The scaling of optimal supply networks: implications for biological and geophysical systems

Workshop III: Transports Systems Geography, Geosciences, and Networks

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Outline

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- Q: How do optimal solutions scale with system size?

Outline

Animal power

Fundamental biological and ecological constraint:

$$\boxed{P = c\,M^{\alpha}}$$

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Prefactor c depends on body plan and body temperature:

Birds	39–41 $^{\circ}C$
Eutherian Mammals	36–38 $^{\circ}C$
Marsupials	34–36 $^{\circ}C$
Monotremes	30–31 $^{\circ}C$

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- Lognormal fluctuations:
  Gaussian fluctuations in $\log P$ around $\log cM^{\alpha}$.
- Stefan-Boltzmann relation for radiated energy:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma \varepsilon S T^4$$

The prevailing belief of the church of quarterology

$$\boxed{\alpha = 3/4}$$

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Huh?

Related putative scalings:

- number of capillaries $\propto M^{3/4}$
- time to reproductive maturity $\propto M^{1/4}$
- heart rate $\propto M^{-1/4}$
- cross-sectional area of aorta $\propto M^{3/4}$
- population density $\propto M^{-3/4}$

History

1840's: Sarrus and Rameaux first suggested $\alpha = 2/3$.

History

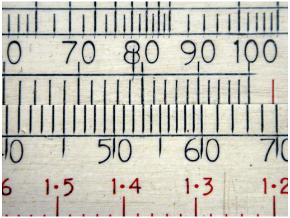
1883: Rubner found $\alpha \simeq 2/3$.

History

1930's: Brody, Benedict study mammals.
Found $\alpha \simeq 0.73$ (standard).

History

1932: Kleiber analyzed 13 mammals.
Found $\alpha = 0.76$ and suggested $\alpha = 3/4$.



History

1950/1960: Hemmingsen
Extension to unicellular organisms.
$\alpha = 3/4$ assumed true.

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1964: Troon, Scotland:
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$\alpha = 3/4$ made official ...

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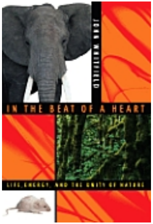

In the Beat of a Heart: Life, Energy, and the Unity of Nature—by John Whitfield

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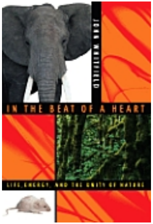

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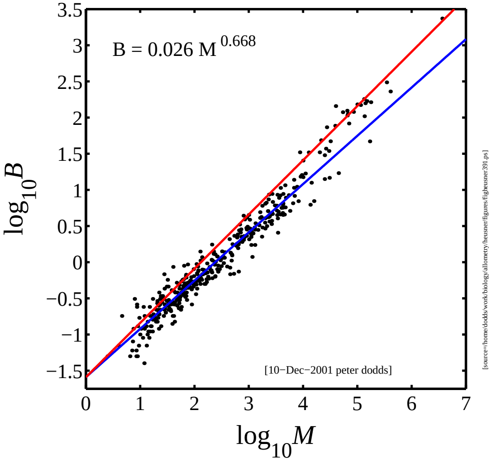
But—much controversy...

Some data on metabolic rates



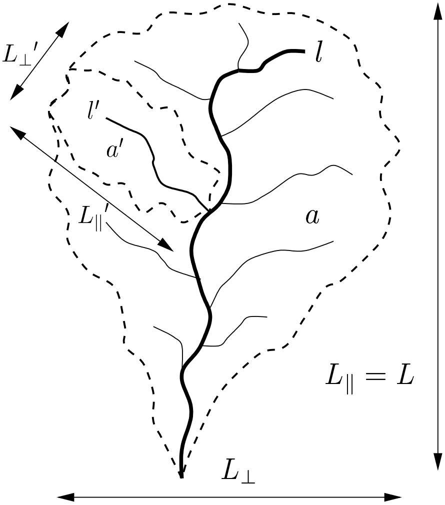
- Heusner's data (1991) [5]
- 391 Mammals
- blue line: 2/3
- red line: 3/4.
- $(B = P)$

Outline

History: Metabolism
History: River networks
Earlier theories
Geometric argument
Blood networks
River networks
Conclusion

References

Basic basin quantities: a, l, $L_{\parallel}$, $L_{\perp}$:



- a = drainage basin area
- ℓ = length of longest (main) stream
- $L = L_{\parallel}$ = longitudinal length of basin

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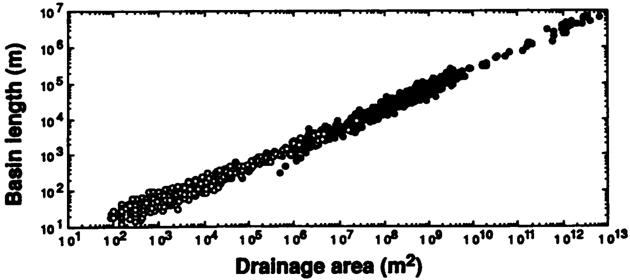
- A catch: studies done on small scales.

