Dynamical systems perturbed by big jumps: meta–stability and non–local search*

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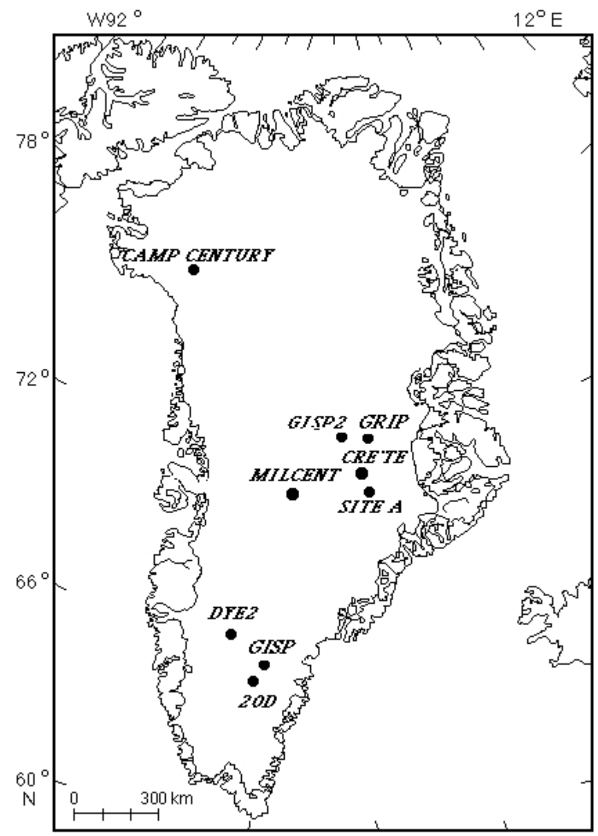
1. Motivation

Greenland ice-core data allows to reconstruct Earth's climate up to 200.000 years before present.

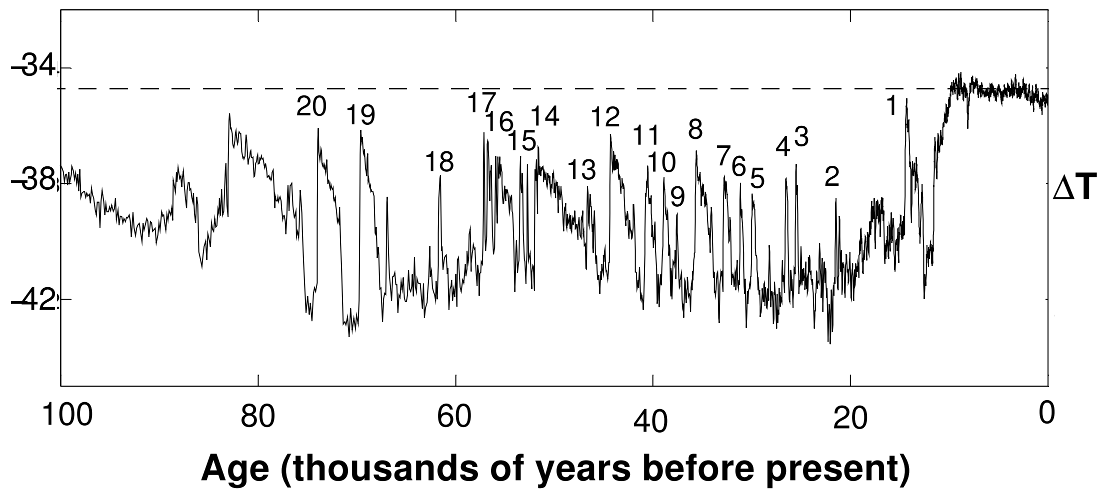
International projects:
GRIP (3028 m), GISP2 (3053.44 m), NGRIP (3084.99 m)



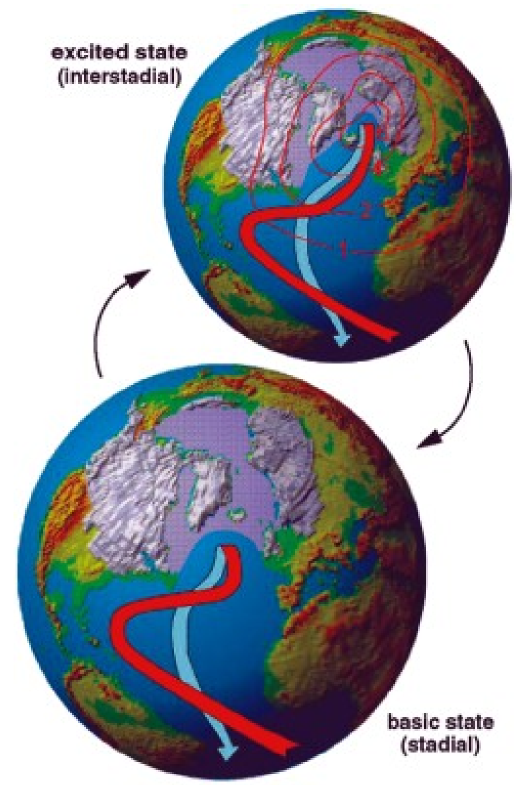
2. Paleo proxy data. Dansgaard–Oeschger events

Paleo data proxies: oxygen isotopes, dust, volcanic markers etc.

Global climate during the last glacial ($\sim$120 000 – 10 000 b. p.) has experienced at least 20 abrupt and large–amplitude shifts (Dansgaard–Oeschger events).



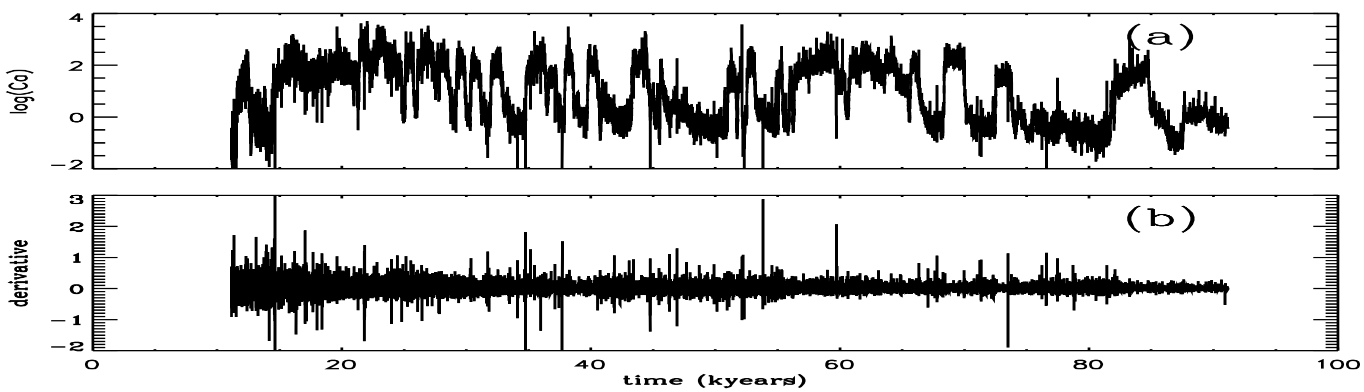
- rapid warming by 5 – 10 °C within at most a few decades
- plateau phase with slow cooling lasting several centuries
- rapid drop to cold stadial conditions



Simulations: Ganopolsky&Rahmstorf, Potsdam Institute for Climate Impact Research

3. Paleo proxy data: detailed look.

The calcium (Ca) signal from the GRIP ice–core:
about 80,000 data points from 11 kyr to 91 kyr before present.



