Convergent difference schemes for the Monge-Ampere equation, the Pucci equation, and the convex envelope

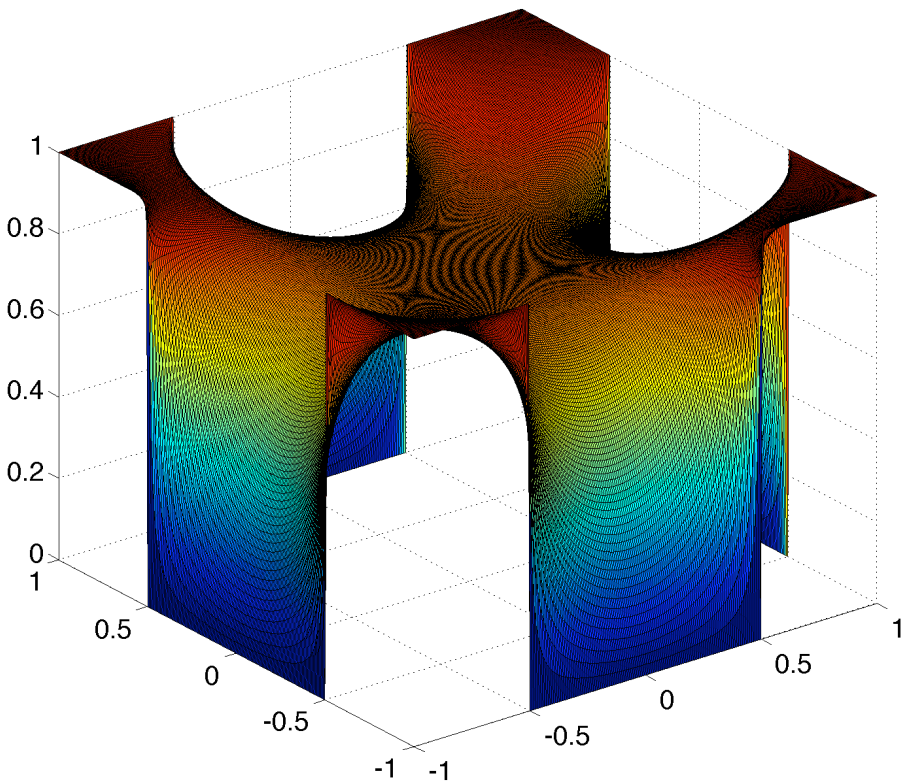
Adam Oberman
Simon Fraser University

Monge-Ampere

$$det(D^2u) = u_{xx}u_{yy} - u_{xy}^2 = f, \qquad \text{u convex}, f \geq 0$$

Fully Nonlinear Pucci Equation

$$\alpha\lambda_+ + \lambda_- = 0 \text{ in } \Omega,$$



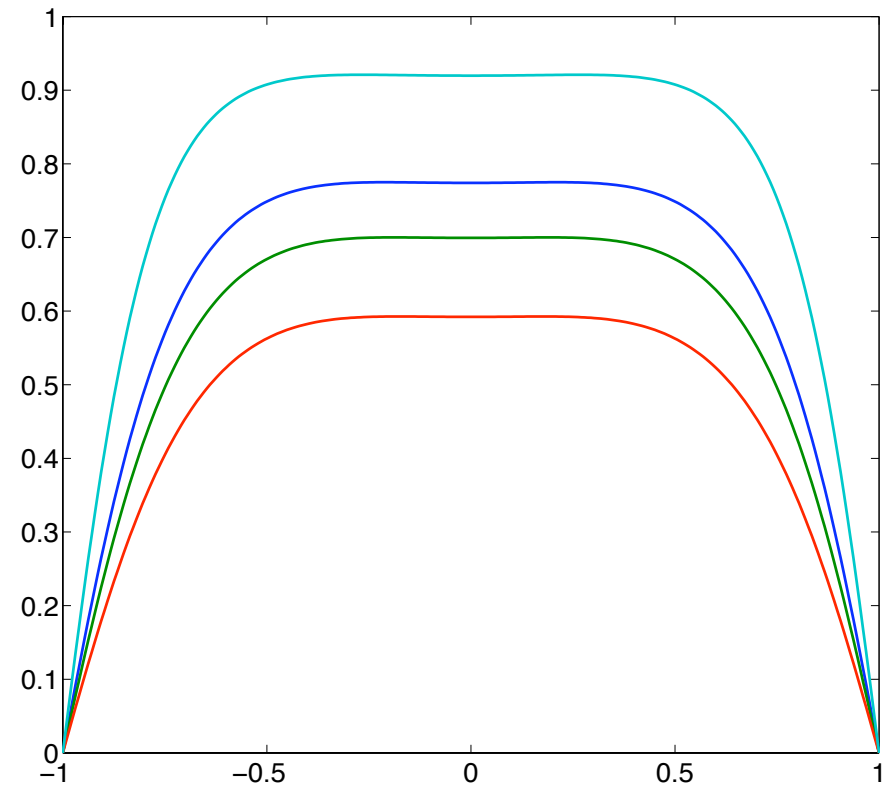


FIGURE 4. Surface plot of the Pucci solution, for $\alpha = 3$, $n = 256$. Plot of the midline of the solutions, increasing with $\alpha = 2, 2.5, 3, 5$, $n = 256$.

