

# Geodesic evolution laws - a level set approach

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# Outline

PDEs on surfaces

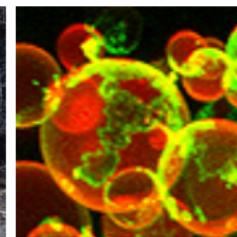
Geometric evolution laws

Geodesic evolution laws

# Motivation

## How to solve problems on surfaces?

- ▶ computer graphics (weathering, ...)
- ▶ image processing (texture synthesis, ...)
- ▶ geometry (splines, geodesics, ...)
- ▶ biophysics (pattern formation, biomembranes, ...)
- ▶ materials science (thin films, colloidal materials, ...)
- ▶ ...



# Motivation

## Example

$$u_t - \Delta_\Gamma u = f \quad \text{on } \Gamma \times I$$

- ▶ parametric finite elements
- ▶ level set approach
- ▶ phase field approach
- ▶ ...

# Parametric finite elements

$$u_t - \Delta_\Gamma u = f \quad \text{on } \Gamma \times I$$

solve on triangulated surface

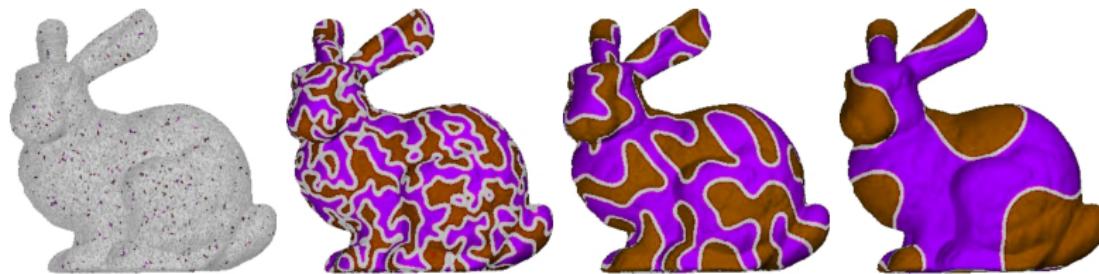
Dziuk. Num. Math. (1991), Dziuk, Elliott. IMA J. Appl. Math. (2006)