Typical interjump time: 1000 – 2000 years, mean waiting time ∼1470 years

What triggers the transitions?

Langevin equation for climate dynamics

$$dX(t) = -U'(X(t))\,dt + \varepsilon\,dL(t)$$

U – double-well potential, wells correspond to the climate states.

P. Ditlevsen (*Geophys. Res. Lett. 1999*): histogram analysis of the noise.
Noise L has α–stable component with $\alpha \approx 1.75$.

4. The random perturbation L. Lévy processes

L is a Lévy process:

- independent increments
  $L(s) \perp L(t) - L(s)$
- stationary increments
  $L(t) - L(s) \stackrel{d}{=} L(t-s)$
- $L(0) = 0$
- stochastically continuous (no fixed jumps)
- right–continuous with left limits

B is a Brownian motion:

- independent increments
- stationary increments
- $B(0) = 0$
- $B(t) - B(s) \sim \mathcal{N}(0, t-s)$
- continuous paths

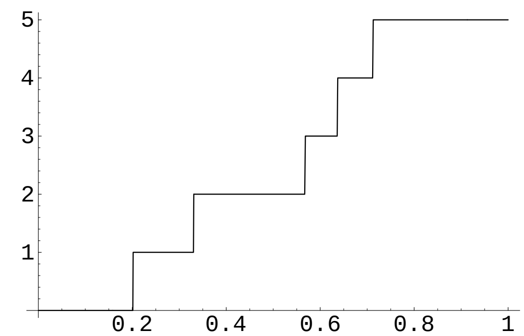
Description via Fourier transform: $\varphi_t(\lambda) = \mathbf{E} e^{i\lambda L(t)}$

$$L(t+s) = L(t) + (L(t+s) - L(t)) \stackrel{d}{=} L(t) + L'(s)$$

$$\varphi_{t+s}(\lambda) = \varphi_s(\lambda) \cdot \varphi_t(\lambda) \quad \Rightarrow \quad \varphi_t(\lambda) = \mathbf{E} e^{i\lambda L(t)} = e^{t\Phi(\lambda)}$$

5. Examples

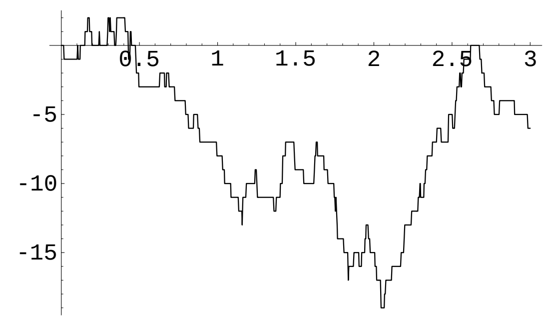
Poisson process, intensity $c > 0$



$$\mathbf{E}e^{i\lambda L(t)} = \sum_{k=1}^{\infty} e^{i\lambda k} e^{-ct} \frac{(ct)^k}{k!} = e^{c(e^{i\lambda}-1)}$$

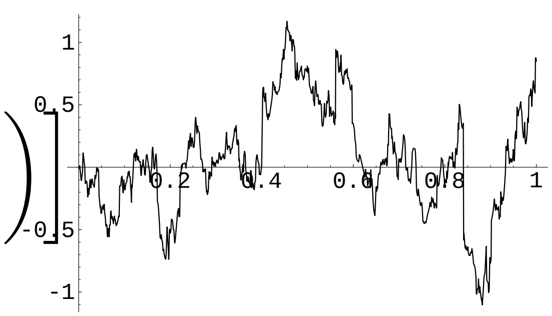
$$= \exp\left[t \int (e^{i\lambda y} - 1) c\delta_1(dy)\right]$$

Compound Poisson process, $c > 0$, jumps $\sim \sigma$



$$\mathbf{E}e^{i\lambda L(t)} = \exp\left[t \int (e^{i\lambda y} - 1) c\sigma(dy)\right]$$

General case, Lévy–Hinchin Formula



$$\mathbf{E}e^{i\lambda L(t)} =$$
$$\exp\left[t\left(-\frac{d}{2}\lambda^2 + i\mu\lambda + \int (e^{i\lambda y} - 1 - i\lambda y\, \mathbb{I}\{|y| \le 1\})\nu(dy)\right)\right]$$

$d \ge 0$, $\mu \in \mathbb{R}$, $\nu(\{0\}) = 0$, $\int (y^2 \wedge 1)\nu(dy) < \infty$

6. α–stable Lévy processes (Lévy flights)

Jump measure $\nu(y) = \dfrac{1}{|y|^{1+\alpha}}$, $\alpha \in (0, 2)$.

$$\mathbf{E} e^{i\lambda L(t)} = \exp\left(t \int (e^{i\lambda y} - 1 - i\lambda y\, \mathbb{I}\{|y| \le 1\}) \frac{dy}{|y|^{1+\alpha}} \right)$$

Markov process with a generator

$$Af(x) = \int [f(x+y) - f(x) - yf'(x)\mathbb{I}\{|y| \le 1\}] \frac{dy}{|y|^{1+\alpha}},$$

$$Af(x) = -(-\Delta)^{\alpha/2} f(x) \qquad \text{— fractional Laplacian}$$

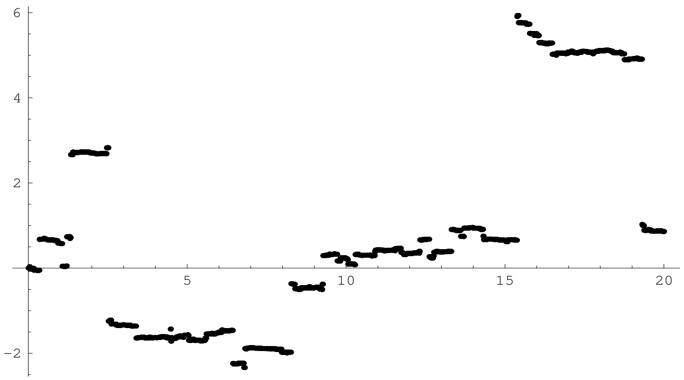
L is a purely jump process.

Control of jumps: $J(t) = L(t) - L(t-)$ the jump size at time $t > 0$.

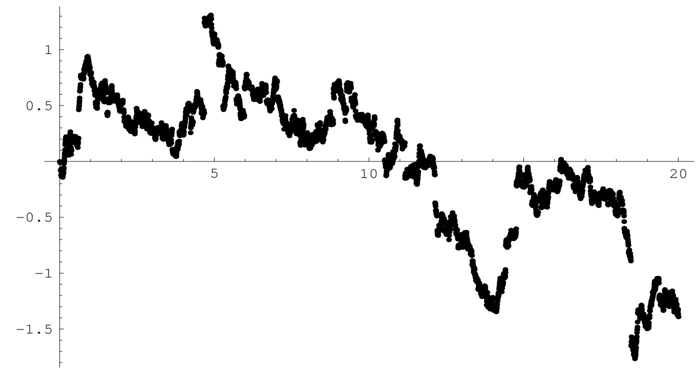
$$N(t, A) = \sharp\{s \in (0, t] : J(s) \in A\} \sim \textbf{Poisson}\left(t \int_A \frac{dy}{|y|^{1+\alpha}} \right)$$

For example: $\quad \sharp\{s \in (0, t] : J(s) \ge 1\} \sim \textbf{Poisson}\left(t \int_1^\infty \frac{dy}{|y|^{1+\alpha}} \right)$

7. α–stable Lévy processes II



$\alpha = 0.75$



$\alpha = 1.75$

Length of sample paths: finite for $\alpha \in (0, 1)$ and infinite for $\alpha \in [1, 2)$.