Large-scale networks

(1992) Montgomery and Dietrich [7]:

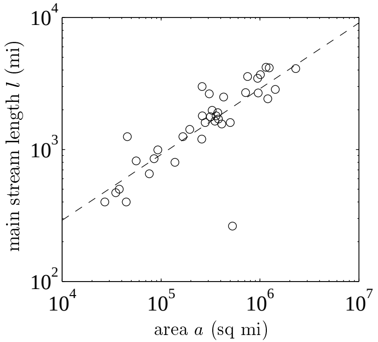


- Composite data set: includes everything from unchanneled valleys up to world's largest rivers.
- Estimated fit:

$$L \simeq 1.78 a^{0.49}$$

- Mixture of basin and main stream lengths.

World's largest rivers only:



- Data from Leopold (1994) [6, 2]
- Estimate of Hack exponent: $h = 0.50 \pm 0.06$

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- So we need another dimension...
- Obviously, a bit silly.

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- Idea is that organismal shapes scale allometrically with 1/4 powers (like nails and trees...)
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- Metabolism and shape never properly connected.

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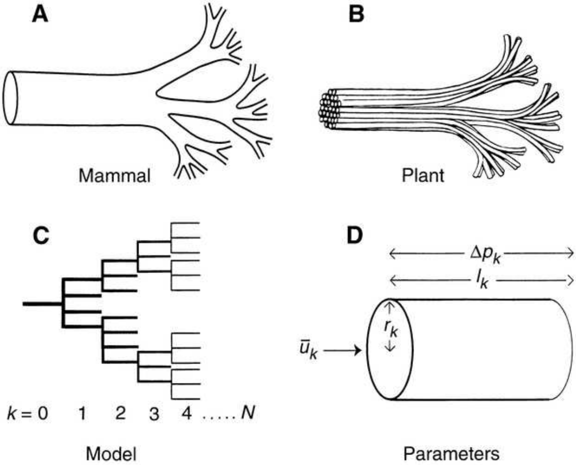
- 1997: West *et al.* [11] use a network story to find $3/4$ scaling.



Nutrient delivering networks:

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- hierarchical network
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Claims:

- $P \propto M^{3/4}$
- networks are fractal
- quarter powers everywhere

Impedance measures:

Poiseuille flow (outer branches):

$$Z = \frac{8\mu}{\pi} \sum_{k=0}^{N} \frac{\ell_k}{r_k^4 N_k}$$

Pulsatile flow (main branches):

$$Z \propto \sum_{k=0}^{N} \frac{h_k^{1/2}}{r_k^{5/2} N_k}$$

Not so fast . . .

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Do find:

- Murray's cube law (1927) for outer branches:

$$r_0^3 = r_1^3 + r_2^3$$

- Impedance is distributed evenly.
- Can still assume networks are fractal.

Connecting network structure to α

1. Ratios of network parameters:

$$R_n = \frac{n_{k+1}}{n_k}, \; R_\ell = \frac{\ell_{k+1}}{\ell_k}, \; R_r = \frac{r_{k+1}}{r_k}$$

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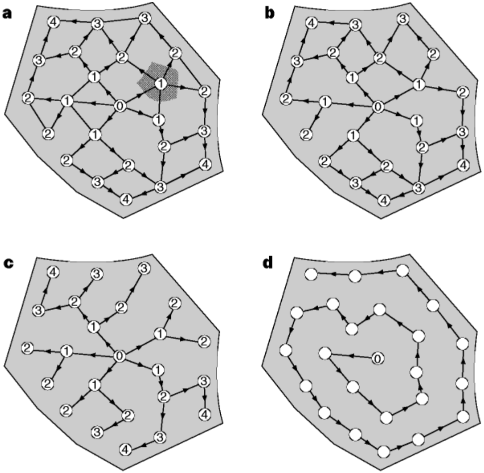
Data from real networks

