** $\nabla \cdot u = 0$

- **initial data:** $u_0(x) = u(x, 0)$

- **mild solutions:** $u = e^{\nu t \Delta} u_0 - \int_0^t e^{\nu (t-s)\Delta} \mathcal{P} \nabla \cdot (u \otimes u)(s) ds$

$$+ \int_0^t e^{\nu(t-s)\Delta} \mathcal{P} g(s) ds$$

$\mathcal{P}$: **Leray projection onto divergence free vector fields**

LeJan and Sznitman (1997): Fourier NS

$$\begin{aligned}\widehat{u}(\xi,t) &= e^{-\nu|\xi|^2 t}\widehat{u}_0(\xi) + \int_{s=0}^{t} (2\pi)^{-3/2} e^{-\nu|\xi|^2 s} \\ &\qquad \Big\{ -i\mathbf{P}_\xi \int \widehat{u}(\eta, t-s)\, \xi \cdot \widehat{u}(\xi-\eta, t-s) d\eta \\ &\qquad\qquad + \mathbf{P}_\xi\, \widehat{g}(\xi, t-s) \Big\}\, ds\end{aligned}$$

Theorem: (LJ-S, 1997) Suppose that

$$\sup_{\xi} |\xi|^2 |\widehat{u}_0(\xi)| < (2\pi)^{3/2}\nu/2 \quad \text{and} \quad \sup_{\xi\neq 0, t\geq 0} 2|\widehat{g}(\xi,t)| \leq (2\pi)^{3/2}\nu/4.$$

Then there is a unique mild solution $\{u(x,t) : x \in \mathbf{R}^3, t \geq 0\}$ to the Navier-Stokes equations with

$$\sup_{\xi\neq 0, t\geq 0} |\xi|^2 |\widehat{u}(\xi,t)| \leq (2\pi)^{3/2}\nu.$$

KEY: Weight $\widehat{u}$ by $|\xi|^2$

The crucial magical realization:

$$\text{for } h(\xi) = |\xi|^{-2}, \qquad h * h(\xi) = c\,|\xi|\,h(\xi),$$

This kernel is used to give Markov spatial transition probability

$$dH_\xi(\eta) = \frac{h * h(\xi)}{|\xi| h(\xi)} = \frac{c|\xi|}{|\eta|^2 |\xi - \eta|^2}$$

P_ξ **projection matrix** $\qquad w \otimes_\xi z = -i(e_\xi \cdot z)\,\mathrm{P}_\xi w$

Rewriting the Fourier NSE in terms of

$$\chi(\xi,t) = |\xi|^2 \widehat{u}(\xi,t) \qquad \text{and} \qquad \phi(\xi,t) = \frac{2}{\nu}\widehat{g}(\xi,t)$$

$$\chi(\xi,t) = e^{-\nu t|\xi|^2}\chi_0(\xi) + \int_0^t \nu|\xi|^2 e^{-\nu|\xi|^2 s}$$

$$\Big\{\tfrac{1}{2}\, c_\nu \int_{\mathbf{R}^3} \chi(\eta, t-s) \otimes_\xi \chi(\xi-\eta, t-s)\, dH_\xi(\eta)$$

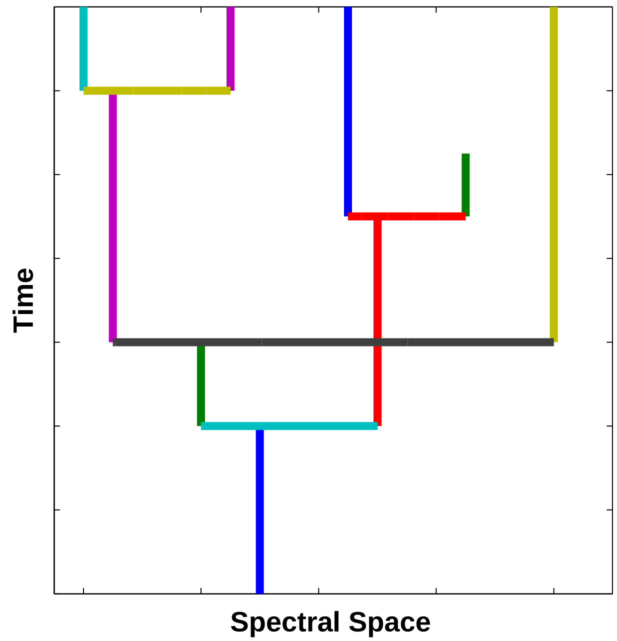
$$+\tfrac{1}{2}\phi(\xi, t-s)\Big\} ds$$