Self-similarity: $(L(ct))_{t\geq 0} = (c^{1/\alpha} L(t))_{t\geq 0}$. Hurst parameter $\mathbb{H} = \dfrac{1}{\alpha} \geq \dfrac{1}{2}$.

Heavy tails: $\mathbf{P}(L(t) \geq u) \approx \dfrac{c}{u^\alpha}$, $u \to \infty$.

Explicit form of the **Fourier transform** $\mathbf{E} e^{i\lambda L(t)} = e^{-c(\alpha)|\lambda|^\alpha t}$,

$\alpha = 1$ Cauchy process $\dfrac{1}{\pi}\dfrac{1}{1+x^2}$

$\alpha = 2$ Brownian motion $\dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

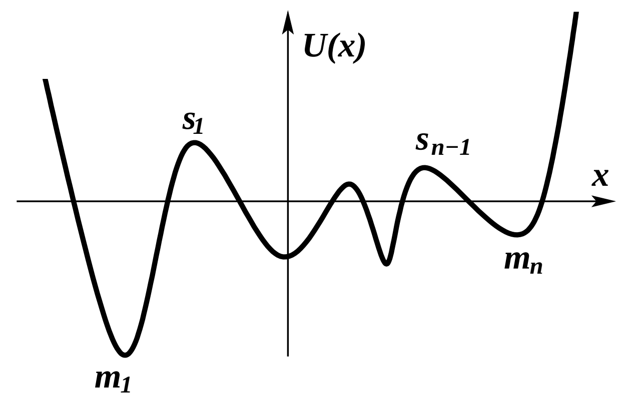
8. Object of study. Simple system with Lévy perturbation

Small noise ($\varepsilon \downarrow 0$) asymptotics of solutions of SDE

$$X_x^\varepsilon(t) = x - \int_0^t U'(X_x^\varepsilon(s))\,ds + \varepsilon L(t), \quad \varepsilon \downarrow 0.$$

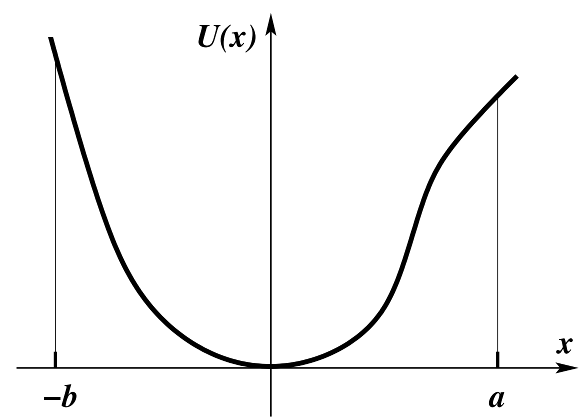
• L — α–stable (symmetric) Lévy process (maybe $+$ Brownian motion)

Regular n–well potential
(smooth, $U''(m_i), U''(s_i) \neq 0$,
$|U'(x)| > |x|^{1+\delta}$, $x \to \pm\infty$)



Meta–stable behaviour
Transitions between the wells

Regular one–well potential
(smooth, $U''(0) > 0$)



Exit time:
$\sigma_x(\varepsilon) = \inf\{t \geq 0 \,:\, X_x^\varepsilon(t) \notin [-b, a]\}$
$a, b < \infty$ ($b = \infty$)

P. Imkeller & I. Pavlyukevich
Stoch. Proc. Appl. 116, 2006; J. Phys. A: Math. Gen. 39, 2006; *ESAIM: P&S*, 2008

9. What is known

Freidlin, Wentzell (Random Perturbations of Dynamical Systems, 1979):
Gaussian perturbations:

$$\hat{X}_x^\varepsilon(t) = x - \int_0^t U'(\hat{X}_x^\varepsilon(s))\, ds + \varepsilon W(t), \quad \varepsilon \downarrow 0.$$

Perturbation: εW — standard Brownian motion of small amplitude (in $\mathbb{R}^d$).

Locally infinitely divisible perturbations leading to Gaussian
$\varepsilon L^\varepsilon$ with jump part

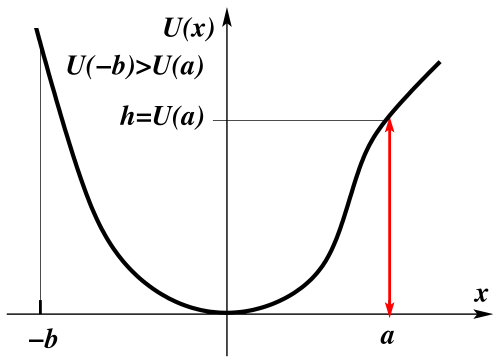
$$A^\varepsilon f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \Big[f(x + \varepsilon y) - f(x) - \langle \varepsilon y, \nabla f(x) \rangle \Big] \frac{\nu(x, dy)}{\varepsilon}$$

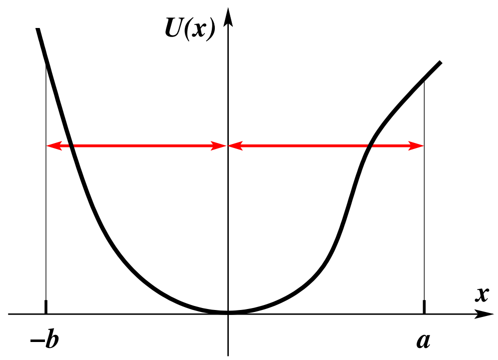
Lévy measure ν has all exponential moments.
For example, $\varepsilon L^\varepsilon(t) = \varepsilon \pi^{1/\varepsilon}(t) - t$, π — standard Poisson process.

Effects of heavy tails
Godovanchuk (Theor. Prob. Appl. 26, 1982):
'Large deviations' for Markov processes with heavy jumps.

Samorodnitsky, Grigoriou (Stoch. Proc. Appl. 105, 69–97, 2003):
Tails of solutions of SDEs driven by Lévy processes with power tails.

