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CONVECTION, OPTIMAL TRANSPORT, COUPLED MONGE-AMPERE SYSTEMS AND MAGNETIC RELAXATION MODELS

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A NAVIER-STOKES BOUSSINESQ 'NSB' MODEL

Let D be a smooth bounded domain $\mathbf{D} \subset \mathbf{R}^3$ in which moves an incompressible fluid of velocity $\mathbf{v(t,x)}$ at $\mathbf{x} \in \mathbf{D}$, $\mathbf{t} \geq \mathbf{0}$, subject to:

$$\mathbf{NSB}: \quad \epsilon(\partial_{\mathbf{t}}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \mathbf{Kv} + \nabla\mathbf{p} = \mathbf{y} \qquad \nabla\cdot\mathbf{v} = \mathbf{0}$$

where $\boxed{\mathbf{Kv} = \alpha\mathbf{v} - \nu\boldsymbol{\Delta}\mathbf{v}}$ **with** $\alpha \geq \mathbf{0}$, $\epsilon > \mathbf{0}$, $\nu > \mathbf{0}$ **and** $\mathbf{v} = \mathbf{0}$ **along** $\partial\mathbf{D}$.

The force field y is subject to the advection equation

$$\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G(x,y)}$$

where G is a given smooth function with bounded derivatives.

CONVECTION THEORY

CLASSICAL CONVECTION THEORY corresponds to the special case

$$\mathbf{K} = -\mathbf{\Delta}, \quad \mathbf{G} = \mathbf{0}, \quad \mathbf{y} // \mathbf{e_3}, \quad \mathbf{y} = \eta \mathbf{e_3}, \quad \eta = \eta(\mathbf{t}, \mathbf{x}) \in \mathbf{R}$$

namely:

$$\epsilon(\partial_{\mathbf{t}} \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \mathbf{\Delta v} + \nabla \mathbf{p} = \eta \mathbf{e_3}, \quad \nabla \cdot \mathbf{v} = \mathbf{0}, \quad \partial_{\mathbf{t}} \eta + (\mathbf{v} \cdot \nabla)\eta = \mu \mathbf{\Delta} \eta$$

with $\mu \geq \mathbf{0}$.

For $\mu = \mathbf{0}$, global existence of weak solutions in 3D follows from Leray/Diperna-Lions theory, while global existence of smooth solutions in 2D follows from Hou-Li 2005 and Chae 2006. (See also recent work by Danchin-Paicu.)

THREE LIMITS OF THE NS BOUSSINESQ MODEL

While keeping unchanged

$$\partial_t \mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y}) \qquad \nabla\cdot\mathbf{v} = \mathbf{0}$$

and dropping inertia terms, we consider three possible limit regimes:

STOKES – BOUSSINESQ SB : $\epsilon = \alpha = \mathbf{0},\ \nu = \mathbf{1} \quad \Rightarrow \quad -\Delta\mathbf{v} + \nabla\mathbf{p} = \mathbf{y}$

(the limit $\epsilon \to 0$ can be rigorously justified, YB 2007)

DARCY – BOUSSINESQ DB : $\epsilon = \nu = \mathbf{0},\ \alpha = \mathbf{1} \quad \Rightarrow \quad \mathbf{v} + \nabla\mathbf{p} = \mathbf{y}$

(the limit $\epsilon \to 0$ can be rigorously justified, YB 2007)

HYDROSTATIC – BOUSSINESQ HB : $\epsilon = \nu = \alpha = \mathbf{0} \quad \Rightarrow \quad \nabla\mathbf{p} = \mathbf{y}$

(here the rigorous justification of the limit $\epsilon \to 0$ seems widely open!)