Obstacle problem for convex envelope

$$\max\{u(x) - g(x), -\lambda_1[u](x)\} = 0,$$

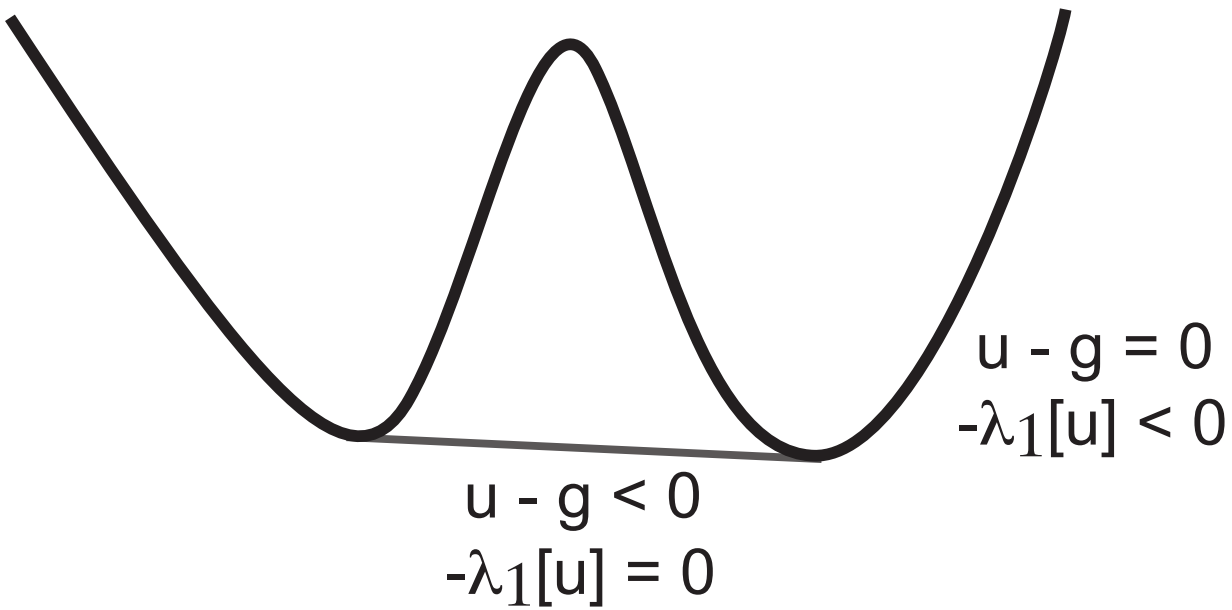
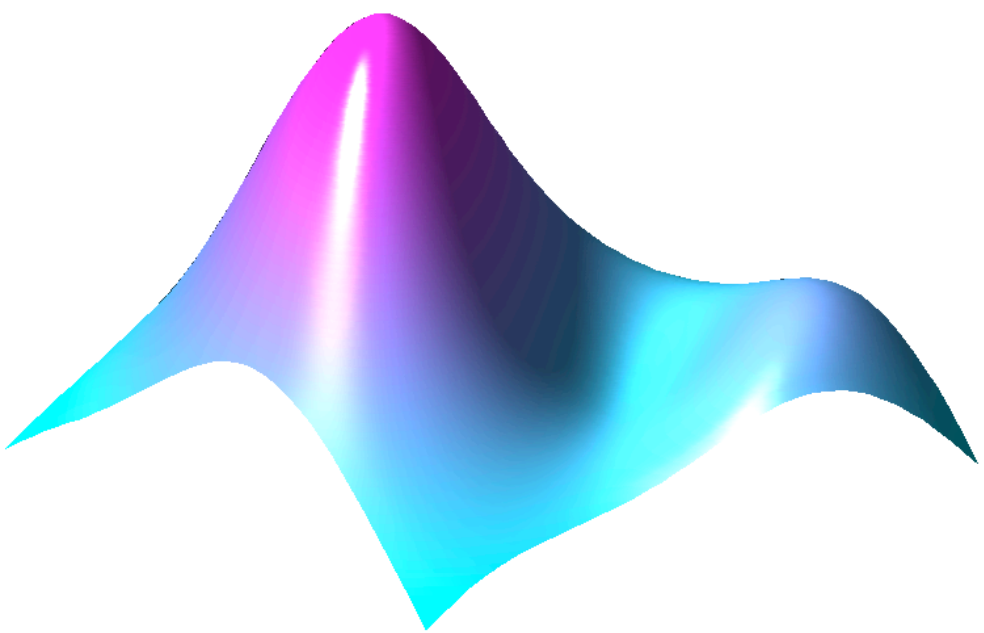
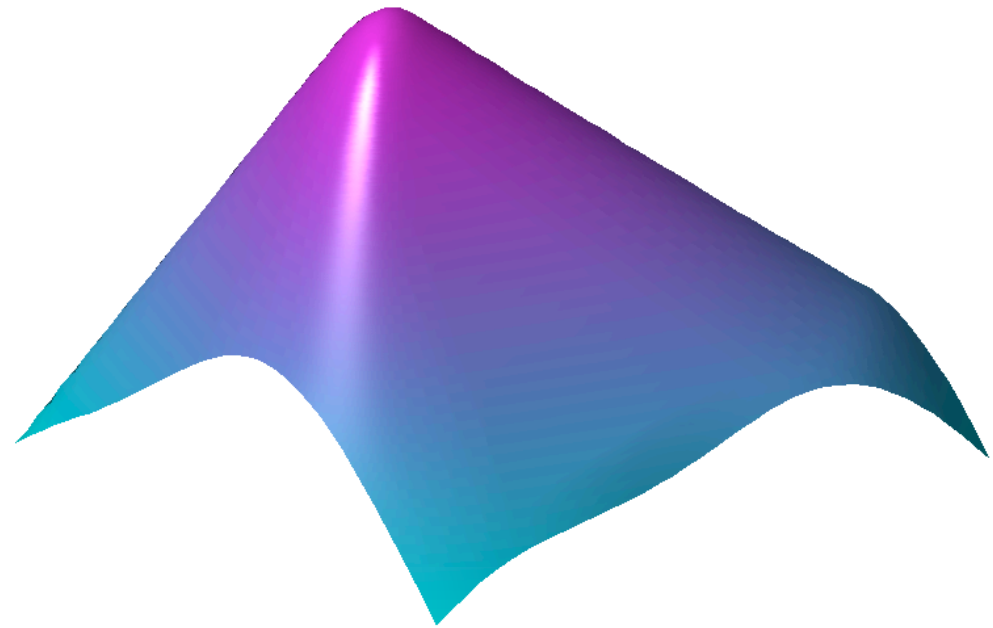