10. First exit: Gaussian vs. Lévy





$$\hat{X}_x^\varepsilon(t) = x - \int_0^t U'(\hat{X}_x^\varepsilon(s))\,ds + \varepsilon W(t) \qquad X_x^\varepsilon(t) = x - \int_0^t U'(X_x^\varepsilon(s))\,ds + \varepsilon L(t)$$

Large deviations (Freidlin–Wentzell):

$$\mathbf{P}(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma}_x(\varepsilon) < e^{(2h+\delta)/\varepsilon^2}) \to 1 \qquad \mathbf{P}(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma_x(\varepsilon) < \tfrac{1}{\varepsilon^{\alpha+\delta}}) \to 1$$

Mean exit time (Kramers, Day, Bovier):

$$\mathbf{E}\hat{\sigma}_x(\varepsilon) \approx \frac{\varepsilon\sqrt{\pi}}{U'(a)\sqrt{U''(0)}} \exp\left(\frac{2h}{\varepsilon^2}\right) \qquad \mathbf{E}\sigma_x(\varepsilon) \approx \frac{\alpha}{\varepsilon^\alpha}\left[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}\right]^{-1}$$

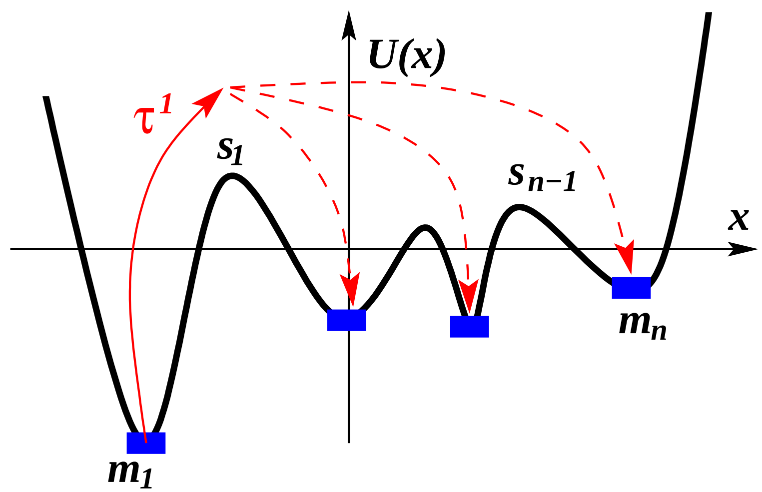
Exponential exit (Day, Bovier)

$$\mathbf{P}\left(\frac{\hat{\sigma}_x(\varepsilon)}{\mathbf{E}\hat{\sigma}_x(\varepsilon)} > u\right) \to \exp(-u) \qquad \mathbf{P}\left(\frac{\sigma_x(\varepsilon)}{\mathbf{E}\sigma_x(\varepsilon)} > u\right) \to \exp(-u)$$

11. Transitions

For small $\Delta > 0$ denote $\mathbf{B}_i = \{y : |y - m_i| \leq \Delta\}$ and

$$\tau_x^i(\varepsilon) = \inf\{t \geq 0 : X_x^\varepsilon(t) \in \cup_{j\neq i} B_j\}$$



Theorem 1. *For $x \in B_i$ the following holds as $\varepsilon \to 0$:*

$$\mathbf{P}(\varepsilon^\alpha \tau_x^i(\varepsilon) > u) \to e^{-q_i u},$$

$$\mathbf{E}\tau_x^i(\varepsilon) \approx \frac{1}{\varepsilon^\alpha q_i}, \qquad q_i = \int_{\mathbb{R}\setminus(s_{i-1},s_i)} \frac{dy}{|y - m_i|^{1+\alpha}},$$

$$\mathbf{P}(X^\varepsilon(\tau_x^i(\varepsilon)) \in B_j) \to \frac{q_{ij}}{q_i}, \qquad q_{ij} = \int_{(s_{j-1},s_j)} \frac{dy}{|y - m_i|^{1+\alpha}}, \quad i \neq j.$$

12. Meta–stability

Theorem 2. *Let $x \in (s_{j-1}, s_j)$ and $t > 0$. Then*

$$X_x^\varepsilon\left(\frac{t}{\varepsilon^\alpha}\right) \xrightarrow{d} Y_{m_j}(t),$$

where Y is a Markov chain on $\{m_1, \ldots, m_n\}$ with a generator $Q = (q_{ij})$, $q_{ii} = -q_i$.

Remark. Y has the invariant measure

$$\pi(dy) = \sum_{j=1}^{n} \pi_j \delta_{m_j}(dy),$$
$$\pi_j > 0,$$

where $Q^T\pi = 0$.

13. Meta–stable behaviour. Gaussian case

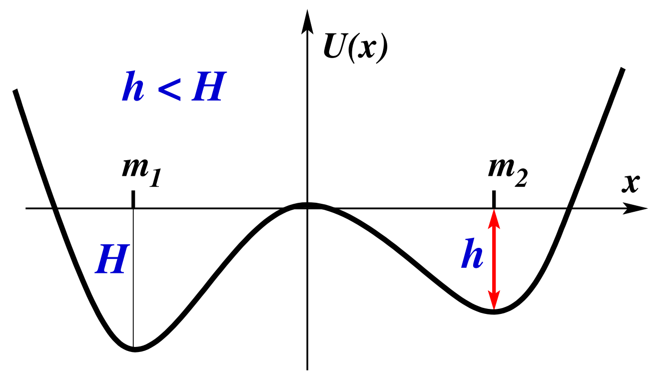
$$\hat{X}_x^\varepsilon(t) = x - \int_0^t U'(\hat{X}_x^\varepsilon(s))\,ds + \varepsilon W(t)$$

Different life times for each well: $\mathbf{E}\tau_x^i(\varepsilon) \sim \exp(V_i/\varepsilon^2)$

$\Rightarrow n-1$ critical exponents $0=\lambda_0<\lambda_1<\cdots<\lambda_{n-1}$ such that for $\Lambda\in(\lambda_i,\lambda_{i+1})$ the process $\hat{X}_x^\varepsilon\left(te^{\Lambda/\varepsilon^2}\right)$ converges to $\mu(\Lambda,x)$

Double–well potential: critical time scale:

$$\lambda_\varepsilon \approx \frac{2\pi}{\sqrt{U''(0)|U''(m_2)|}}\exp(2h/\varepsilon^2)$$



Convergence of fin. dim. distrib.:
(Kipnis, Newman; Mathieu)

$$\hat{X}^\varepsilon(\lambda_\varepsilon t)\to\hat{Y}(t)$$

Generator of $\hat{Y}$

$$\begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix}, \text{ and } \hat{Y}(0)=\begin{cases} m_1, & \text{if } x<0,\\ m_2, & \text{if } x>0.\end{cases}$$

14. Doubts and questions

$$X_x^\varepsilon(t) = x - \int_0^t U'(X_x^\varepsilon(s))\,ds + \varepsilon L(t), \quad \varepsilon \downarrow 0.$$

- Why α–**stable** (non-Gaussian, heavy-tail) noise?
- Why **small** noise?

15. A natural system with small noise

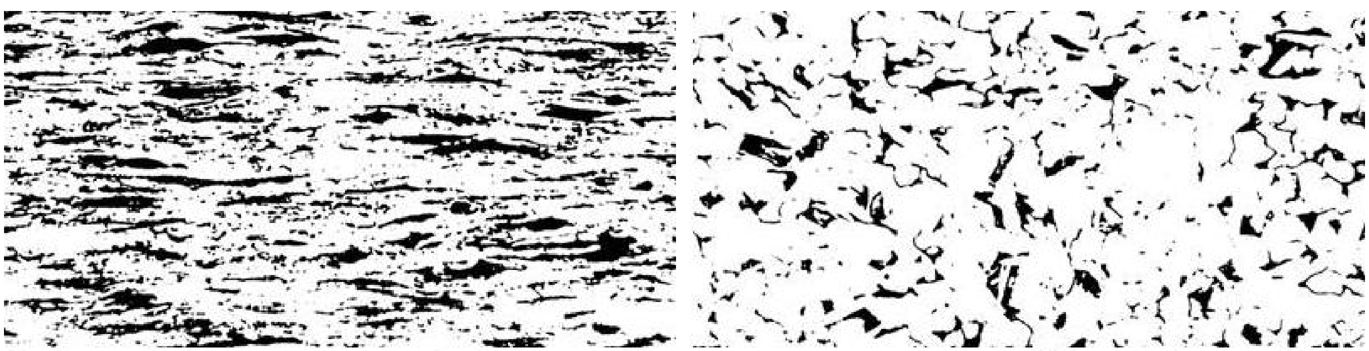
Is there a Gaussian system with *a priori* small noise?

