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Intro

Goal: Prove convergence results for difference schemes.

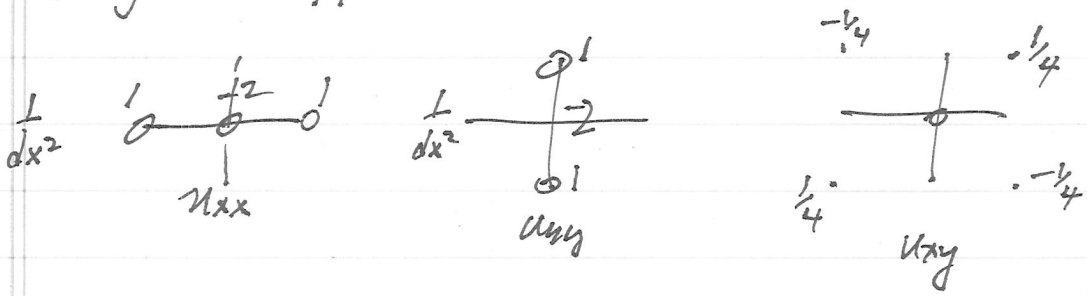
What about
- fast solvers?
- higher accuracy?

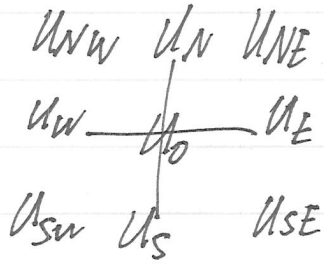
Monotone schemes
- Right class for proving convergence to viscosity solutions.
- Good contraction properties in L^∞
- Use as a base for more efficient schemes (e.g. for HJ eqns, WENO, ENO)
- provably convergent around singularities / for degenerate eqns.

Quick & dirty for M-A

$$u_{xx}u_{yy} - u_{xy}^2 = f \quad , \quad u \text{ convex.}$$

Do finite differences





Solve quadratic expression

$$(u_E + u_W - 2u_0)(u_N + u_S - 2u_0)$$

$$- \left(\frac{u_{NE} + u_{SW} - u_{SE} - u_{NW}}{4}\right)^2 = f$$

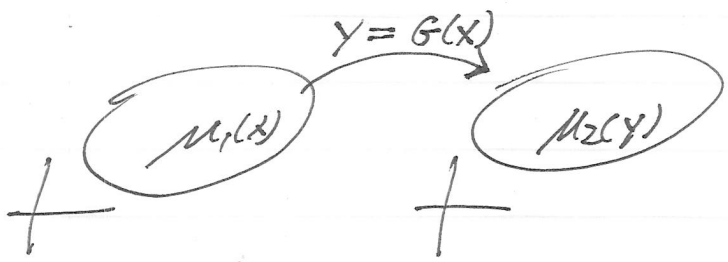
for u_0. Take smaller of 2 roots. (for convexity).
iterate.

<Seems to work well.>
<No proof>
<Not exactly convex>

Systematic methods for
fully nonlinear, elliptic eqns which are
functions of eigenvalues of the Hessian.

<u>Introduce / Motivate the equations</u>

1. <u>Monge-Ampère.</u>



$$\det(G'(x)) = \frac{\mu_1(x)}{\mu_2(y)}$$

If G is gradient of convex function $G = \nabla u$

$$\Rightarrow \quad \det(D^2 u(x)) = f(x, \nabla u(x)).$$

Need u convex, $f > 0$.