THE ANGENENT-HAKER-TANNENBAUM MODEL

As $G = 0$, the Darcy-Boussinesq and Stokes-Boussinesq models coincide with the Angenent-Haker-Tannenbaum model

$$\mathbf{AHT}: \quad \partial_t y + (v \cdot \nabla) y = 0, \quad Kv + \nabla p = y, \quad \nabla \cdot v = 0$$

We (formally) get

$$\int_D |y(t,x) - x|^2 dx + 2 \int_0^t \int_D (Kv \cdot v)(\theta, x) \, dx d\theta = \int_D |y(0,x) - x|^2 dx$$

So, we expect, as $t \to +\infty$, $v \to 0$, $(y, p) \to (y^\infty, p^\infty)$, so that

$$y^\infty = \nabla p^\infty$$

is a curl-free 'rearrangement' of the given initial vector field $y(t = 0, \cdot)$.
This was the original goal of the AHT model,
in order to solve the polar factorization problem arising in optimal transport theory (cf. Y.B. CPAM 1991).
Existence of local or global smoth solutions is proven by Agenent-Haker-Tannenbaum in SIAM J. Math. Analysis 2003.

THE HYDROSTATIC BOUSSINESQ (HB) EQUATIONS
1) A CONVEXITY PRINCIPLE

The Hydrostatic Boussinesq 'HB' system

$$\mathbf{HB}: \quad \partial_t y + (v \cdot \nabla) y = G(x, y), \quad \nabla \cdot v = 0, \quad \nabla p = y$$

looks strange since there is no direct equation for v.
Let us consider, for simplicity, the case of 2 space variables $x = (x_1, x_2)$ and write $v = (-\partial_1 \theta, \partial_2 \theta)$, where $\theta(t, x_1, x_2)$ is a 'stream function' for v. Taking the 2D curl of the evolution equation in $y = (\partial_1 p, \partial_2 p)$, we find:

$$\partial_{11} p \, \partial_{22} \theta + \partial_{22} p \, \partial_{11} \theta - 2 \partial_{12} p \, \partial_{12} \theta = \partial_1 (G_2(x, \nabla p)) - \partial_2 (G_1(x, \nabla p))$$

which is a well posed

LINEAR ELLIPTIC EQUATION IN θ WHEN $D^2_x p(t, x) > 0$

So, a natural **SOLVABILITY CONDITION FOR THE HB SYSTEM** is

$p(t, x)$ is a CONVEX function of $x \in D$.

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THE HYDROSTATIC BOUSSINESQ (HB) EQUATIONS
2) EVOLUTION OF 'OBSERVABLES' IN BOUSSINESQ SYSTEMS

For each suitable test function f, we define the 'observable'

$$\mathbf{t} \to \rho_{\mathbf{f}}(\mathbf{t}) = \int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

where y is solution to one of the Boussinesq systems **NSB,SB,DB,HB**

Since $\partial_{\mathbf{t}}\mathbf{y} + (\mathbf{v}\cdot\nabla)\mathbf{y} = \mathbf{G}(\mathbf{x},\mathbf{y})$ where $\nabla\cdot\mathbf{v} = \mathbf{0}, \quad \mathbf{v}//\partial\mathbf{D},$

we always get, for each suitable test function f,

$$\frac{\mathbf{d}}{\mathbf{dt}}\int_{\mathbf{D}} \mathbf{f}(\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx} = \int_{\mathbf{D}} (\nabla\mathbf{f})(\mathbf{y}(\mathbf{t},\mathbf{x}))\cdot\mathbf{G}(\mathbf{x},\mathbf{y}(\mathbf{t},\mathbf{x}))\mathbf{dx}$$

THE HYDROSTATIC BOUSSINESQ (HB) EQUATIONS
3) 'OBSERVABLES' AND THE CONVEXITY PRINCIPLE

The Hydrostatic Boussinesq 'HB' model
requires the field y to be a gradient $y = \nabla p$

Under the A PRIORI CONVEXITY ASSUMPTION

$p(t, x)$ is a CONVEX function of $x \in D$ (D being supposed to be convex),

the field y is COMPLETELY DETERMINED
by the knowledge of all observables $t \rightarrow \rho_f(t) = \int_D f(y(t, x))dx$

NB: This is a typical result of OPTIMAL TRANSPORT THEORY

YB, CRAS Paris 1987 and CPAM 1991, Smith and Knott, JOTA 1987, cf. Villani, Topics in optimal transportation, AMS, 2003, see also papers, lecture notes and books by Rachev-Rüschendorf, Evans, Caffarelli, Urbas, Gangbo-McCann, Otto, Ambrosio-Savaré, Villani, Trudinger-Wang and many others contributions...