FIGURE 1. Illustration of the equation

Convex Envelope





$$\max\{u(x) - g(x), -\lambda_1[u](x)\} = 0.$$

Here $\lambda_1[u](x)$ is the smallest eigenvalue of the Hessian $D^2u(x)$.

Local conditions for monotonicity

- The equation or scheme is *degenerate elliptic* if a local structure condition holds

$$F^i[u] \equiv F^i\left(u_i, \left.\frac{u_i - u_j}{|x_i - x_j|}\right|_{j=N(i)}\right), \quad i = 1, \ldots N.$$

Definition 2. The scheme F is *degenerate elliptic* if each component F^i is nondecreasing in each variable.

Lemma 2 (Exercise in [3]). The function $F(x, r, p, X)$ is degenerate elliptic if and only if whenever x is a nonnegative local maximum of $u - v$, for $u, v \in C^2$, $F[u](x) \geq F[v](x)$.

Lemma 3. The scheme F is degenerate elliptic if and only if whenever x_i is a nonnegative maximum of $u - v$, for u, v grid functions, $F^i[u] \geq F^i[v]$.

Explicit solution method

Definition 4 (The explicit Euler map). For $\rho > 0$, define $S_\rho : \mathbb{R}^N \to \mathbb{R}^N$ by

$$S_\rho(u) = u - \rho F[u]. \tag{7}$$

This map is the explicit Euler discretization, with time step ρ, of the ordinary differential equation $du/dt + F[u] = 0$.

Definition 5 (Nonlinear CFL condition). Let F be a Lipschitz continuous, degenerate elliptic scheme, with Lipschitz constant K. The nonlinear Courant-Friedrichs-Lewy condition for the Euler map S_ρ is

$$\rho \leq \frac{1}{K}. \tag{CFL}$$

Theorem 7 (The Euler map is a contraction)**.**

e.g

$$dt(G^{MA}) = \frac{h^2}{2}(\max(\lambda_+^{h,d\theta}, 0) + \max(\lambda_-^{h,d\theta}, 0))^{-1}. \tag{5.4}$$

The resulting allowable time step in this case can be significantly less restrictive than the one given by the the dimensional scaling $O(h^4)$.

Semi-Discrete Scheme

$$\lambda_1[u](x) = \min_{|v|=1} v^T D^2 u\, v. \tag{4}$$

Substitute the standard centered difference approximation to the second derivative in the direction of v, to arrive at the semi-discrete scheme

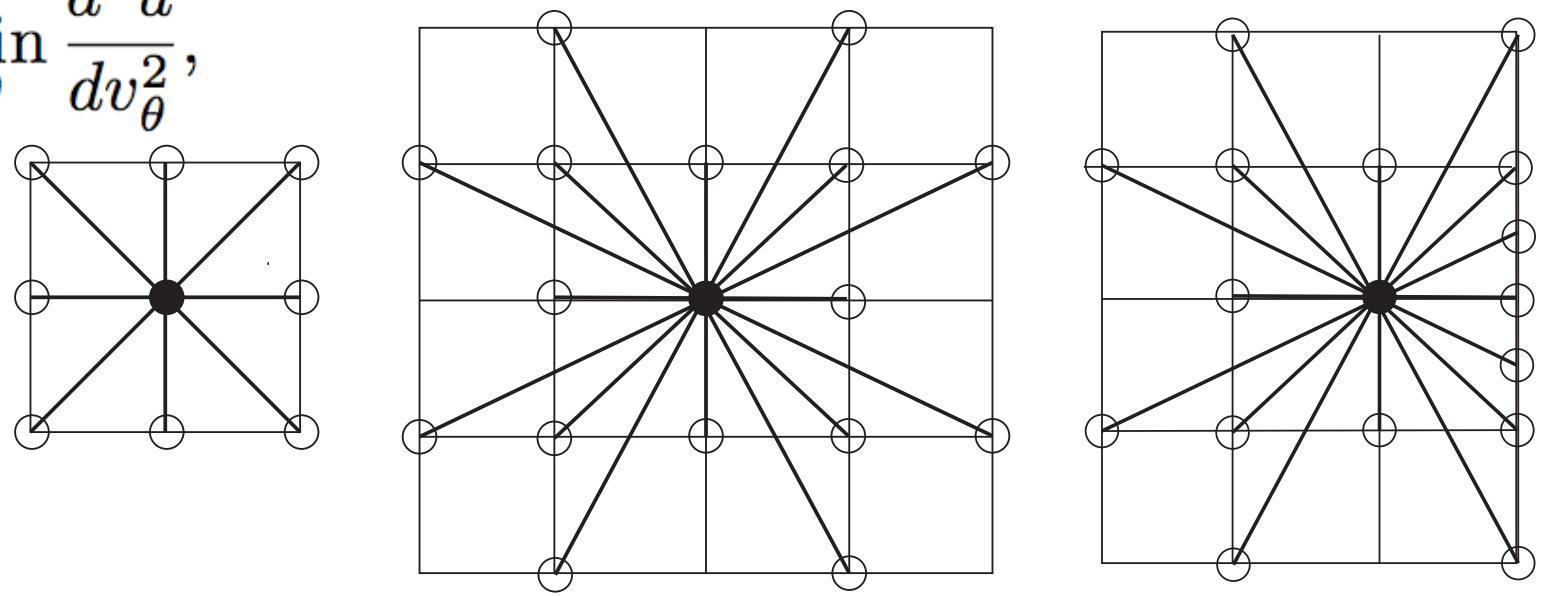
$$\lambda_1^h[u](x) \equiv \min_{|v|=1} \frac{u(x+hv) - 2u(x) + u(x-hv)}{h^2}.$$