# Cahn-Hilliard equation on surface

$$u_t = \nu \Delta_\Gamma \mu$$

$$\mu = -\gamma \Delta_\Gamma u + \gamma^{-1} G'(u) + \alpha \gamma u_t$$

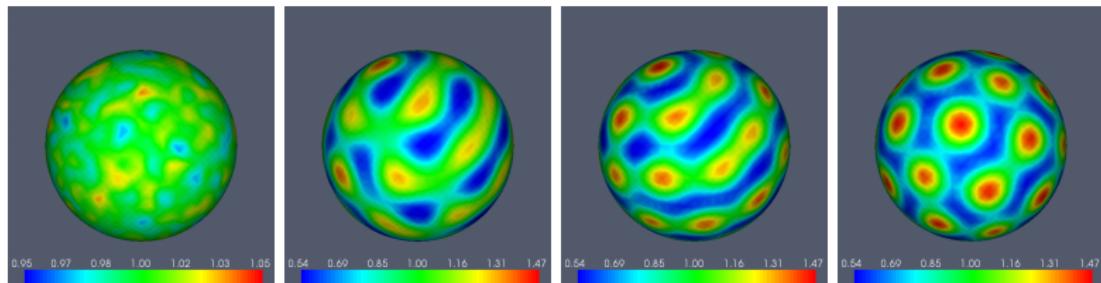
$$G(u) = 18u^2(1 - u^2)$$



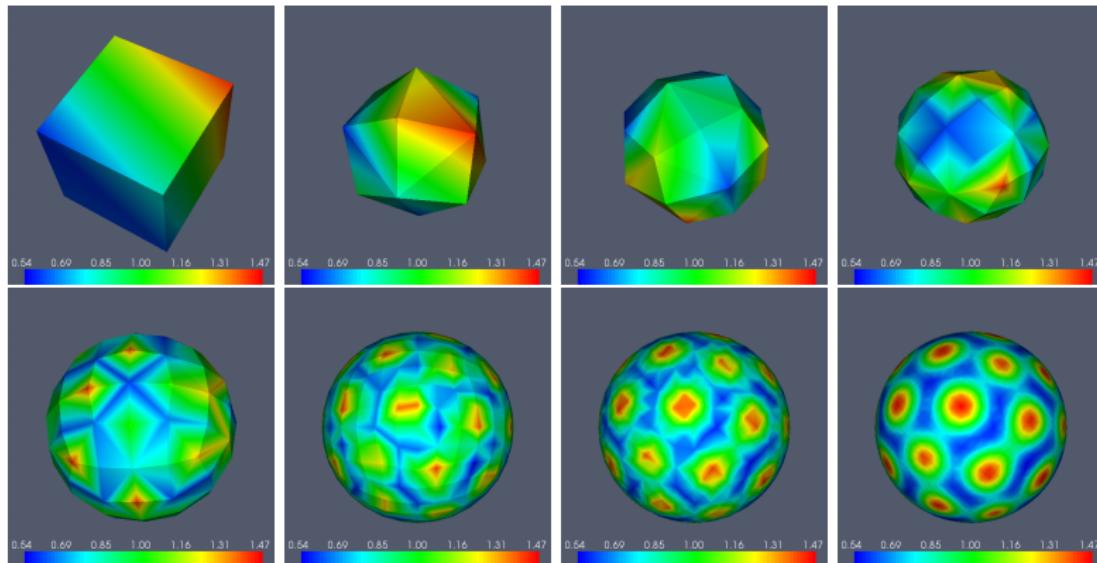
# Reaction-diffusion equation - Schnakenberg model

$$\begin{aligned}\partial_t u &= \Delta_\Gamma u + \gamma f(u, v) \\ \partial_t v &= d\Delta_\Gamma v + \gamma g(u, v)\end{aligned}$$

$$f(u, v) = a - u + u^2v, \quad g(u, v) = b - u^2v$$



# Multigrid solver



Landsberg, Voigt (in review)

# Level set approach

$$u_t - \Delta_\Gamma u = f \quad \text{on } \Gamma \times I$$

$$\tilde{u}_t - \frac{1}{|\nabla\phi|} \nabla \cdot (|\nabla\phi| P \nabla \tilde{u}) = \tilde{f} \quad \text{in } \Omega \times I$$

$$P = I - \frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}$$

Bertalmio, Cheng, Osher, Sapiro. J. Comput. Phys. (2001)

# Phase field approach

$$u_t - \Delta_\Gamma u = f \quad \text{on } \Gamma \times I$$

$$B(\phi)\tilde{u}_t - \nabla \cdot (B(\phi)\nabla \tilde{u}) = B(\phi)\tilde{f} \quad \text{in } \Omega \times I$$

$$B(\phi) = \phi^2(1-\phi)^2, \quad \phi(x) = \frac{1}{2}(1 - \tanh(\frac{3}{\epsilon}(|x| - 1)))$$

Rätz, AV. Comm. Math. Sci. (2006)

# general second order PDE

$$u_t - \nabla_\Gamma \cdot (A \nabla_\Gamma u) + b \cdot \nabla_\Gamma u + cu = f \quad \text{on } \Gamma \times I$$

$$A = A(u, \nabla_\Gamma u, x, t), b = b(u, \nabla_\Gamma u, x, t), c = c(u, \nabla_\Gamma u, x, t)$$