Conditional expectation:

$$
\begin{aligned}
\chi(\xi,t) = \qquad & \mathbf{E}_\xi\Big(\chi_0(\xi)\mathbf{1}[S_\xi > t] \\
& +\Big\{\sigma_\xi \, c_\nu \, \chi(\eta, t - S_\xi) \otimes_\xi \chi(\xi - \eta, t - S_\xi) \\
& +(1-\sigma_\xi)\, \phi(\xi, t - S_\xi)\Big\}\mathbf{1}[S_\xi \le t]\Big)
\end{aligned}
$$

Motivates formulation of stochastic process and iterative functional

$$
\Xi(\xi,t) = \begin{cases} \chi_0(\xi) & S > t \\ c_\nu \Xi(\eta, t - S) \otimes_\xi \Xi(\xi - \eta, t - S) & S \le t, \sigma = 1 \\ \phi(\xi, t - S_\xi) & S \le t, \sigma = 0 \end{cases}
$$



$$\Xi(\xi, t) = \begin{cases} \chi_0(\xi) \\ c_\nu \Xi \otimes_\xi \Xi \\ \phi(\xi, t - S_\xi) \end{cases}$$

Markov walk on tree

PROPOSITION: If $\mathbf{E}_\xi |\Xi(t)| < \infty$, then

$$\hat{u}(\xi, t) = c_\nu |\xi|^{-2} \mathbf{E}_\xi \Xi(t)$$

provides a mild solution to the Navier-Stokes equations.

Non-negative Fourier multiplier majorizing kernel:

$$h * h(\xi) \le c|\xi| h(\xi)$$

With $\chi = \frac{\widehat{u}}{h}$ and $\phi = \frac{|\xi|^2 \widehat{g}}{h}$, the probabilistic construction gives:

PROPOSITION: If $\mathbf{E}_\xi |\Xi(t)| < \infty$, then $\widehat{u}(\xi, t) = c_\nu h(\xi) \mathbf{E}_\xi \Xi(t)$ provides a mild solution to the Navier-Stokes equations.

THEOREM(BCDGOOTW, 2004): If both

$$\sup_\xi \frac{|\widehat{u}_0(\xi)|}{h(\xi)} \le c_\nu \quad \text{and} \quad \sup_{\xi,t} \frac{|\widehat{g}(\xi,t)|}{|\xi|^2 h(\xi)} \le c_\nu,$$

then there is a unique mild solution $\{u(x,t) : x \in \mathbf{R}^3, t \ge 0\}$ to the Navier-Stokes equations with

$$\sup_{\xi \ne 0, t \ge 0} h(\xi) |\widehat{u}(\xi, t)| \le c_\nu.$$

Fourier multiplier majorizing kernels in $\mathbf{R}^3$: $h * h(\xi) \le |\xi| h(\xi)$

Le-Jan Snitzman kernels:

$$h_0(\xi) = \frac{c}{|\xi|^2}, \qquad h_1(\xi) = \frac{ce^{-\alpha|\xi|}}{|\xi|}$$

A couple of BCDGOOTW kernels:

$$h_{\alpha,\beta}(\xi) = c|\xi|^{\beta-2} e^{-\alpha|\xi|^\beta}, \qquad \alpha > 0, \quad \beta \in [0,1]$$

Here $0 < \beta < 1$ allows smooth compactly supported initial data.

$$h(\xi) = c_\nu \int_{t>0} (\pi t)^{-3} \prod_{j=1}^{3} \left(\frac{1}{1 + (\frac{\xi_j}{t})^2}\right) dt$$

Note: the class of majorizing kernels is log-convex.