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THE HYDROSTATIC BOUSSINESQ (HB) EQUATIONS
4) A GLOBAL EXISTENCE RESULT

YB 2007, inspired by G. Loeper, SIAM J. Math. Anal. 2006

THEOREM

Assume $G(x,y)$ to be a smooth function with bounded first derivatives.
Let C be the convex cone of all maps $y \in L^2(D, R^3)$
such that $y(x) = \nabla p(x)$ a.e. in D for some CONVEX convex lsc function p.

Then, for each $y_0 \in C$, there is $(t \to y(t,\cdot)) \in C^0(R_+, L^2(D, R^3))$
valued in the cone C such that $y(t=0,\cdot) = y_0$ and

$$\frac{d}{dt}\int_D f(y(t,x))dx = \int_D (\nabla f)(y(t,x)) \cdot G(x, y(t,x))dx,$$

(for all test functions f) which we call a
SOLUTION WITH CONVEX POTENTIAL TO THE HB SYSTEM

$$\partial_t y + (v \cdot \nabla)y = G(x,y), \quad \nabla \cdot v = 0, \quad y = \nabla p$$

SOME COUPLED MONGE-AMPERE SYSTEMS
1) DERIVATION FROM THE HB SYSTEM

Under the **POTENTIAL CONVEXITY** assumption, the HB system

$$\mathbf{HB}: \quad \partial_t y + (v \cdot \nabla) y = G(x, y), \qquad y = \nabla p, \qquad \nabla \cdot v = 0$$

is (formally) equivalent to the coupled Monge-Ampère system

$$\mathbf{CMA}: \quad \partial_t \rho + \nabla \cdot (\rho w) = 0, \quad w = G(\nabla p^*(t, x), x), \quad \rho = \det(D^2 p^*(t, x))$$

where p^* is the **LEGENDRE-FENCHEL** transform

$$p^*(t, x) = \sup_{\tilde{x} \in D} \; x \cdot \tilde{x} - p(t, \tilde{x})$$

Indeed, using the change of variable $x = \nabla p(t, \tilde{x}) \iff \tilde{x} = \nabla p^*(t, x)$,

$$\frac{d}{dt} \int f(x) \det(D^2 p^*(t, x)) dx - \int \nabla f(x) \cdot G(\nabla p^*(t, x), x) \det(D^2 p^*(t, x)) dx$$

$$= \frac{d}{dt} \int_D f(y(t, \tilde{x})) d\tilde{x} - \int_D (\nabla f)(y(t, \tilde{x})) \cdot G(\tilde{x}, y(t, \tilde{x})) d\tilde{x} = 0$$

SOME COUPLED MONGE-AMPERE SYSTEMS
2) TWO EXAMPLES

Example 1: Setting $G(x, y) = (y_2 - x_2, x_1 - y_1, 0)$ we recover Hoskins' SEMI-GEOSTROPHIC equations.

Then, the CONVEXITY PRINCIPLE for the HB system EXACTLY corresponds to the CULLEN-PURSER PRINCIPLE.

cf. Cullen-Norbury-Purser 1991, Benamou-Brenier 1998, Cullen-Gangbo 2001, Loeper 2006.

Example 2: With $G(x, y) = \frac{y-x}{\beta}$ where $\beta > 0$ is a constant, Setting

$$\boxed{\nabla p^*(t, x) = x - \beta \nabla \psi(t, x)}$$ we get

$$\boxed{\text{CMA}: \ \partial_t \rho + \nabla \cdot (\rho w) = 0, \quad w = \nabla \psi(t, x), \quad \rho = \det(I - \beta D^2 \psi(t, x))}$$

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SOME COUPLED MONGE-AMPERE SYSTEMS
3) FULLY NONLINEAR CHEMOTAXIS

The resulting system can be seen as a FULLY NON-LINEAR CHEMOTAXIS model.