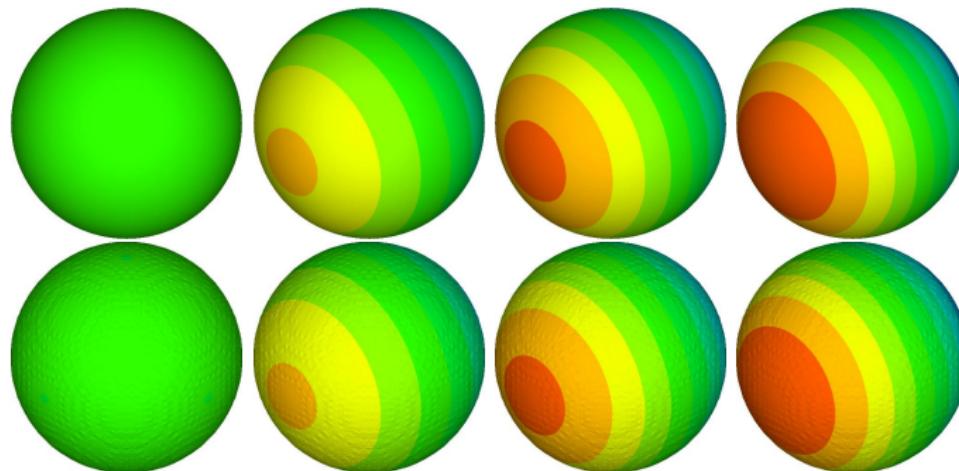
$$B(\phi)\tilde{u}_t - \nabla \cdot (B(\phi)\tilde{A} \nabla \tilde{u}) + B(\phi)\tilde{b} \cdot \nabla \tilde{u} + B(\phi)\tilde{c}\tilde{u} = B(\phi)\tilde{f} \quad \text{in } \Omega \times I$$

$\tilde{A}, \tilde{b}, \tilde{c}$  extensions

# Benchmark

$$u_t - \Delta_\Gamma u = f \quad \text{on } \Gamma \times I, \quad f(x) = 2x_1$$

$$B(\phi)\tilde{u}_t - \nabla \cdot (B(\phi)\nabla \tilde{u}) = B(\phi)\tilde{f} \quad \text{in } \Omega \times I, \quad \tilde{f}(x) = \frac{2x_1}{|x|^2}$$



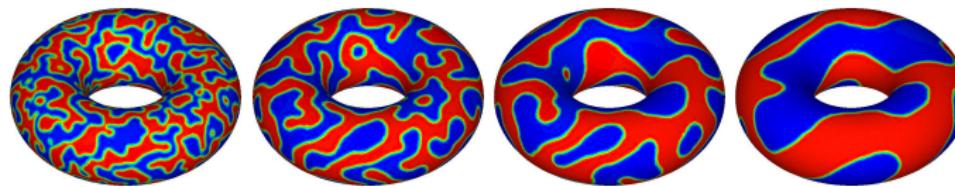
# Cahn-Hilliard equation on surface

## Phase field approximation

$$B(\phi)\tilde{u}_t = \nu \nabla \cdot (B(\phi) \nabla \tilde{\mu})$$

$$B(\phi)\tilde{\mu} = -\gamma \nabla \cdot (B(\phi) \nabla \tilde{u}) + \gamma^{-1} B(\phi) G'(\tilde{u}) + \alpha \gamma B(\phi) \tilde{u}_t$$

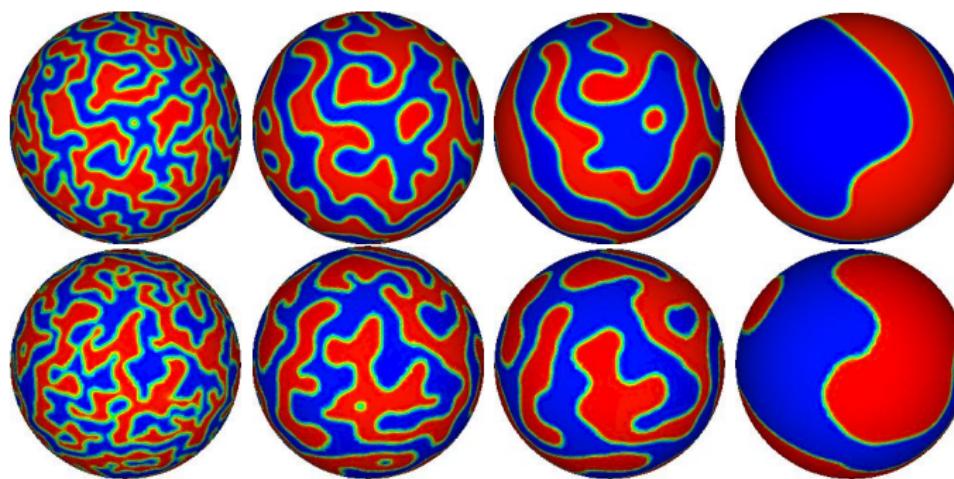
$$G(u) = 18u^2(1-u^2), B(\phi) = \phi^2(1-\phi)^2$$