Network	R_n	R_r^{-1}	R_ℓ^{-1}	$-\frac{\ln R_r}{\ln R_n}$	$-\frac{\ln R_\ell}{\ln R_n}$	α
West *et al.*	–	–	–	1/2	1/3	3/4
rat (PAT)	2.76	1.58	1.60	0.45	0.46	0.73
cat (PAT) (Turcotte *et al.* [10])	3.67	1.71	1.78	0.41	0.44	0.79
dog (PAT)	3.69	1.67	1.52	0.39	0.32	0.90
pig (LCX)	3.57	1.89	2.20	0.50	0.62	0.62
pig (RCA)	3.50	1.81	2.12	0.47	0.60	0.65
pig (LAD)	3.51	1.84	2.02	0.49	0.56	0.65
human (PAT)	3.03	1.60	1.49	0.42	0.36	0.83
human (PAT)	3.36	1.56	1.49	0.37	0.33	0.94

Simple supply networks



- Banavar et al., Nature, (1999) [1]
- Flow rate argument
- Ignore impedance
- Very general attempt to find most efficient transportation networks

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- Such a pachyderm would be rather miserable.

Outline

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- Assume sinks are invariant.
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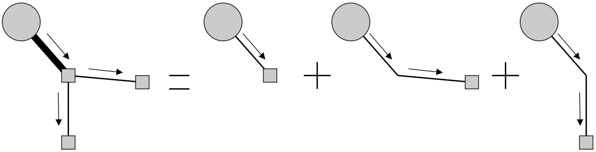

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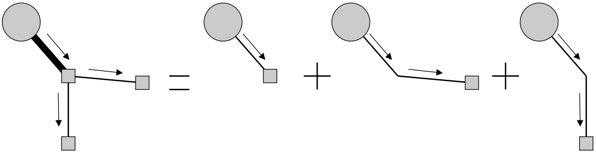

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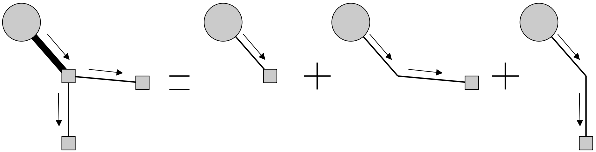
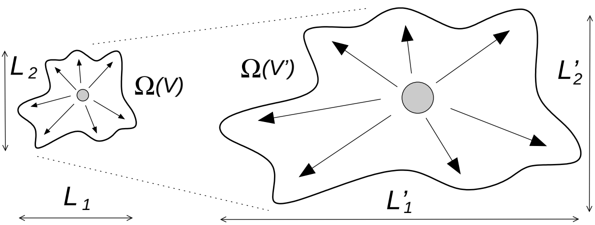


- Q: how does the number of sustainable sinks N_{sinks} scale with volume V for the most efficient network design?
- Or: what is the highest α for $N_{\text{sinks}} \propto V^{\alpha}$?

Geometric argument

- Allometrically growing regions:



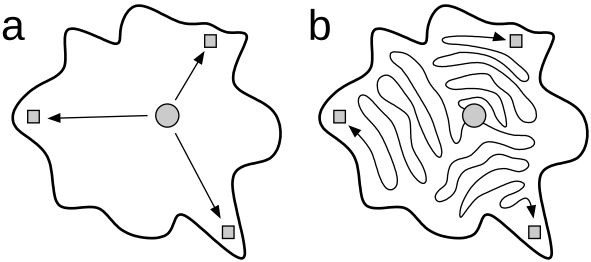
- Have d length scales which scale as

$$L_i \propto V^{\gamma_i} \text{ where } \gamma_1 + \gamma_2 + \ldots + \gamma_d = 1.$$

- For isometric growth, $\gamma_i = 1/d$.
- For allometric growth, we must have at least two of the $\{\gamma_i\}$ being different

Geometric argument

- Best and worst configurations (Banavar et al.)



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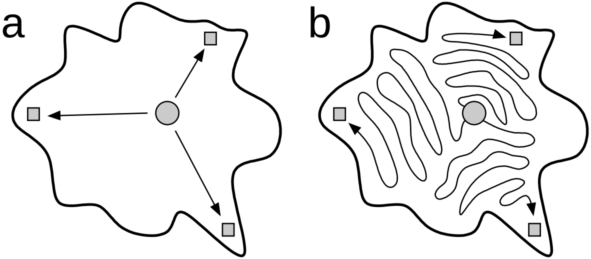

- Rather obviously:
  min $V_{\text{net}} \propto \sum$ distances from source to sinks.

Minimal network volume:

Real supply networks are close to optimal:

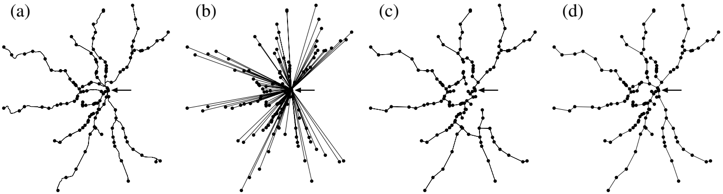

Figure 1. (a) Commuter rail network in the Boston area. The arrow marks the assumed root of the network. (b) Star graph. (c) Minimum spanning tree. (d) The model of equation (3) applied to the same set of stations.