<u>Operator</u> $\lambda_1[u] \le \lambda_2[u]$ eigenvalues of $D^2 u$

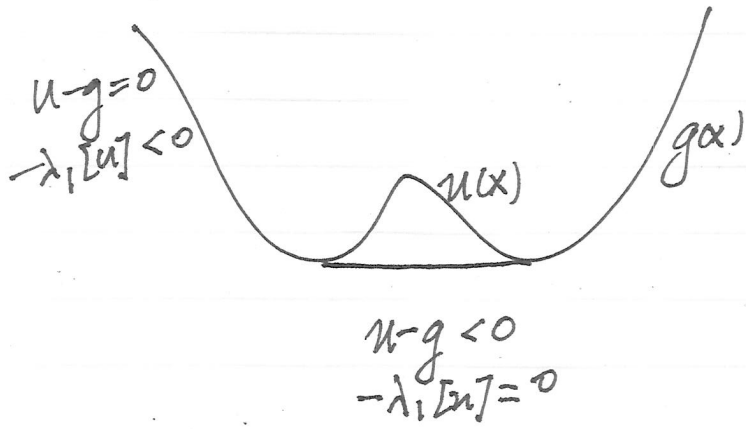
$$\lambda_1[u](x)\, \lambda_2[u](x) = f$$

for convexity

$$\lambda_1^+ \lambda_2^+ = f.$$

forces $\lambda_1, \lambda_2 \geq 0$.

2. Convex Envelope

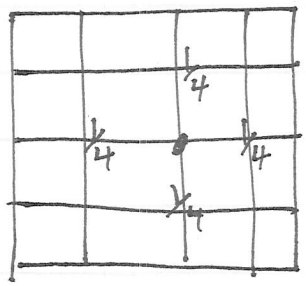


$$\Rightarrow \quad \max\{u(x)-g(x),\ -\lambda_1[u](x)\}=0$$

~~Simplex~~ Operator

$\lambda_1[u]$.

3. Pucci Operators



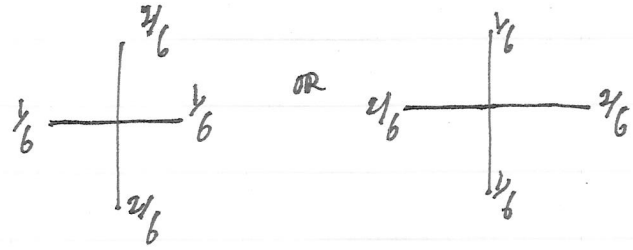
$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

random walk

g = payoff on boundary

$u(x)$ = expected value starting from

choose random walk



in order to max / min expected payoff.

$$L_1[u] = \frac{2u_{xx} + u_{yy}}{\sqrt{5}} \qquad L_2[u] = \frac{u_{xx} + 2u_{yy}}{\sqrt{5}}$$

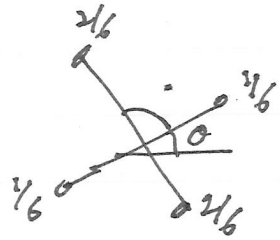
payoff if made optimal choice at each step:

$$F_M[u] = \max(L_1[u], L_2[u])$$

$$F_m[u] = \min(L_1[u], L_2[u])$$

solves
$$\begin{cases} F_m[u] = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Now imagine can choose entire orientation



$$L_\theta = R_\theta \begin{pmatrix} 2/\sqrt{5} & \\ & 1/\sqrt{5} \end{pmatrix} R_{-\theta}$$

$$F_M[u] = \max_\theta L_\theta[u]$$

$$F_m[u] = \min_\theta L_\theta[u]$$

Get Pucci operators, can be written as

$$F_M[u] = a_M \lambda_1[u] + b_M \lambda_2[u]$$
$$F_m[u] = a_m \lambda_1[u] + b_m \lambda_2[u]$$

a, b related to coeffs $2/\sqrt{5}$, $1/\sqrt{5}$

Summary

Function of eigenvalues

Convex/Concave envelopes $\lambda_1[u]$, $\lambda_2[u]$ (along with obstacle problem) no b.c. (or — Dirichlet)

Pucci $a\lambda_1[u] + b\lambda_2[u]$ (Dirichlet) $a, b > 0$.

Monge Ampère $\lambda_1^+[u]\lambda_2^+[u] = f$ (Along with Mass transport b.c.)

All nonlinear elliptic. All non-decreasing fns of λ_1, λ_2.