Indeed, Assuming $|\beta| << 1$, the MONGE-AMPERE becomes

$$\rho = \mathbf{det}(\mathbf{I} - \beta \mathbf{D}^2 \psi(\mathbf{t}, \mathbf{x})) = \mathbf{1} - \beta \boldsymbol{\Delta} \psi + \mathbf{O}(\beta^2)$$

which approximates the CHEMOTAXIS model (without viscosity) considered by Jäger and Luckhaus Trans. AMS 1992:

$$\partial_{\mathbf{t}} \rho + \nabla \cdot (\rho \mathbf{w}) = \mathbf{0}, \quad \mathbf{w} = \nabla \psi(\mathbf{t}, \mathbf{x}), \quad \rho = \mathbf{1} - \beta \boldsymbol{\Delta} \psi(\mathbf{t}, \mathbf{x}))$$

SOME COUPLED MONGE-AMPERE SYSTEMS
4) CONVEXITY PRINCIPLE AND ENTROPY CONDITION

In one space variable the approximation is exact:

$$\partial_t \rho + \partial_x(\rho \mathbf{w}) = 0, \quad \rho = 1 - \beta \partial_x \mathbf{w}$$

In that case, the system can be reduced to the inviscid BURGERS equation

$$\partial_t \mathbf{w} + \partial_x(\mathbf{w}^2/2) = \frac{\mathbf{w}}{\beta}$$

and the Kruzhkov-Oleinik ENTROPY CONDITION condition EXACTLY fits with the CONVEXITY PRINCIPLE we used for the HB system.

A STRINGY DARCY-BOUSSINESQ MODEL:
1) THE DB SYSTEM IN LAGRANGIAN COORDINATES

The Darcy-Boussinesq DB system reads

$$\partial_t y + (v \cdot \nabla) y = G(x, y), \quad v + \nabla p = y, \quad \nabla \cdot v = 0$$

Let us introduce the time dependent volume preserving map

$a \in D \Rightarrow X(t, a) \in D$ defined by $\partial_t X(t, a) = v(t, X(t, a)), \;\; X(t = 0, a) = a$

Setting $Y(t, a) = y(t, X(t, a)),$ we get

$$\partial_t Y(t, a) = G(X(t, a), Y(t, a)), \quad \partial_t X(t, a) + (\nabla p)(t, X(t, a)) = Y(t, a)$$

A STRINGY DARCY-BOUSSINESQ MODEL: 2) THE DB/AHT SYSTEM AS A GRADIENT FLOW

In the special case $G = 0$, the DB system can be interpreted also as a Angenent-Haker-Tannenbaum AHT model. In Lagrangian coordinates, we get $Y(t, a) = Y(t = 0, a) = Y_0(a)$ (since $G = 0$) and, therefore,

$$\partial_t X(t, a) + (\nabla p)(t, X(t, a)) = Y_0(a)$$

Following Angenent-Haker-Tannenbaum, this model should be understood as the GRADIENT FLOW of the functional

$$X \in VPM(D) \to \frac{1}{2} \int_D |X(a) - Y_0(a)|^2 da$$

where VPM(D) is the set of all VOLUME PRESERVING MAPS of D embedded in the Hilbert space $L^2(D, R^d)$ and p is the Lagrange multiplier of the volume preserving constraint.

A STRINGY DARCY-BOUSSINESQ MODEL:
3) STRINGY GENERALIZATION OF THE DB MODEL

A natural generalization of the DB model amounts to consider the set of 'strings' of volume preserving maps

$$\mathbf{X}: \ \mathbf{s} \in [\mathbf{0}, \mathbf{1}] \to \mathbf{X}(\mathbf{s}, \cdot) \in \mathbf{VPM}(\mathbf{D})$$

and the corresponding gradient flow for the Dirichlet functional

$$\mathbf{J}[\mathbf{X}] = \frac{1}{2} \int_0^1 \int_D |\partial_s \mathbf{X}(s, a)|^2 \mathbf{dads}$$