Simulated annealing (Kirkpatrick et al., Geman&Geman, Černy, $\sim$ 1985):

$$\hat{Z}(t) = z - \int_0^t U'(\hat{Z}(s))\, ds + \int_0^t \sigma(s) dW(s),$$

$\sigma(t)$ — 'temperature', $\sigma(t) \to 0$ as $t \to \infty$.

ANNEALING: gradual cooling of steel (copper, glass etc.) in order to induce softenness, to relieve internal stresses, to refine the crystalline structure.



Carbon steel: before annealing fully annealed at 920 °C

I. Pavlyukevich: *Stoch. Proc. Appl.*, 2008; *J. Phys. A: Math. and Theor.*, 2007

16. Gaussian diffusion: long–time behaviour

Stochastic optimisation: look for a **global minimum** m^* of U.

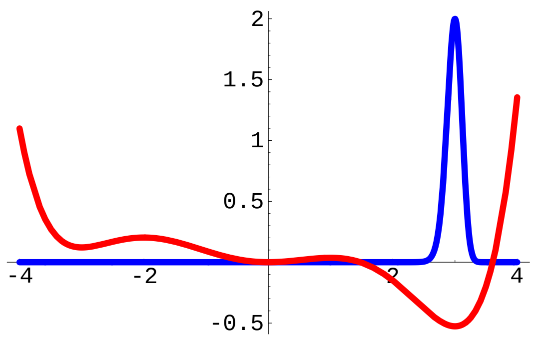
$$\hat{X}(t) = x - \int_0^t U'(\hat{X}(s))\,ds + \varepsilon W(t)$$

Generator $A_\varepsilon f = \frac{\varepsilon^2}{2}\Delta f - U'f'$

Invariant measure: $A_\varepsilon^*\mu = 0$

$\mu_\varepsilon(dx) = c(\varepsilon)e^{-2U(x)/\varepsilon^2}dx$

$\mu_\varepsilon \Rightarrow \delta_{m^*}$, $\varepsilon \to 0$



The spectrum $\{-\Lambda_k^\varepsilon, \Psi_k^\varepsilon\}_{k\geq 0}$ of A_ε is discrete.

SPECTRAL GAP

$\Lambda_0^\varepsilon = 0$, $\Psi_0^\varepsilon(x) = 1$

$\Lambda_1^\varepsilon \sim \exp(-\Theta/\varepsilon^2)$ (Friedman, Day, Bovier et al.)

$$\mathbf{E}_x f(\hat{X}^\varepsilon(t)) = \langle f, 1\rangle_{L^2(\mu_\varepsilon)} 1(x) + e^{-\Lambda_1^\varepsilon t}\langle f, \Psi_1^\varepsilon\rangle_{L^2(\mu_\varepsilon)}\Psi_1^\varepsilon(x) + \cdots$$

$$\mathrm{Law}(\hat{X}^\varepsilon(t)) \to \mu_\varepsilon, \quad t \to \infty$$

17. Gaussian simulated annealing

$$\hat{Z}_z(t) = z - \int_0^t U'(\hat{Z}_z(s))\,ds + \int_0^t \sigma(s)dW(s), \quad \sigma(t) = \Big(\frac{\theta}{\ln(\lambda+t)}\Big)^{1/2}.$$

INTUITION: $\hat{Z}(t) \approx \hat{X}^{\sigma(t)}(t)$

"FOURIER EXPANSION":

$$\mathbf{E}_{0,z} f(\hat{Z}(t)) \approx \langle f, 1\rangle_{L^2(\mu_{\sigma(t)})} 1 + e^{-\Lambda_1^{\sigma(t)} t} \langle f, \Psi_1^{\sigma(t)}\rangle_{L^2(\mu_{\sigma(t)})} \Psi_1^{\sigma(t)} + \cdots$$

CONVERGENCE:

$$\mu_{\sigma(t)} \Rightarrow \delta_{m^*},$$

$$\Lambda_1^{\sigma(t)} t \sim e^{-\Theta/\sigma(t)^2} t = \frac{t}{(t+\lambda)^{\Theta/\theta}} \to \begin{cases} \infty, & \theta > \Theta, \\ 0, & \theta < \Theta, \end{cases} \quad t \to +\infty,$$

$$\theta > \Theta \quad \Rightarrow \quad Z(t) \to m^*.$$

This works, see Chiang&Hwang&Sheu, Holley&Kusuoka&Stroock and others

HOWEVER: very slow cooling, local search, unknown critical rate Θ...

18. Modifications: big jumps

Szu and Hartley (*Phys. Lett. A 122, 1987*): "fast simulated annealing"

Discrete schema with Cauchy jumps ξ_k

$X_{kh} = X_{(k-1)h} + U'(X_{(k-1)h})h + \frac{\xi_k}{(k-1)h}$

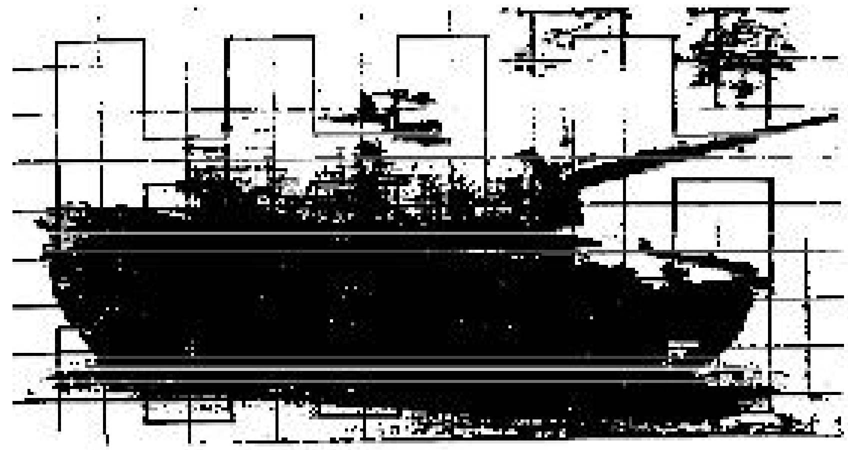
$\Delta U = U(X_{kh}) - U(X_{(k-1)h})$

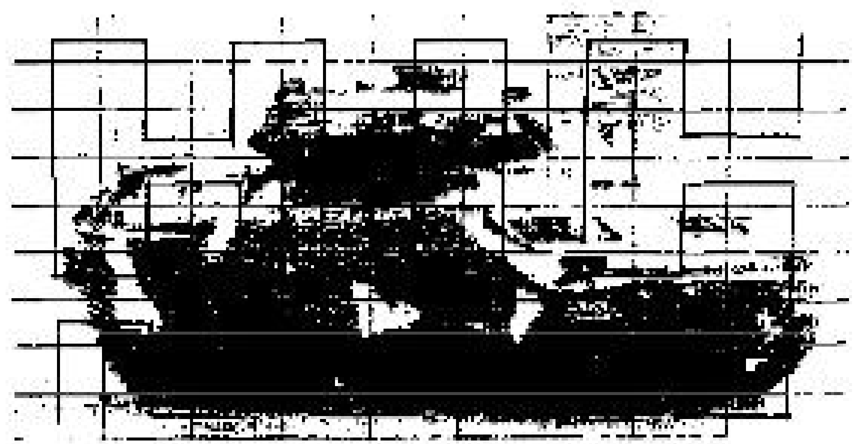
Metropolis acceptance probability:
Accept always if $\Delta U \le 0$
Accept with probability $\exp(-\Delta U \cdot kh)$ if $\Delta U > 0$

Non-local search
Fast algebraic cooling

Convergence ???
Probably works ???