Inserting the last equation into (PDE) and solving for $u(x)$ gives

$$u(x) = \min_{|v|=1} \left(g(x), \frac{u(x+hv) + u(x-hv)}{2} \right) \tag{5}$$

Discretization

$$\lambda_-[u](x) = \min_\theta \frac{d^2u}{dv_\theta^2},$$



$$\lambda_-^{dx,d\theta}[u](x_0) = \min_i \frac{u(x_0+v_i) - 2u(x_0) + u(x_0-v_i)}{|v_i|^2}$$

$$\lambda_+^{dx,d\theta}[u](x_0) = \max_i \frac{u(x_0+v_i) - 2u(x_0) + u(x_0-v_i)}{|v_i|^2}.$$

Approximate Convexity

Proposition 1. *Let $u(x)$ be a twice-continuously differentiable function defined on $\mathbb{R}^n$. Let $\{v_i\}_{i=1}^k$ be a set of direction vectors, with directional resolution $d\theta$. Assume $d\theta \leq \pi/4$. If*

$$\frac{d^2u}{dv_i^2} \geq 0, \quad i = 1, \ldots, k, \tag{6}$$

then

$$\frac{\lambda_1}{\lambda_n} \geq -\tan^2(d\theta). \tag{7}$$

where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of the Hessian of u. If instead

$$\frac{d^2u}{dv_i^2} \geq \sin^2(d\theta)\lambda_n, \quad i = 1, \ldots, k. \tag{8}$$

Then u is convex.