Majorizing kernel formulation:

Go to a scalar *comparison equation*

$$U(\xi,t) = e^{-\nu|\xi|^2 t}U_0(\xi) + c\int_{s=0}^{t} |\xi| e^{-\nu|\xi|^2 s} U * U(\xi, t-s)ds$$

Comparison equation: set $U(\xi,t) = |\widehat{u}(\xi,t)|$

The steady state solution:

$$U_\infty(\xi) = c|\xi|^{-1} U_\infty * U_\infty(\xi)$$

gives candidates for the majorizing kernels.

Comparison equation: Blomker, Romito, Tribe (2007)

Physical space analogue

$$
\begin{aligned}
u(x,t) &= \int_{\mathbf{R}^3} u_0(x-y)K(y,2\nu t)dy \\
&+ \int_{s=0}^{t}\int_{\mathbf{R}^3} \Big\{ \Big[\frac{|z|}{4\nu s} K(z,2\nu s)\mathbf{b}_1(z; u(x-z,t-s), u(x-z,t-s)) \\
&- \Big(\frac{1}{4\pi^2 \nu s |z|^4} \int_{\{y:|y|\le|z|\}} |y|^2 K(y,2\nu s)dy \Big) \\
&\qquad \mathbf{b}_2(z; u(x-z,t-s), u(x-z,t-s)) \Big] \\
&+ \Big[K(z,2\nu s)\mathbf{P}_z \\
&- \frac{1}{4\pi|z|^3} \int_{\{y:|y|\le|z|\}} K(y,2\nu s)dy(I - 3e_z e_z^t) \Big] g(x-z,t-s) \Big\} dzds
\end{aligned}
$$

$$K(y,t) = (2\pi t)^{-3/2} e^{-|y|^2/2t}$$

$$e_y = y/|y|, \quad \mathbf{P}_y = I - e_y e_y^t$$

$$\mathbf{b}_1(y; u, v) = (u \cdot e_y)\mathbf{P}_y v + (v \cdot e_y)\mathbf{P}_y u$$

$$\mathbf{b}_2(y; u, v) = (\mathbf{b}_1(y; u, v) + u \cdot (I - 3 e_y e_y^t) v \, e_y)/2$$

$$\mathbf{c}_1 = \mathbf{P}_z \qquad \mathbf{c}_2 = (I - 3 e_z e_z^t)/2$$

Lemma:

$$|\mathbf{b}_k(y; u, v)| \leq |u||v| \qquad \text{and} \qquad \mathbf{b}_k(y; u, v) = h^2 \, \mathbf{b}_k(y; \frac{u}{h}, \frac{v}{h})$$

Scalar comparison steady state equation:

$$U(x) = c \int_{y \in \mathbf{R}^3} |y|^{-2} U^2(x-y)\, dy$$

Motivation for majorizing kernel:

$$h(x) \geq c \int_{y \in \mathbf{R}^3} |y|^{-2} h^2(x-y)\, dy$$

Introduce majorizing kernel pairs $(h, \tilde{h})$**:**

(A) $\int_{\mathbf{R}^3} h^2(x-y)|y|^{-2}dy \leq h(x)$ h scales u, dominates $|u|$

and

$(\tilde{A})$ $\int_{\mathbf{R}^3} \tilde{h}(x-y)|y|^{-1}dy \leq h(x)$ $\tilde{h}$ scales g, dominates $|g|$

$\chi(x,t) = \dfrac{u(x,t)}{h(x)}$ **scaled solution** $\chi_0(x) = \chi(x,0)$ **initial data**

$$\varphi(x,t) = \frac{g(x,t)}{\tilde{h}(x)} \quad \textbf{scaled forcing}$$

Building blocks of probabilistic representation:
spatial and temporal transition densities

spatial: built from the majorizing kernels and based on

$$\frac{h^2(x-y)|y|^{-2}}{\int_{\mathbf{R}^3} h^2(x-y)|y|^{-2}dy} \quad \text{iterative term}$$

and

$$\frac{\tilde{h}(x-y)|y|^{-1}}{\int_{\mathbf{R}^3} \tilde{h}(x-y)|y|^{-1}dy} \quad \text{forcing term}$$