# Cahn-Hilliard equation on surface

## Comparison

parametric finite elements - phase field approach



# Cahn-Hilliard equation on surface

Gradient flow of free energy

$$E(u) = \int_{\Omega} \gamma^{-1} B(\phi) G(u) + \gamma \frac{1}{2} B(\phi) |\nabla u|^2$$

$$\partial_t E(u) \leq 0$$

# Mean curvature flow

## numerical approaches

$$\nu = -\kappa$$

- ▶ parametric finite elements
- ▶ phase field method
- ▶ level set formulation

$$u_t + \nu |\nabla u| = 0, \quad \nu = -\kappa = -\nabla \cdot \frac{\nabla u}{|\nabla u|}$$

# Surface diffusion

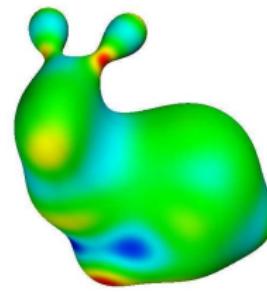
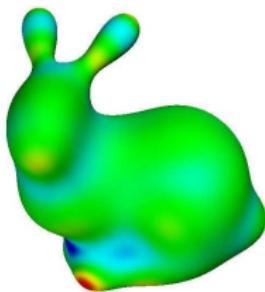
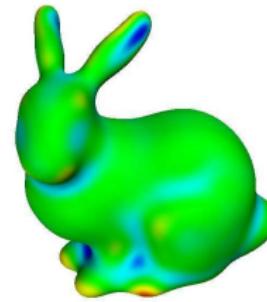
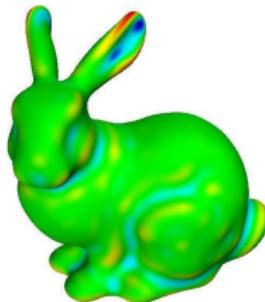
## numerical approaches

$$v = \Delta_\Gamma \kappa$$

- ▶ parametric finite elements
- ▶ phase field method
- ▶ level set formulation

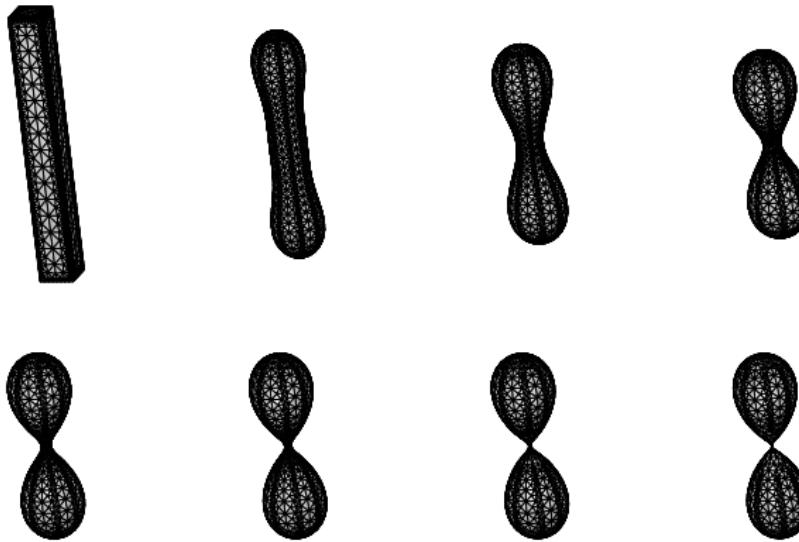
$$u_t + v|\nabla u| = 0, \quad v = \Delta_\Gamma \kappa = \frac{1}{|\nabla u|} \nabla \cdot (|\nabla u| P_\Gamma \nabla \kappa)$$