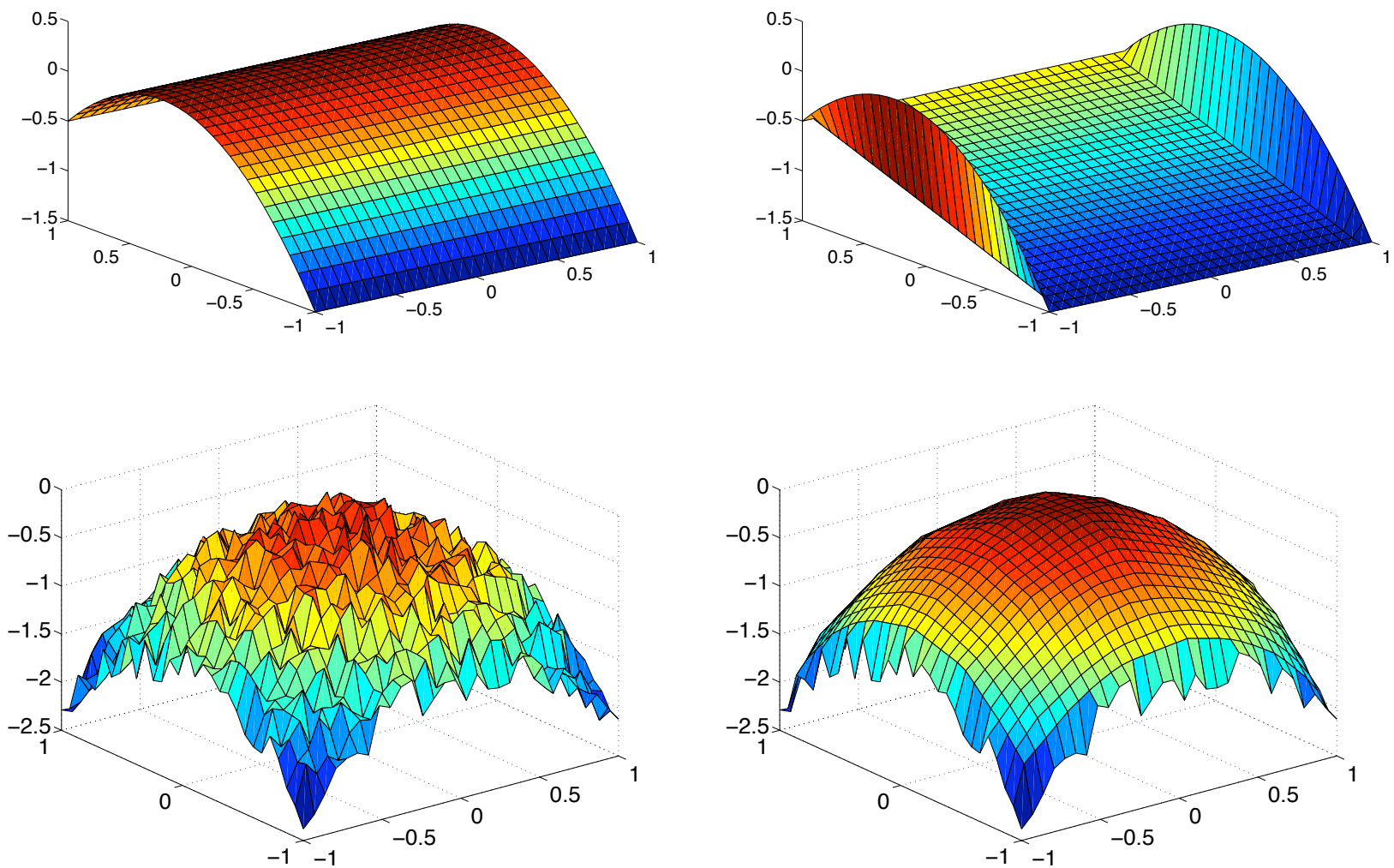
Uniformly Convex Envelope

$$\max\{u(x) - g(x), -\lambda_1[u](x) + \epsilon\} = 0.$$

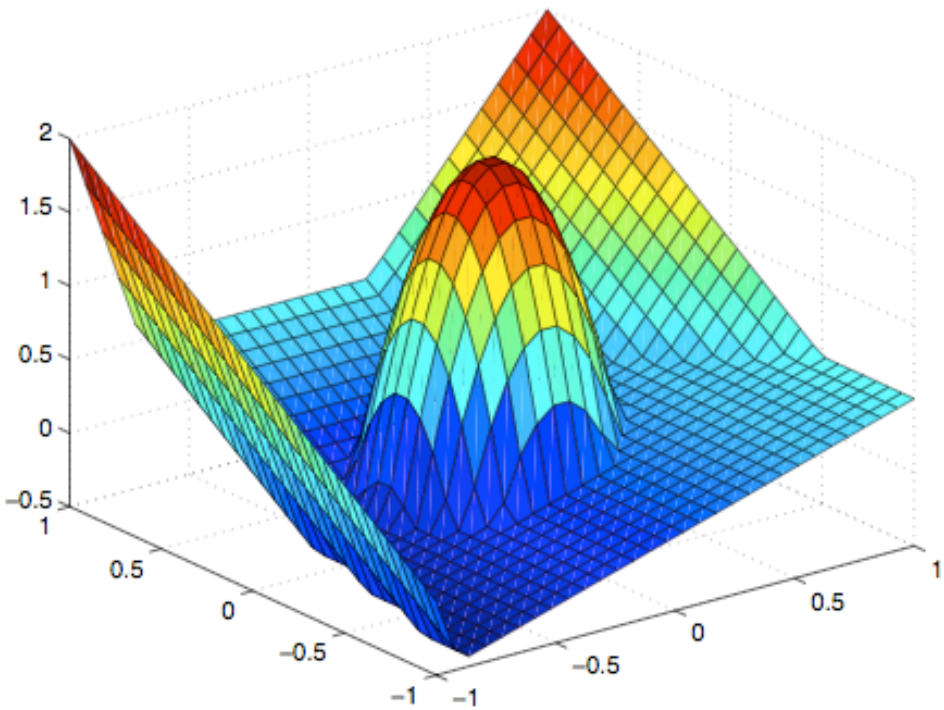
$$u(x) = \sup\{v(x) \mid v \in P^\epsilon, v(y) \leq g(y) \text{ for all } y \in \mathbb{R}^n\}.$$

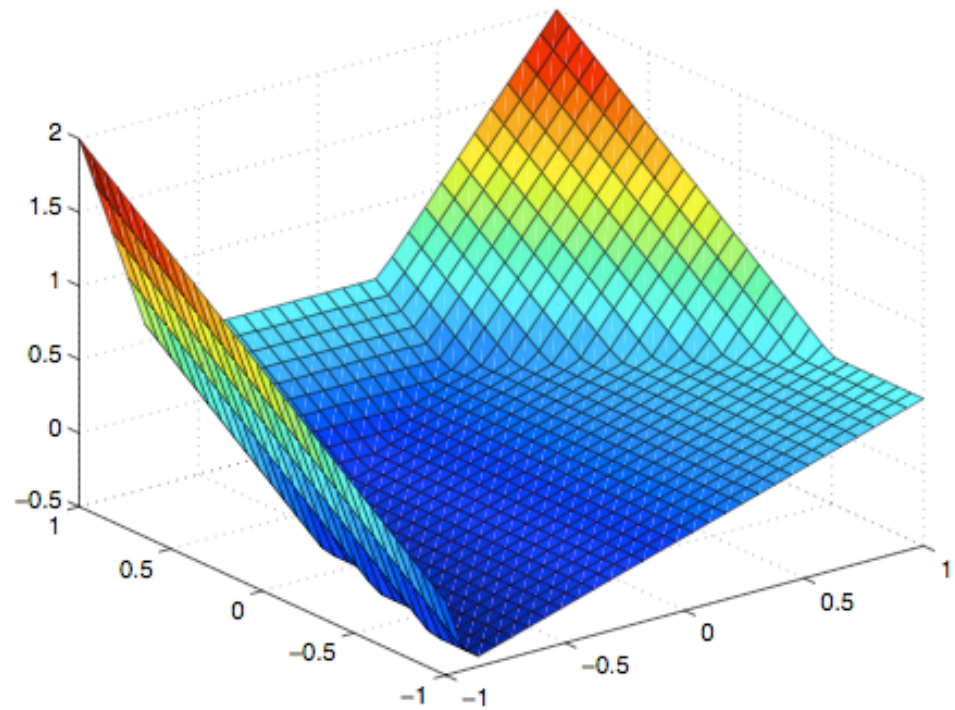
P^ϵ is the set of quadratic functions $q(x)$, with $\lambda_1[q] \geq \epsilon$.