The actual spatial transition densities:

$$f(y,z|x) = \frac{|y|^{-1}|z|^{-4}h^2(x-z)\mathbf{1}[|z|>|y|]}{2\pi \int_{\mathbf{R}^3} |z|^{-2}h^2(x-z)dz} \quad \text{(branch point)}$$

$$\tilde{f}(y,z|x) = \frac{|y|^{-1}|z|^{-3}\tilde{h}(x-z)\mathbf{1}[|z|>|y|]}{2\pi \int_{\mathbf{R}^3} |z|^{-1}\tilde{h}(x-z)dz} \quad \text{(input forcing)}$$

Temporal transition densities:

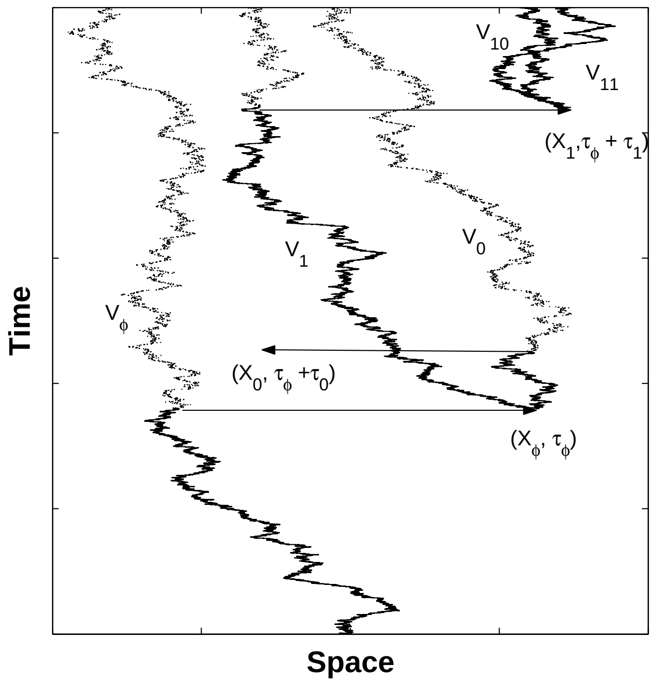
densities for waiting times between branches; conditional on spatial location

$$f_0(s|z) = c_\nu s^{-5/2}|z|^3 e^{-|z|^2/4\nu s}$$

$$f_1(s|y) = c_\nu s^{-3/2}|y| e^{-|y|^2/4\nu s}$$

If the kernel h is excessive

$$J(y,t|x) = \frac{h(y)K(x-y, 2\nu t)}{h(x)} \quad \text{h-Brownian motion}$$



- V_v**'s** h**-Brownian motion**
- X_v**'s Markov branching rw**
- τ_v**'s waiting times**
- σ_v**'s 0 or 1**

Stochastic iterative functional Υ on tree; start with leaves and work down branches:

$$\Upsilon_v(t) = \chi_0(V_v(t)) + \begin{cases} m(X_{\bar{v}})\, \mathrm{B}_v(\Upsilon_{v*0}(t-\tau_v), \Upsilon_{v*1}(t-\tau_v)) \\ \qquad \textbf{if} \quad \sigma_v = 1,\ \tau_v \le t \\ \tilde{m}(X_{\bar{v}})\, \mathrm{c}_v \varphi(X_v, t-\tau_v) \quad \textbf{if} \quad \sigma_v = 0,\ \tau_v \le t \end{cases}$$

- V_v**'s** h**-Brownian motion**
- X_v**'s Markov branching rw**
- τ_v**'s waiting times**
- σ_v**'s 0 or 1**