# Surface diffusion - parametric finite elements



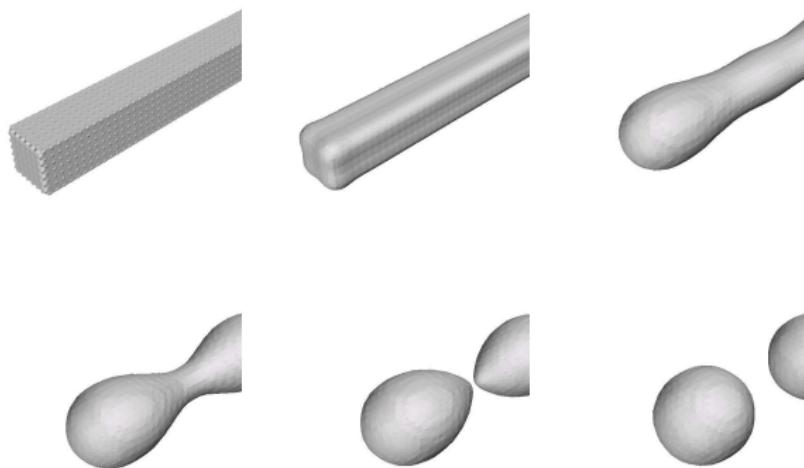
Vey, Voigt, Compu. Vis. Sci. (2007)

# Surface diffusion - parametric finite elements



Bänsch, Morin, Nochetto, J. Comput. Phys. (2006)

# Surface diffusion - phase field approximation



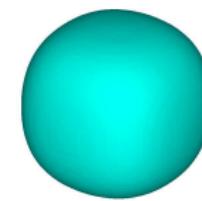
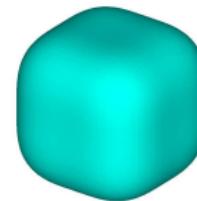
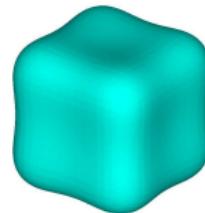
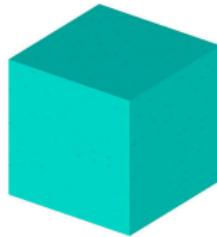
Rätz, Ribalta, Voigt, J. Comput. Phys. (2006), Rätz PhD Thesis (2007)

**error in pinch-off time 7.8 %**

# Surface diffusion - level set method

## weak formaultion

$$\begin{aligned} - \int_{\Omega} \frac{\phi^{m+1} - \phi^m}{\tau_m} \psi \, d\Omega &= - \int_{\Omega} \nu \|\nabla \phi^n\| \mathbf{P}_{\nabla \phi} \nabla \mu^{m+1} \cdot \nabla \psi \, d\Omega \\ \int_{\Omega} \mu^{m+1} \psi \, d\Omega &= - \int_{\Omega} \frac{\nabla \phi^{m+1}}{\|\nabla \phi^m\|} \cdot \nabla \psi \, d\Omega \end{aligned}$$



Burger, Haußer, Stöcker, Voigt, J. Comput. Phys. (2007)

# Level set formulation

Evolution equations on surf.

$$u_t + v_g |\nabla_\Gamma u| = 0$$

Geodesic mean curvature flow:

$$v_g = -\kappa_g = -\nabla_\Gamma \cdot \frac{\nabla_\Gamma u}{|\nabla_\Gamma u|}$$

Geodesic surface diffusion:

$$v_g = \Delta_g \kappa_g$$

$$= \frac{1}{|\nabla_\Gamma u|} \nabla_\Gamma \cdot (|\nabla_\Gamma u| P_g \nabla_\Gamma \kappa_g)$$

$\Delta_g$ : "geodesic" Laplacian

Evolution equations in  $\mathbb{R}^2$

$$u_t + v |\nabla u| = 0$$

Mean curvature flow:

$$v = -\kappa = -\nabla \cdot \frac{\nabla u}{|\nabla u|}$$

Surface diffusion:

$$v = \Delta_\Gamma \kappa$$

$$= \frac{1}{|\nabla u|} \nabla \cdot (|\nabla u| P_\Gamma \nabla \kappa)$$

$\Delta_\Gamma$ : surface Laplacian

# Level set formulation

Simplification through signed distance function  
assume  $\Gamma$  implicitly given by signed distance function

$$|\nabla\psi| = 1.$$

simplifications:

- ▶  $R = \mathcal{I} - \frac{\nabla\psi}{|\nabla\psi|} \otimes \frac{\nabla\psi}{|\nabla\psi|} = \mathcal{I} - \nabla\psi \otimes \nabla\psi$
- ▶  $\nabla_\Gamma \cdot F = \frac{1}{|\nabla\psi|} \nabla \cdot (|\nabla\psi| F) = \nabla \cdot F$

# Geodesic mean curvature flow

## Geodesic mean curvature flow

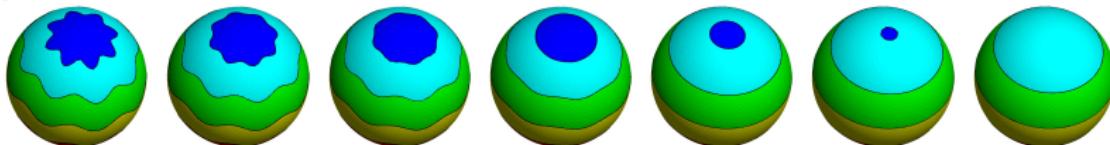
$$v_g = -\kappa_g = -\nabla_\Gamma \cdot \mathbf{n}_g = -\nabla_\Gamma \cdot \frac{\nabla_\Gamma u}{|\nabla_\Gamma u|} = -\nabla \cdot \frac{P_\Gamma \nabla u}{|P_\Gamma \nabla u|}$$

## Weak formulation for geodesic mean curvature flow

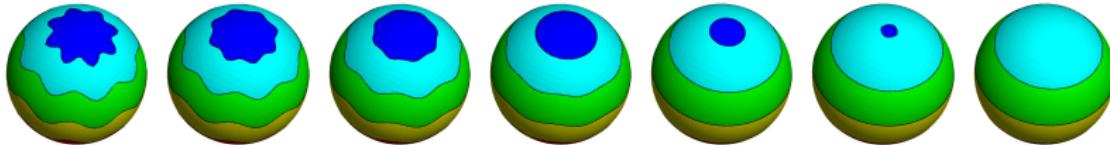
$$\int_{\Omega} \frac{u_t}{|P_\Gamma \nabla u|} \eta \, dx = - \int_{\Omega} \frac{P_\Gamma \nabla u}{|P_\Gamma \nabla u|} \cdot \nabla \eta \, dx$$