Monotone Scheme:
robust to noise, discontinuities.



Monotone Scheme:
invariant on convex polyhedra





A Hybrid Scheme

(1) Compute numerically $\|D^2u\| = \sqrt{u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2}$.

(2) Pass the result, multiplied by the spatial resolution, h, to a threshold function, $s(t)$, to obtain $T(x) = s(h\|D^2u(x)\|)$. The threshold function should be increasing with values in $[0, 1]$. The following function was used.

$$s(t) = \begin{cases} 0 & t \le t_1 \\ \log(t/t_1)/\log(t_2/t_1) & t_1 \le t \le t_2 \\ 1 & t_2 \le t \end{cases}$$

with $t_1 = .1, t_2 = .5$.

(3) Propagate the threshold values to adjacent neighbors, so that

$$\tilde{T}(x) = \max_{|x-y|\le h} T(y)$$

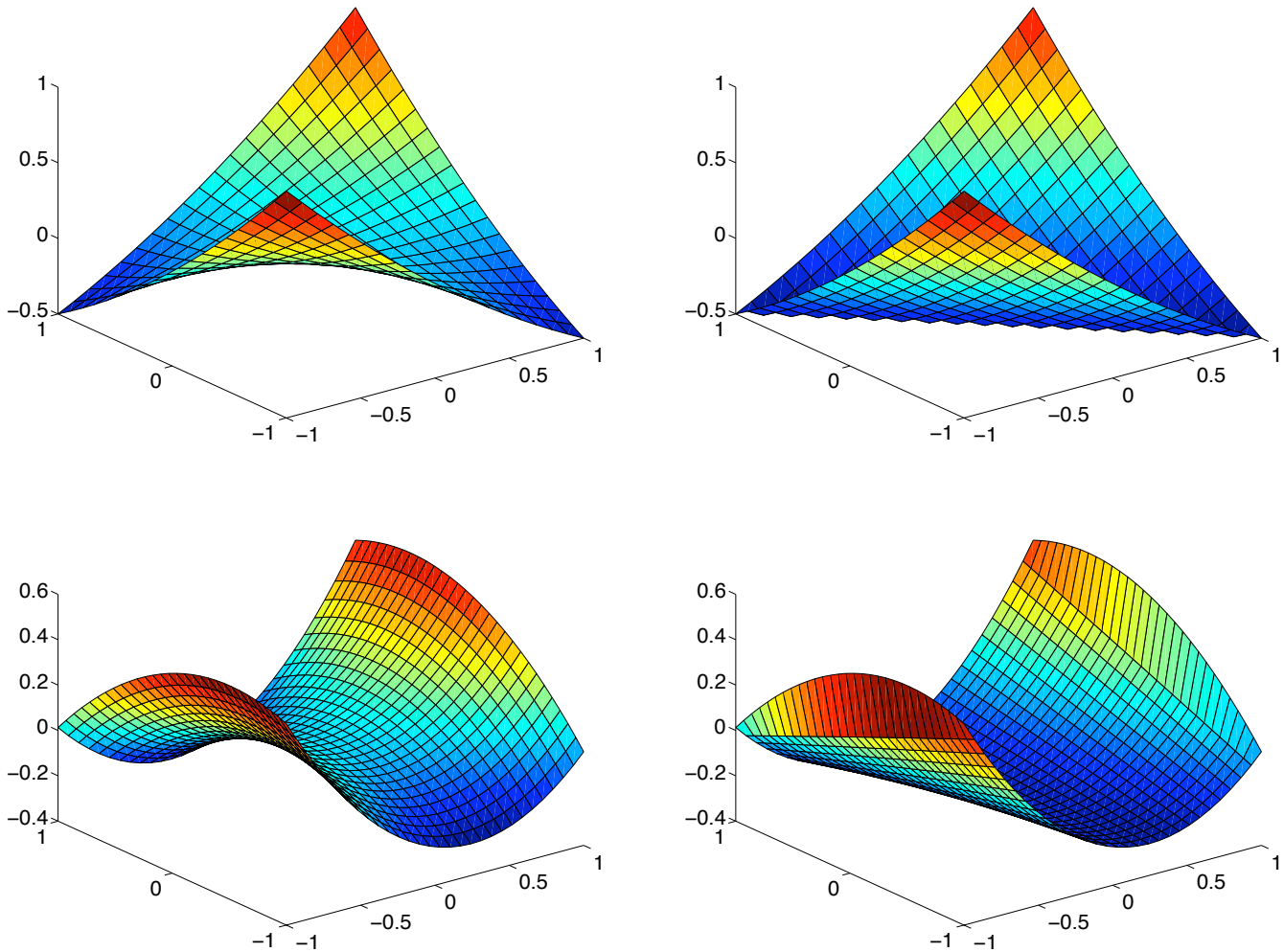
This means that $\tilde{T}(x)$ is a slightly non-local function of x.

(4) Update the values using a convex combination of the monotone scheme and the finite difference scheme, with weight $\tilde{T}(x)$

$$u^H(x) = \tilde{T}(x)u^M(x) + (1 - \tilde{T}(x))u^Q(x). \tag{17}$$

Robust to Singularities, and Higher Accuracy.