- B_v **randomized** b_k**'s**
- c_v **randomized:**

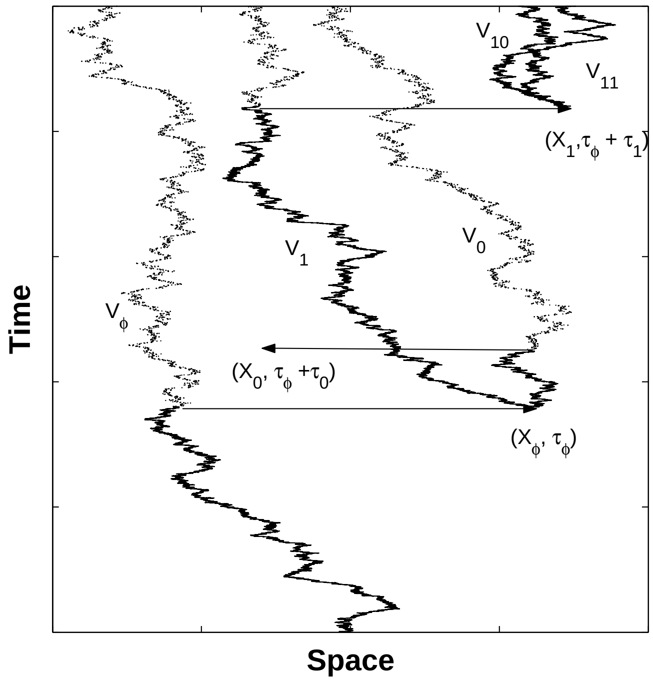
$$\mathrm{P}_{Z_v} \quad \textbf{or} \quad -(I - 3e_{Z_v}e_{Z_v}^t)/2$$

Normalization multipliers:

$$m(x) = c_\nu \, \frac{\int |y|^{-2} h^2(x-y)dy}{h(x)}$$

$$\tilde{m}(x) = c_\nu \, \frac{\int |y|^{-1} \tilde{h}(x-y)dy}{h(x)}$$

Since $(h, \tilde{h})$ is a majorizing kernel pair, the magnitude of the multipliers is controlled.



PROPOSITION: If $\mathbf{E}_x|\Upsilon(t)| < \infty$, then

$$u(x,t) = h(x)\mathbf{E}_x\Upsilon(t)$$

provides a mild solution to the Navier-Stokes equations.

Theorem: If h is excessive, $(h, \tilde{h})$ is a majorizing kernel pair, and

$$\frac{|u_0(x)|}{h(x)} \quad \text{and} \quad \frac{|g(x,t)|}{\tilde{h}(x)}$$

are both uniformly small enough, then

$$u(x,t) = h(x) E_x \Upsilon(t)$$

is an unique mild solution to the NS equations for all time and bounded in magnitude by $c_\nu h(x)$ for each x and all t.

Proof of theorem: Easy contraction argument by induction on finite binary tree* shows that $|\Upsilon|$ is uniformly bounded if the magnitudes of both χ_0 and φ are uniformly small enough. It follows that $\chi(x,t) = E_x \Upsilon_\emptyset(t)$ exists and is uniformly bounded in magnitude. Uniqueness takes a little bit more. $\square$

finite with probability 1!

Regularity (no forcing): Ladyzhenskaya-Prodi-Serrin condition; If a solution u to the NS equations satisfies

$$\int_0^T \left(\int |u(x,t)|^q dx\right)^r dt < \infty$$

for some $q > 3, r \geq r(q) > 0$, then u is regular on $[0, T]$.

Take a majorizing kernel $h(x)$ proportional to

$$\frac{1}{(1+|x|)^{4(2-\gamma)}} \quad \text{for any} \quad \gamma \in (3/2, 13/8)$$

Corollary: If $g = 0$ and for some $\gamma \in (3/2, 13/8)$

$$\sup_x (1+|x|)^{4(2-\gamma)} |u_0(x)|$$

is finite, then there exists a unique regular solution to the NS equations for $t \leq T(\gamma)$.

Proof: Modifies probability model somewhat since these h's are not excessive.