# Geodesic mean curvature flow

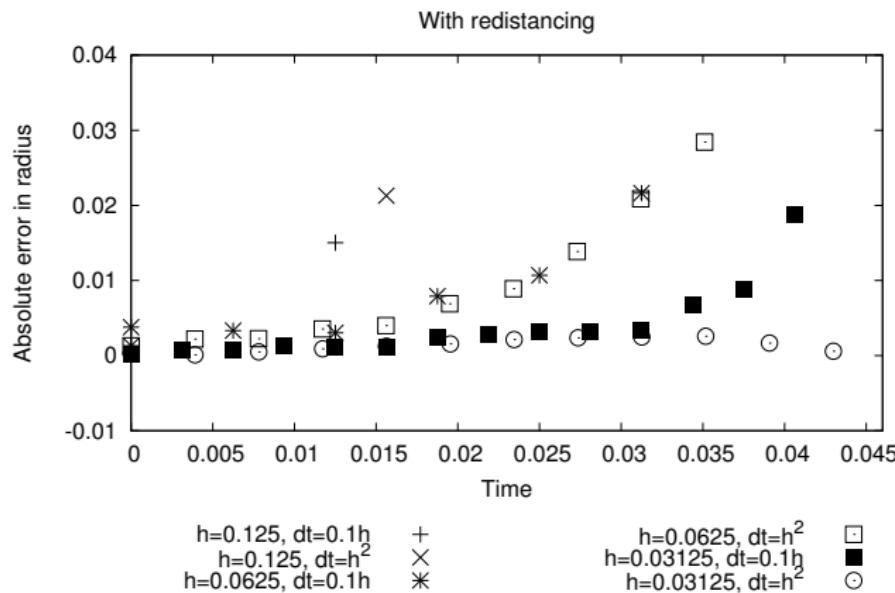
parametric finite elements



level set representation of surface

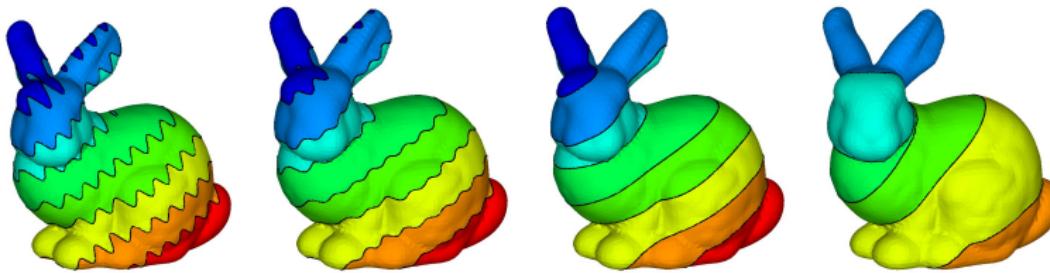


# Geodesic mean curvature flow



# Geodesic mean curvature flow

level set representation of surface



Stöcker, Voigt (in review)

# Geodesic surface diffusion

## Geodesic surface diffusion

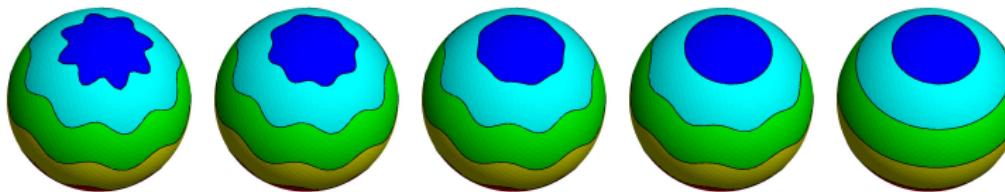
$$\begin{aligned} v_g &= \Delta_g \kappa_g = \frac{1}{|\nabla_\Gamma u|} \nabla_\Gamma \cdot (|\nabla_\Gamma u| P_g \nabla_\Gamma \kappa_g) \\ &= \frac{1}{|P_\Gamma \nabla u|} \nabla \cdot (|P_\Gamma \nabla u| P_g P_\Gamma \nabla \kappa_g) \end{aligned}$$

## Weak formulation for geodesic surface diffusion

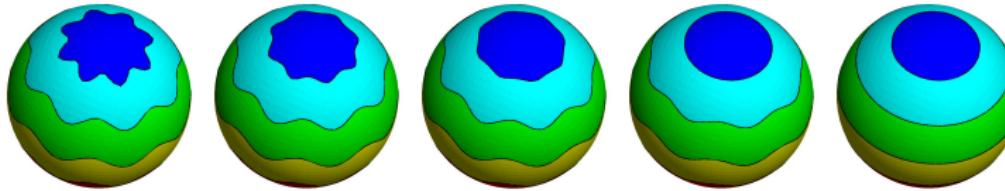
$$\begin{aligned} \int_{\Omega} u_t \eta \, dx &= \int_{\Omega} |P_\Gamma \nabla u| P_g P_\Gamma \nabla \omega \cdot \nabla \eta \, dx \\ \int_{\Omega} \omega \xi \, dx &= - \int_{\Omega} \frac{P_\Gamma \nabla u}{|P_\Gamma \nabla u|} \cdot \nabla \xi \, dx \end{aligned}$$