(2006) Gastner and Newman [3]: "Shape and efficiency in spatial distribution networks"

Minimal network volume:

Approximate network volume by integral over region:

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- General result:

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$$\min V_{\text{net/allo}} \propto \rho V^{1+\gamma_{\text{allo}}}$$

- Isometrically growing volumes require less network volume than allometrically growing volumes:

$$\frac{\min V_{\text{net/iso}}}{\min V_{\text{net/allo}}} \rightarrow 0 \text{ as } V \rightarrow \infty$$

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- For cardiovascular networks, $d = D = 3$.
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- Sink density must ∴ decrease as volume increases:

$$\rho \propto V^{-1/d}.$$

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- For cardiovascular networks, $d = D = 3$.
- Know that volume of blood scales linearly with blood volume [9], $V_{\mathrm{net}} \propto V$.
- Sink density must $\therefore$ decrease as volume increases:

$$\rho \propto V^{-1/d}.$$

- Density of suppliable sinks decreases with organism size.

Blood networks

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- For $d = 3$ dimensional organisms, we have

$$\boxed{P \propto M^{2/3}}$$

Outline

River networks

- View river networks as collection networks.
- Many sources and one sink.
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- It's all okay:
  Landscapes are d=2 surfaces living in D=3 dimension.
- Streams can grow not just in width but in depth...

Hack's law

- Volume of water in river network can be calculated by adding up basin areas
- Flows sum in such a way that

$$V_{\mathrm{net}} = \sum_{\text{all pixels}} a_{\text{pixel } i}$$

- Hack's law again:

$$\ell \sim a^h$$

- Can argue

$$V_{\mathrm{net}} \propto V_{\mathrm{basin}}^{1+h} = a_{\mathrm{basin}}^{1+h}$$

  where h is Hack's exponent.
- $\therefore$ minimal volume calculations gives

$$\boxed{h = 1/2}$$

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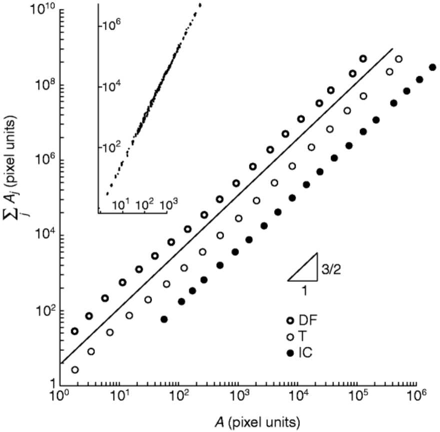

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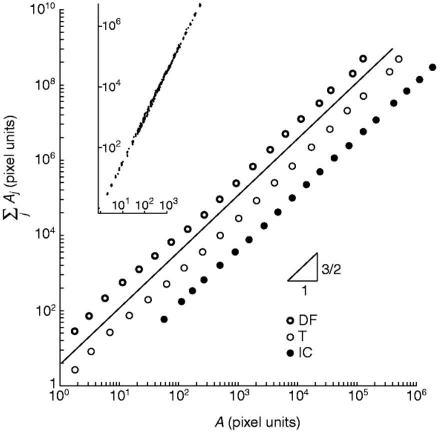

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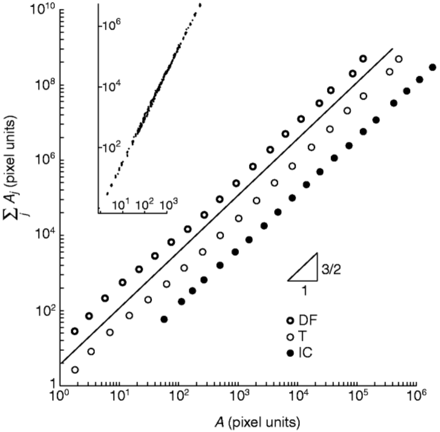

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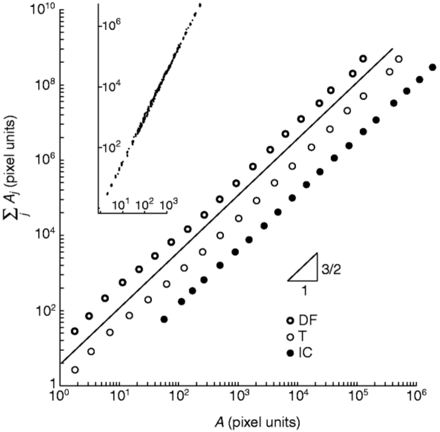

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