Applications to image recognition $\longrightarrow$





19. Simulated annealing with α–stable processes

Analogously to Gaussian simulated annealing:

$$Z(t) = z - \int_0^t U'(Z(s))\,ds + \int_0^t \frac{dL(s)}{(\lambda+s)^\theta}, \quad \lambda>0, \theta>0.$$

Convergence of $Z(t)$ as $t\to\infty$?

Time-homogeneous case:

$$X^\varepsilon\left(\frac{t}{\varepsilon^\alpha}\right) \to Y(t)$$

Y — Markov chain on $\{m_1,\dots,m_n\}$ with generator $Q=(q_{ij})$

$$q_{ij} = \int_{(s_{j-1},s_j)} \frac{dy}{|y-m_i|^{1+\alpha}}, \quad q_{ii} = -q_i = -\int_{\mathbb{R}\setminus(s_{i-1},s_i)} \frac{dy}{|y-m_i|^{1+\alpha}}$$

$\mathrm{Law}(Y(t)) \to \pi$, $Q^*\pi = 0$

$$Z(t) \approx X^{1/(\lambda+t)^\theta}(t) \approx Y\left(\frac{t}{(t+\lambda)^{\alpha\theta}}\right) \Rightarrow \begin{cases} \pi, & \alpha\theta<1, \\ \text{no convergence}, & \alpha\theta>1. \end{cases}$$

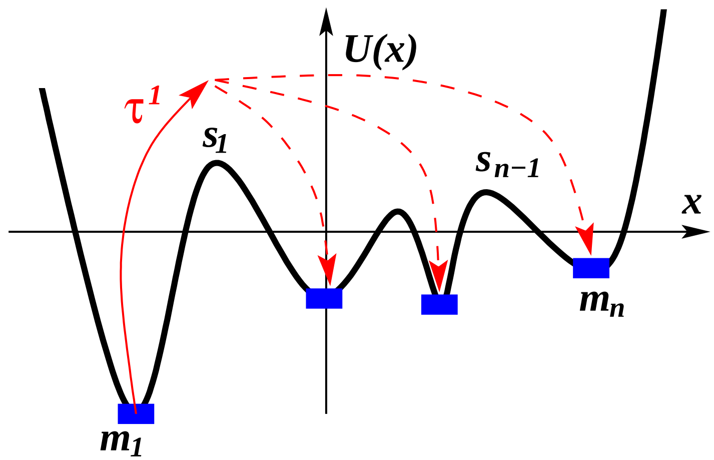
Slow cooling: convergence

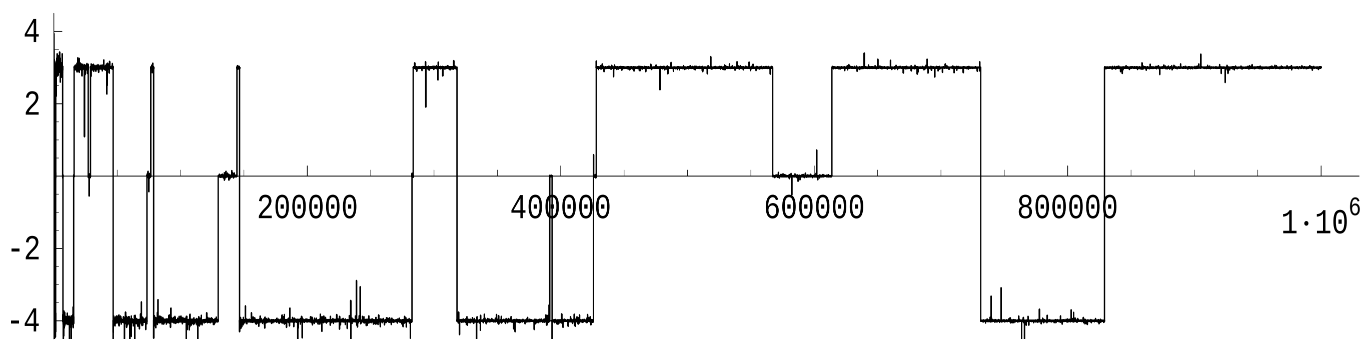
Theorem 3. [slow cooling] *Let $\alpha\theta < 1$. Then*

$$\mathbf{E}\tau^i(\lambda) \approx \frac{\lambda^{\alpha\theta}}{q_i},$$

$$\mathbf{P}(Z(\tau^i(\lambda)) \in B_j) \to \frac{q_{ij}}{q_i}, \quad \lambda \to \infty,$$

$$Z(t) \Rightarrow \pi, \quad t \to \infty.$$

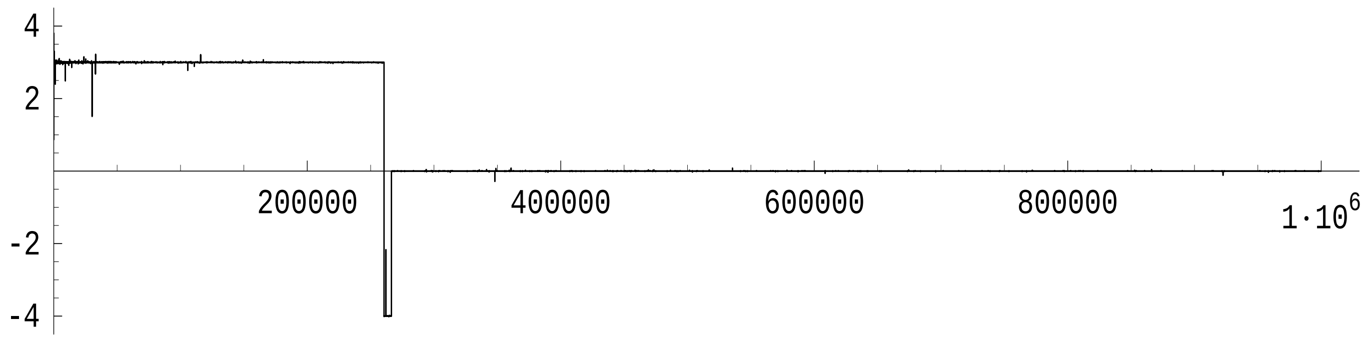




21. Fast cooling: trapping

Theorem 4. [trapping] *Let $\alpha\theta > 1$. Then*

$$\mathbf{P}(\tau^i(\lambda) < \infty) = \mathcal{O}(\lambda^{1-\alpha\theta}), \quad \lambda \to \infty,$$
$$\mathbf{E}\tau^i(\lambda) = \infty.$$



22. Gaussian vs. α–stable simulated annealing

Simulated annealing with α–stable process:

- allows to determine the measure π, i.e. the **sizes** of potential wells
- the search is **non-local**
- polynomially **fast** cooling

BUT: how to find the **global** minimum of U?