Mildly non-convex quadratics



Monge-Ampere

$$det(D^2u) = u_{xx}u_{yy} - u_{xy}^2 = f, \qquad \text{u convex}, f \geq 0$$

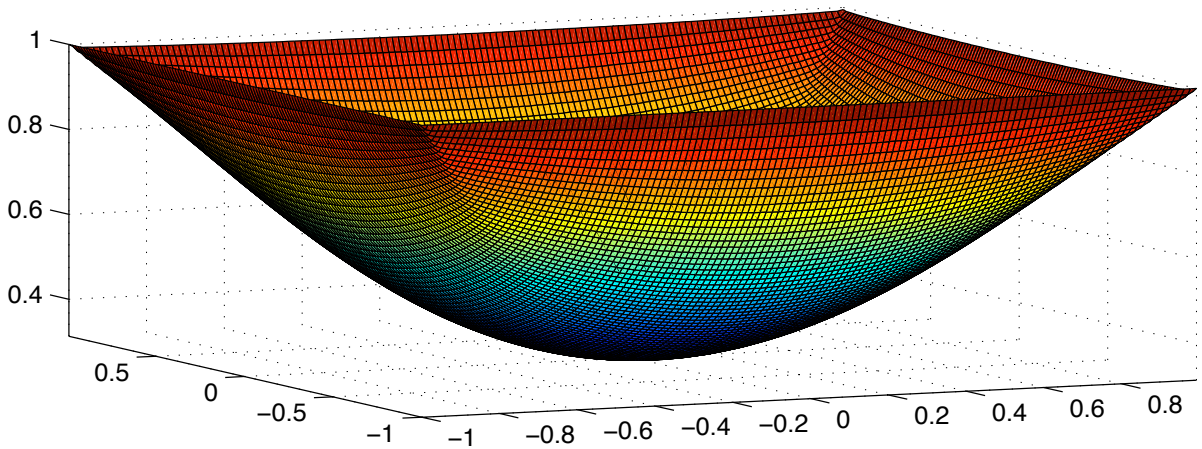
$$\lambda_1[u]\lambda_2[u] = f, \quad 0 \leq \lambda_1 \leq \lambda_2, f \geq 0$$