The resulting equation reads

$$\partial_t \mathbf{X}(t, s, a) + (\nabla p)(t, s, \mathbf{X}(t, s, a)) = \partial_{ss} \mathbf{X}(t, s, a)$$

where p is a Lagrange multiplier for the incompressibility constraint. The (formal) energy balance is:

$$\frac{d}{dt} \int_0^1 \int_D |\partial_s \mathbf{X}(t, s, a)|^2 \mathbf{dads} = -2 \int_0^1 \int_D |\partial_t \mathbf{X}(t, s, a)|^2 \mathbf{dads}$$

A STRINGY DB MODEL

4) EQUATIONS WRITTEN IN EULERIAN COORDINATES

Going back to Eulerian coordinates by setting

$$\partial_t \mathbf{X}(t,s,a) = \mathbf{v}(t,s,\mathbf{X}(t,s,a)), \quad \partial_s \mathbf{X}(t,s,a) = \mathbf{b}(t,s,\mathbf{X}(t,s,a)),$$

we get two differential constraints (since $\mathbf{X}(t,s,\cdot)$ is volume preserving) and a compatibility condition (using $\partial_t\partial_s \mathbf{X} = \partial_s\partial_t \mathbf{X}$):

$$\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = 0\ , \quad \partial_t \mathbf{b} + (\mathbf{v}\cdot\nabla)\mathbf{b} = \partial_s \mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v}$$

Then, the equation $\partial_t \mathbf{X}(t,s,a) + (\nabla p)(t,s,\mathbf{X}(t,s,a)) = \partial_{ss}\mathbf{X}(t,s,a)$

reads, in Eulerian coordinates, $\mathbf{v} + \nabla p = \partial_s \mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{b}$

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A STRINGY DB MODEL

5) INTERPRETATION AS A MAGNETIC RELAXATION MODEL

We can see the stringy DB model, written in Eulerian coordinates,

$$\mathbf{v} + \nabla \mathbf{p} = \partial_{\mathbf{s}} \mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{b} \,, \quad \partial_{\mathbf{t}} \mathbf{b} + (\mathbf{v} \cdot \nabla)\mathbf{b} = \partial_{\mathbf{s}} \mathbf{v} + (\mathbf{b} \cdot \nabla)\mathbf{v} \,, \quad \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{b} = \mathbf{0}$$

as a MAGNETIC RELAXATION MODEL A LA ARNOLD-MOFFATT (see Arnold-Khesin, Springer 1998, Moffatt JFM 1985)

The EQUILIBRIUM STATES formally obtained as $\mathbf{t} \to +\infty$

$$\mathbf{v}(\mathbf{t} = \infty, \mathbf{s}, \mathbf{x}) = \mathbf{0}, \quad \mathbf{b}(\mathbf{t} = \infty, \mathbf{s}, \mathbf{x}) = \mathbf{B}(\mathbf{s}, \mathbf{x}), \quad \mathbf{p}(\mathbf{t} = \infty, \mathbf{s}, \mathbf{x}) = \mathbf{P}(\mathbf{s}, \mathbf{x})$$

are indeed solutions to the EULER EQUATIONS

$$\nabla \mathbf{P} = \partial_{\mathbf{s}} \mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{B}, \quad \nabla \cdot \mathbf{B} = \mathbf{0}$$

THE CROSS-BURGERS EQUATION
1) DERIVATION FROM THE STRINGY DB MODEL

In the special case when D is the unit ball of R^3, we find special solutions of the stringy DB model of form $\boxed{X(t,s,a) = U(t,s)a, \ \ \forall a \in D}$ where $U(t,s)$ is valued in THE ORTHOGONAL GROUP O(3) and subject to:

$$\boxed{\partial_t U(t,s) + S(t,s)U(t,s) = \partial_{ss} U(t,s)}$$

where each $S(t,s)$ is a real symmetric 3×3 matrix.
Introducing for each (t,s) the unique vector $B(t,s) \in R^3$ such that

$$\boxed{\partial_s U(t,s)a = B(t,s) \times (U(t,s)a), \ \ \forall a \in D}$$

we get for $B(t,s)$ what we call THE CROSS-BURGERS EQUATION

$$\boxed{\partial_t B(t,s) + B(t,s) \times \partial_s B(t,s) = \partial_{ss} B(t,s)}$$