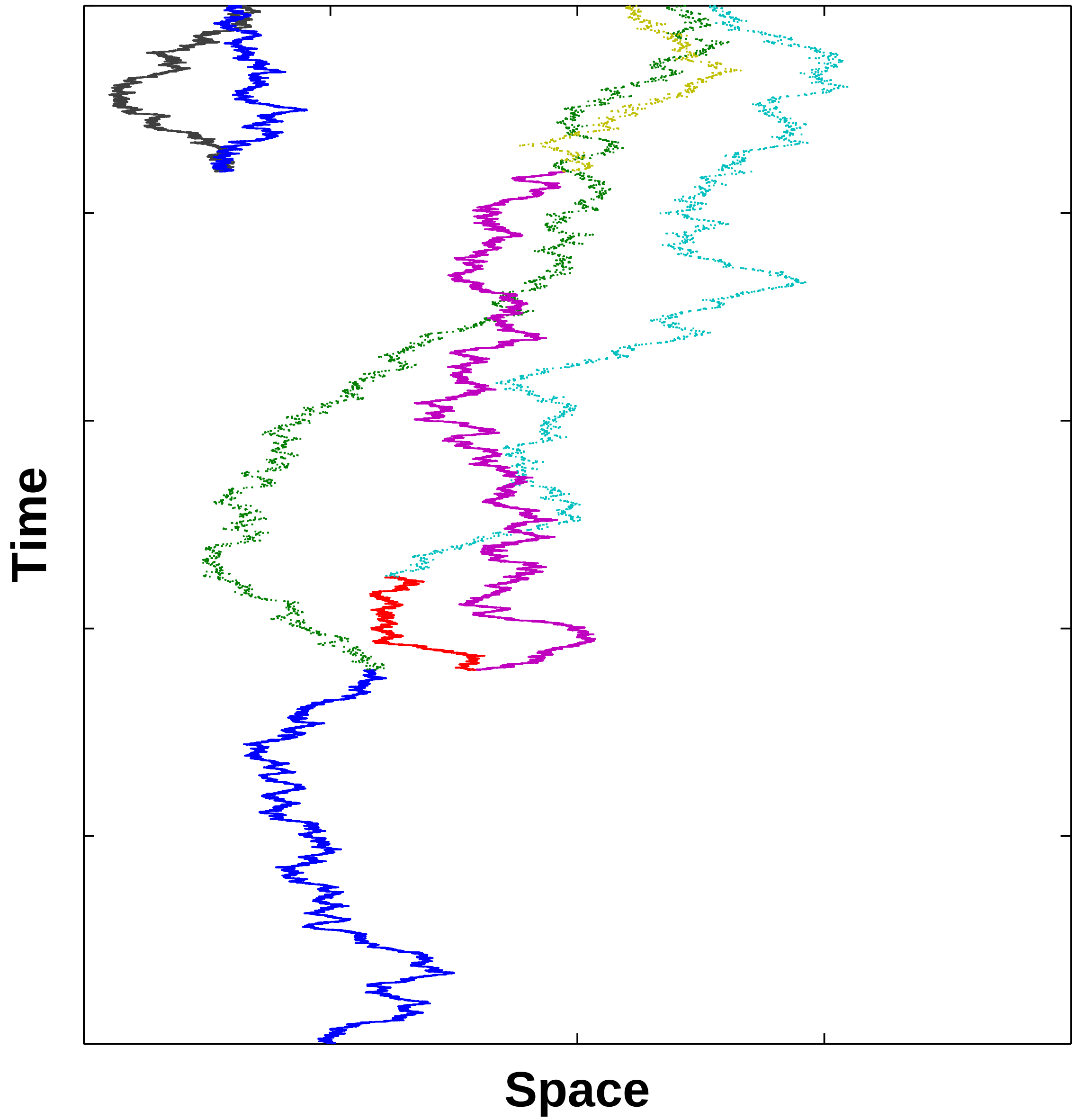
Comments:

- Short time existence for large initial data: speed up branching in time

- Branching is finite with probability 1; controls 'blow-up'

- Incompressibility is partially captured in the transition probabilities

- Family of branching diffusions connected to physical process


Time
Space