To preserve convexity, write as

$$\max(\lambda_1, 0)\max(\lambda_2, 0).$$

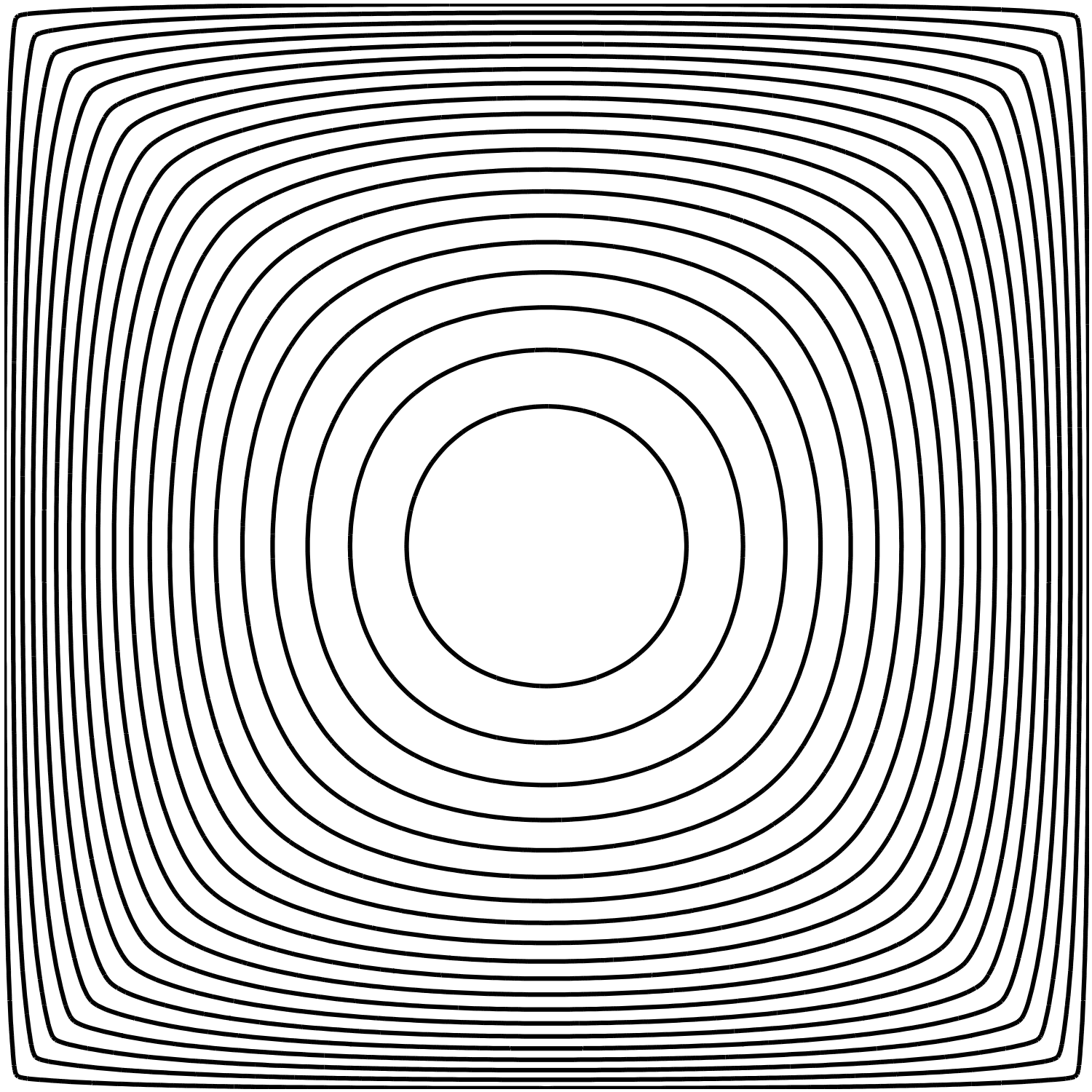
Solution of Monge-Ampere

boundary conditions, constant, f = 1
solution is singular



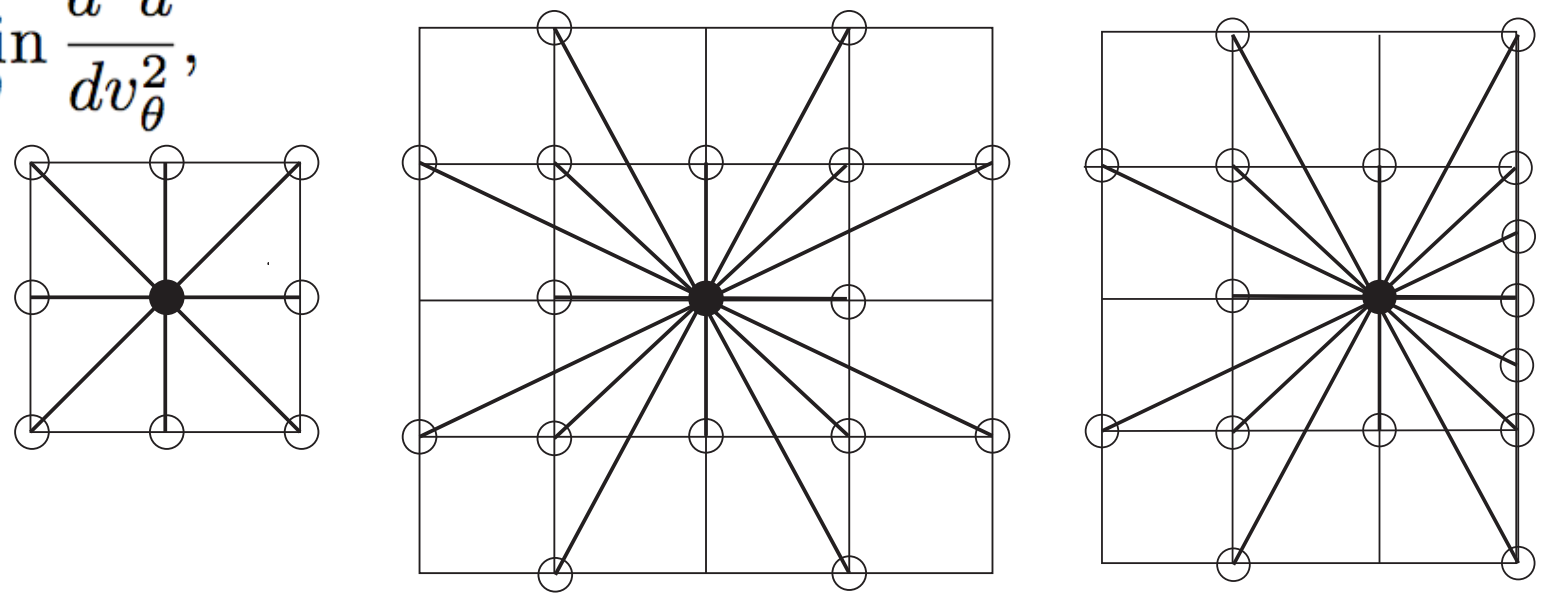
since on boundary

$$u_{xx}u_{yy} = 0$$



Discretization

$$\lambda_-[u](x) = \min_\theta \frac{d^2u}{dv_\theta^2},$$



$$\lambda_-^{dx,d\theta}[u](x_0) = \min_i \frac{u(x_0+v_i) - 2u(x_0) + u(x_0-v_i)}{|v_i|^2}$$

$$\lambda_+^{dx,d\theta}[u](x_0) = \max_i \frac{u(x_0+v_i) - 2u(x_0) + u(x_0-v_i)}{|v_i|^2}.$$

Convergence test.

n	$\|u - u_9^n\|_\infty$	$\|u - u_{17}^n\|_\infty$	$\|u - u_9^n\|_2$	$\|u - u_{17}^n\|_2$
21	1.8×10^{-3}	1.5×10^{-3}	5.2×10^{-5}	4.1×10^{-5}
41	1.3×10^{-3}	5.5×10^{-4}	1.6×10^{-5}	7.2×10^{-6}
81	7.9×10^{-4}	2.4×10^{-4}	3.9×10^{-6}	1.3×10^{-6}
128	5.0×10^{-4}	1.4×10^{-4}	1.3×10^{-6}	4.0×10^{-7}

TABLE 2. Errors for the exact solution of Monge-Ampère , $u(x) = \exp(|x|^2/2)$, $f(x) = (1 + |x|^2)\exp(|x|^2)$, using the 9 and 17 point schemes.

n	8	16	32	64	128	256
min: 9 point	.3159	.3153	.3137	.3132	.3131	.3131
min: 25 point	.3131	.3001	.2966	.3125	.3130	.3131
max error between 9 and 25	0.0049	0.0189	0.0217	0.0157	0.0113	0.0004

TABLE 2. Minimum value of the solution of Monge-Ampère for different grid sizes $n \times n$, using both the 9 and 25 point scheme. Difference between the solutions for fixed spatial resolution and different stencils.