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THE CROSS-BURGERS EQUATION
2) A MAGNETIC REVERSAL PHENOMENON 1

Special solutions of the CROSS-BURGERS EQUATION read

$$\mathbf{B}(t,s) = (f(t)\cos(2\pi s), f(t)\sin(2\pi s), g(t))$$

where

$$\frac{df}{dt} = 2\pi(g - 2\pi)f, \quad \frac{dg}{dt} = -2\pi f^2$$

For $g(t=0) > 4\pi, \quad f(t=0) \neq 0$, we can check that
$g(t=+\infty) = 4\pi - g(0) < 0, \quad f(t=+\infty) = 0$, even when $|f(t=0)| << 1$.

This looks like a magnetic reversal

THE CROSS-BURGERS EQUATION

3) A MAGNETIC REVERSAL PHENOMENON 2

$$\mathbf{g(0) = 36.2830925, \quad f(0) = 0.075, \quad g(+\infty) = -23.7167}$$

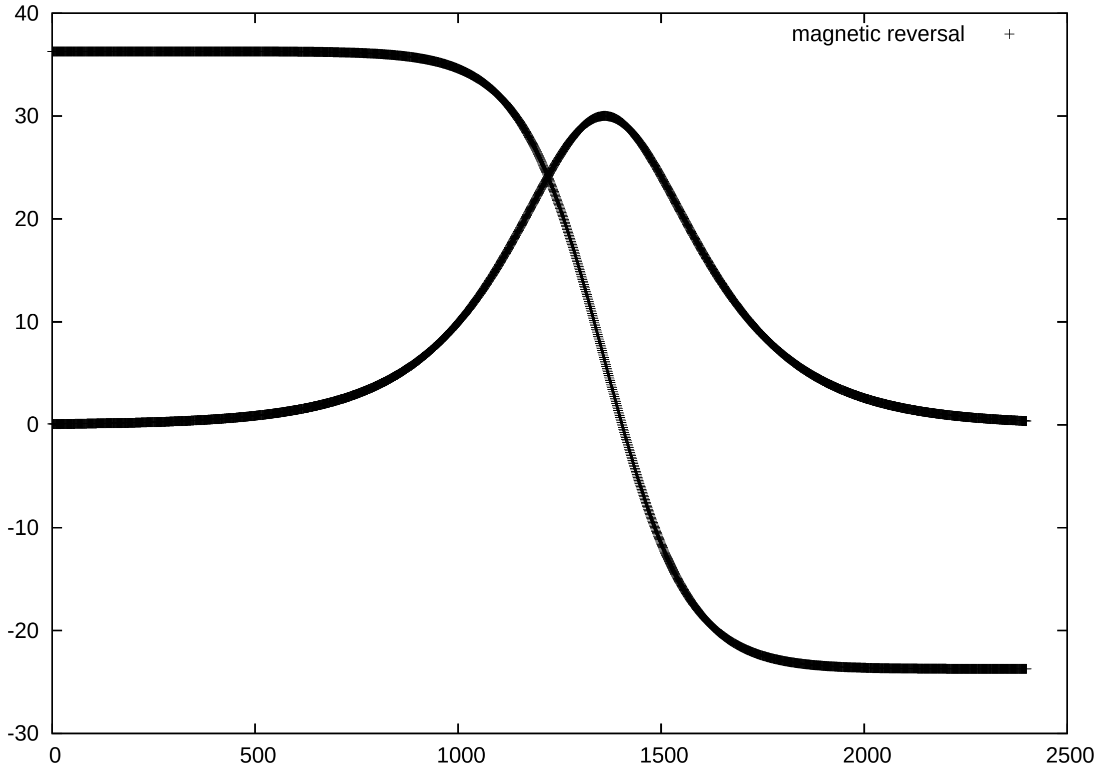

Figure 1: g(t) and f(t) versus t