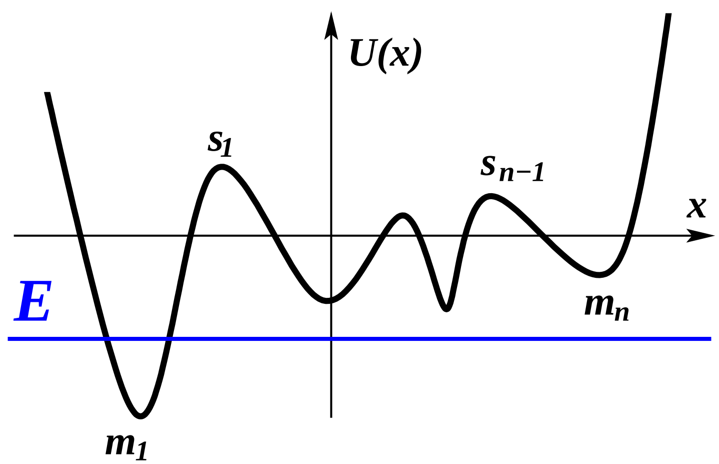
23. Simulated annealing with *stable-like* processes

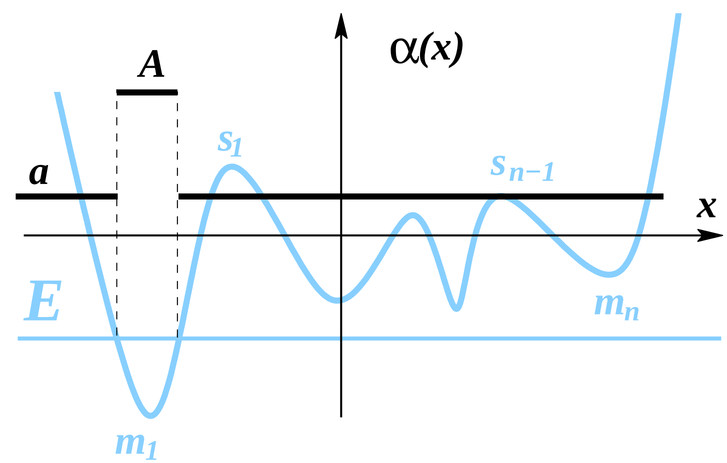
In progress: I.P., *J. Comp. Phys.*, 2007

$$V(t) = v - \int_0^t U'(V(s-))\,ds + \int_0^t \frac{dH(V(s-),s)}{(\lambda+s)^\theta}, \quad \lambda>0, \theta>0$$

H a *stable-like process* with the jump measure

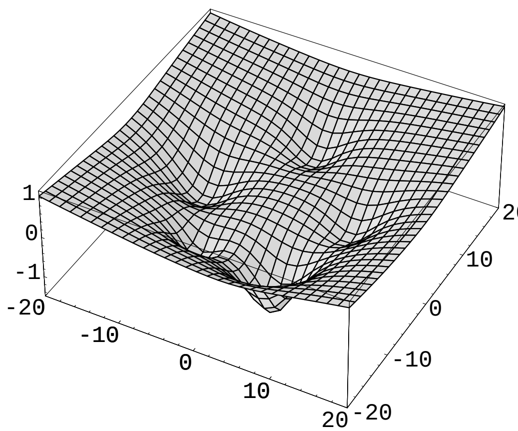
$$\nu_x(dy) = \frac{dy}{|y|^{1+\alpha(x)}}$$





$$\alpha(x) = \begin{cases} A, & U(x) \le E, \quad \textbf{trapping}, \\ a, & U(x) > E, \quad \textbf{exit}, \end{cases} \qquad 0<a<A<2, \quad a\theta < 1 < A\theta.$$

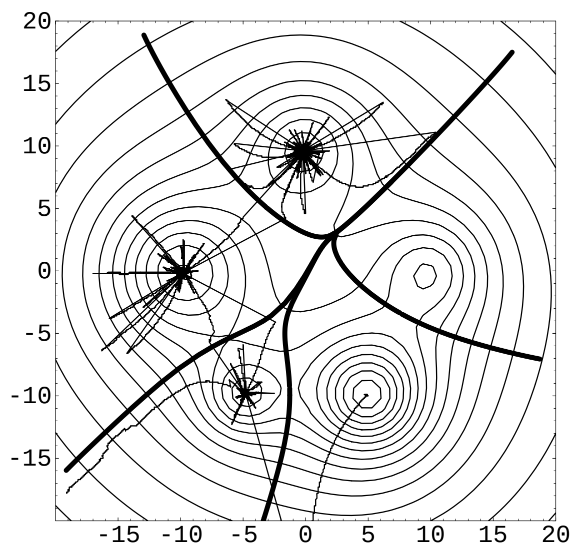
24. An example of non-local search

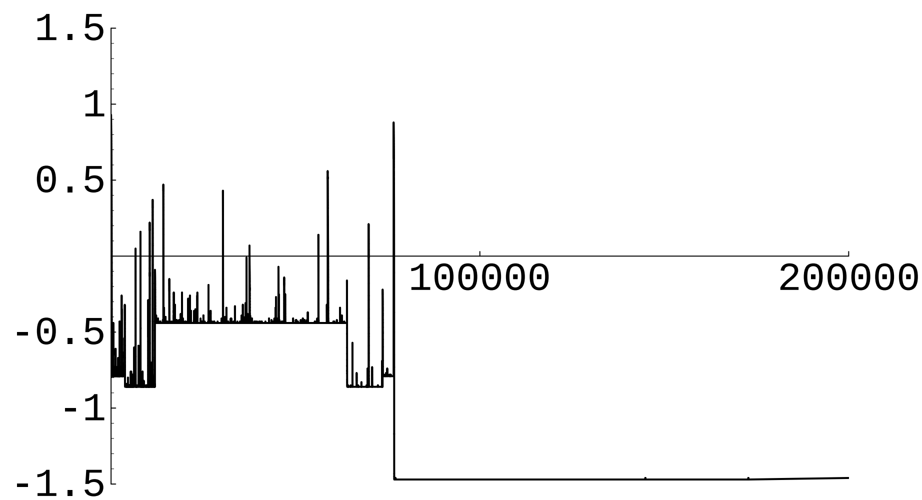


The lowest minmum: $U(4.9, -9.9) = -1.46$
The second lowest minimum: $U(-9.7, -0.1) = -0.85$

$\alpha(x) = \begin{cases} 1.8, \text{ if } U(x) \le -1, \\ 1.1, \text{ if } U(x) > -1, \end{cases}$ $\theta = 0.75.$

Look for a local minimum: $U(m) \le -1$.