# Geodesic surface diffusion

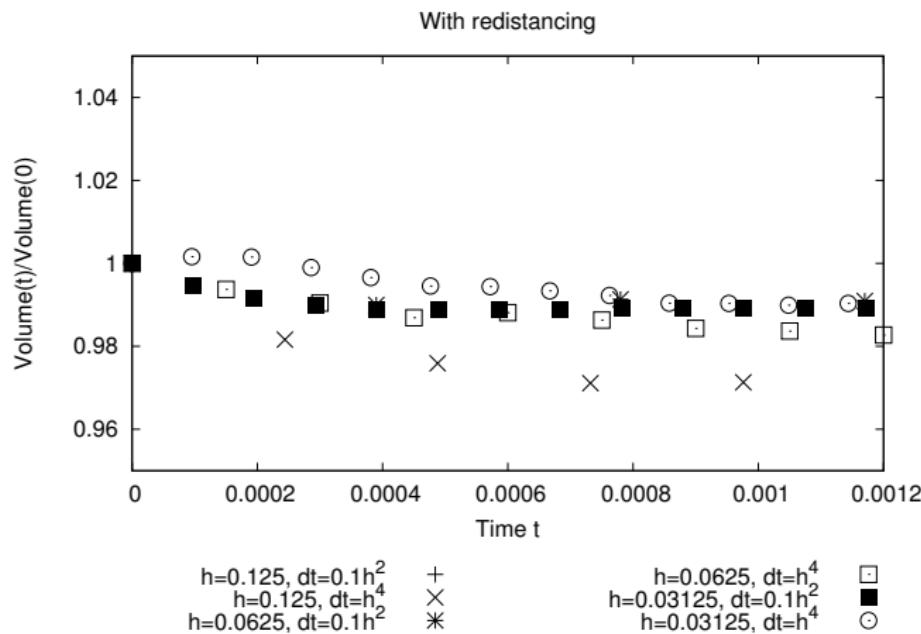
parametric finite elements



level set representation of surface

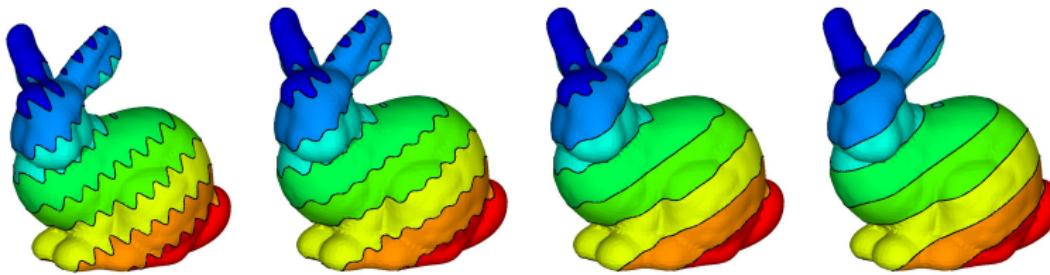


# Geodesic Surface diffusion



# Geodesic surface diffusion

level set representation of surface



Stöcker, Voigt (in review)

# Denoising using ROF

Rudin, Osher, Fatemi, Physica D (1992)

$$u_t = |\nabla u| \left( \nabla \cdot \frac{\nabla u}{|\nabla u|} - 2\lambda(u - u_0) \right)$$

applied on surfaces

$$u_t = |\nabla_{\Gamma} u| \left( \nabla_{\Gamma} \cdot \frac{\nabla_{\Gamma} u}{|\nabla_{\Gamma} u|} - 2\lambda(u - u_0) \right)$$

applied on implicitly defined surfaces

$$u_t = |(I - \nu \otimes \nu) \nabla u| \left( \frac{1}{|\nabla \phi|} \nabla \cdot (|\nabla \phi| \frac{(I - \nu \otimes \nu) \nabla u}{|(I - \nu \otimes \nu) \nabla u|} - 2\lambda(u - u_0)) \right)$$

# Denoising

parametric finite elements



level set representation of surface



# Denoising

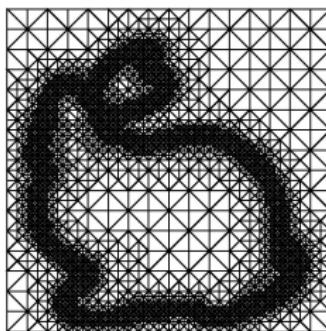
level set representation of surface



Stöcker, Voigt (in review)

## Conclusion

- ▶ parametric finite element are most efficient to solve PDEs on surface - 2d problem, reuse of code, multigrid
- ▶ level set and phase field representations are only efficient if adaptivity or narrow band approaches are used



all algorithms implemented in adaptive finite element toolbox

AMDiS  
adaptive multidimensional simulations