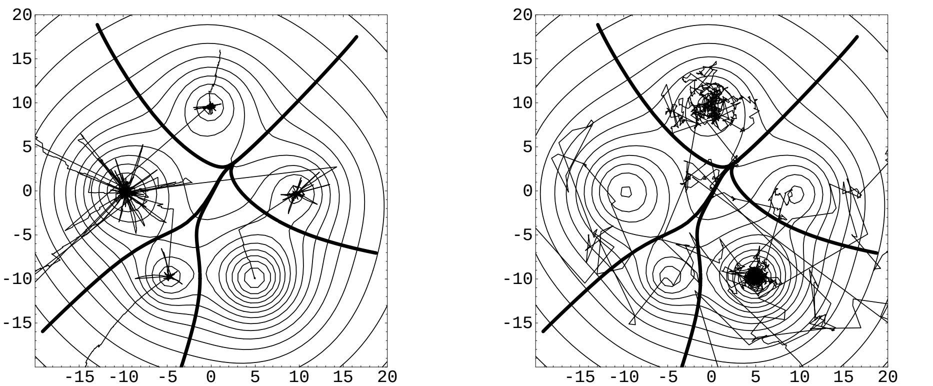


$$V_k = V_{k-1} - \nabla U(V_{k-1})h - \frac{\xi_k^h(V_{k-1})}{(\lambda+(k-1)h)^\theta}$$

$0 \le k \le 2 \cdot 10^6$, $\lambda = 10^4$, $V_0 \in [-20, 20]^2$, $h = 0.1$

Success ratio: 96 out of 100.

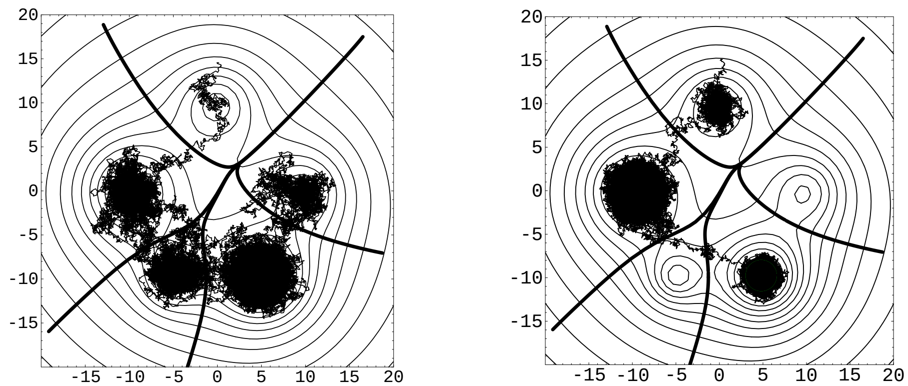
25. Numerical example in $\mathbb{R}^2$



Stable-like simulated annealing

$a = 1.1,\ A = 1.8,\ \theta = 0.75$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
$1 \cdot 10^2$	175458	61	0.0004	90	0.0004
$5 \cdot 10^2$	93273	75	0.0004	96	0.0004
$1 \cdot 10^3$	135081	62	0.0004	93	0.0004
$5 \cdot 10^3$	148972	60	0.0004	93	0.0004
$1 \cdot 10^4$	264070	47	0.0004	85	0.0004



Fast simulated annealing

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^1	586029	67	0.0005	69	0.0004
$5 \cdot 10^1$	5787	97	0.0019	98	0.0006
$1 \cdot 10^2$	1887	100	0.0035	100	0.0008
$5 \cdot 10^2$	4477	99	0.0294	100	0.0038
$1 \cdot 10^3$	7552	70	0.0600	100	0.0112

Gaussian SDE

$\theta = 3$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^3	8749	62	0.1211	76	0.1086
10^4	12768	71	0.1170	79	0.1109
10^5	34961	61	0.1204	81	0.1154
10^6	37262	66	0.0888	86	0.0923

Gaussian simulated annealing

$\theta = 3$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^3	81943	67	0.0786	93	0.0804
10^4	113516	62	0.0715	94	0.0756
10^5	301035	44	0.0685	79	0.0643
10^6	573990	50	0.0662	63	0.0633

26. Numerical example in $\mathbb{R}^4$. Shekel's function

$$S_{10,4}(\mathbf{y}) = -\sum_{i=1}^{10} \frac{1}{c_i + \|\mathbf{y} - \mathbf{a}_i\|^2}, \quad \mathbf{y} \in \mathbf{R}^4.$$

Stable-like simulated annealing

$a = 1.2$, $A = 1.9$, $\theta = 0.6$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^2	50198	39	0.3598	92	0.0887
$5 \cdot 10^2$	46768	52	0.2456	92	0.0866
$1 \cdot 10^3$	43951	63	0.2072	95	0.0810
$5 \cdot 10^3$	63960	56	0.0848	95	0.0595
$1 \cdot 10^4$	77239	53	0.0572	97	0.0515

Fast simulated annealing

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
50	612902	67	0.1385	70	0.0179
100	354639	81	0.2499	82	0.0324
500	57214	7	0.8834	99	0.1470
1000	99846	0	—	99	0.2644
1500	139922	0	—	100	0.4101

Gaussian SDE

$\theta = 10$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^3	37089	1	1.5448	7	1.7751
10^4	26749	10	1.5667	18	1.6223
10^5	24657	52	1.5839	45	1.6339
10^6	26611	59	1.5631	67	1.3583
10^7	53874	54	1.4889	69	1.5545

Gaussian simulated annealing

$\theta = 10$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^3	16889	69	1.4155	79	1.2417
10^4	18722	78	1.2062	90	1.3376
10^5	59266	68	1.0551	89	1.1888
10^6	183003	74	0.9401	84	1.0355
10^7	536875	67	1.1161	72	1.0520

27. Numerical example in $\mathbb{R}^6$. Hartman's function

$$H_{4,6}(\mathbf{y}) = -\sum_{i=1}^{4} c_i \exp\left(-\sum_{j=1}^{6} a_{ij}(y_i - p_{ij})^2,\right), \quad \mathbf{y} \in \mathbf{R}^6.$$

Stable-like simulated annealing

$a = 1.5$, $A = 1.9$, $\theta = 0.6$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
100	20070	13	0.0295	84	0.0044
250	18492	18	0.0257	84	0.0043
500	12841	43	0.0264	80	0.0049
1000	18624	42	0.0155	83	0.0042
5000	29495	73	0.0043	83	0.0025

Fast simulated annealing

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10	1302701	35	0.0044	35	0.0001
50	1525483	17	0.0323	23	0.0029
100	1451393	2	0.0428	26	0.0049
500	1348318	0	—	11	0.0323
1000	865728	0	—	7	0.0388

Gaussian SDE

$\theta = 6$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^6	360969	0	—	0	—
10^7	191639	0	—	0	—
10^8	597533	0	—	0	—

Gaussian simulated annealing

$\theta = 6$

λ	$\langle N_{\text{first}} \rangle$	$k = 5 \cdot 10^4$		$k = 5 \cdot 10^5$	
		N_k	Δ_k	N_k	Δ_k
10^5	668062	0	—	0	—
10^6	193789	0	—	0	—
10^7	466592	0	—	1	0.0430

Pictures

Page 1 `http://www.ngdc.noaa.gov/paleo/globalwarming/gallery/icecore_4.jpg`

`http://ess.geology.ufl.edu/ess/Notes/Paleoclimatology/Paleoclimate Slides/greenland.gif`

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