M_1, M_2 symmetric matrices

$M_1 \leq M_2$ means $x^T M_1 x \leq x^T M_2 x$ for all x

Then $\lambda_1[M_1] \leq \lambda_2[M_2]$

$\lambda_2[M_1] \leq \lambda_2[M_2]$

✓. Existence & Uniqueness of Viscosity solutions

Regularity: Pucci, Yes Caff. Cabré.

Monge-Ampère, Yes when $f > 0$, convex domain.

No when $f = 0$.

Convex Envelope obstacle problem — yes if obstacle.

Comparison

Operator $F(M) = h(\lambda_1(M), \dots, \lambda_n(M))$ $\quad$ M symmetric $n \times n$, $\lambda_1 \le \dots \le \lambda_n$ e.v.

where $h(a_1, \dots, a_n)$ non-decreasing fn.

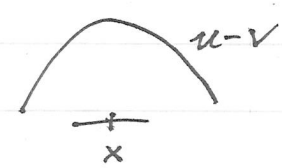
If u solves $\begin{cases} -F(D^2u) = f_1 & \text{in } \Omega \\ u = g_1 & \text{on } \partial\Omega \end{cases}$

v solves $\begin{cases} -F(D^2v) = f_2 & \text{in } \Omega \\ v = g_2 & \text{on } \partial\Omega \end{cases}$

Ω bounded domain.

If $f_1 \le f_2$, $g_1 \le g_2$ then $u \le v$ in Ω

<u>"Proof"</u> Suppose not. On boundary $u \le v$.
So if $u > v$ somewhere in Ω then
there exists x s.t. $u - v$ non-neg. max at x.



Then $D^2u(x) \le D^2v(x)$

so $F(D^2u(x)) \le F(D^2v(x))$

so $\underbrace{-F(D^2u(x))}_{= f_1} \ge \underbrace{-F(D^2v(x))}_{= f_2}$

$\times$

where $M \le N$ means $x^T M x \le x^T N x$ for all x.

Semi-Discrete Numerical Schemes

Rayleigh-Ritz

$$\lambda_1[M] = \min_{|x|=1} x^T M x \quad ,$$

$$\lambda_1[D^2u] = \min_{|x|=1} x^T D^2u\, x = \min_{|v|=1} \frac{d^2u}{dv^2}$$

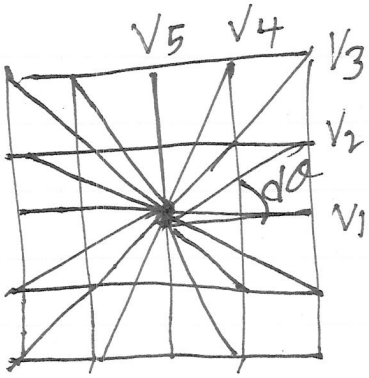
$$\lambda_n[D^2u] = \max_{|v|=1} \frac{d^2u}{dv^2}$$

1. Replace $\frac{d^2u}{dv^2}$ by finite difference $\frac{u(x+\varepsilon v) - 2u(x) + u(x-\varepsilon v)}{\varepsilon^2}$

2. Replace $|v|=1$ by $v_1, \dots, v_K$, vectors directional resolution $d\theta$

$1 \Rightarrow$ Error $O(\varepsilon^2)$

$2 \Rightarrow$ Error $O(d\theta)$



Quantify

$d\theta$ error

Approx Convexity

Prop Let $u(x) \in C^2(\mathbb{R}^n)$. Let $v_1, \dots, v_k$ be direction vectors with directional resolution $d\theta$. (Assume $d\theta < \pi/4$). Then

$$\frac{d^2u}{dv_i^2} \geq 0 \quad , i = 1, \dots, k \implies \frac{\lambda_1}{\lambda_n} \geq -\tan^2(d\theta)$$

where $\lambda_1 \leq \dots \leq \lambda_n$ eigenvalues of D^2u. If

$$\frac{d^2u}{dv_i^2} \geq \sin^2(d\theta)\lambda_n \quad , i = 1, \dots, k \implies u \text{ is convex.}$$

Schemes have error which depends on dx and $d\theta$.

Uniformly Convex envelope

$$\max\{u(x) - g(x), -\lambda_1[u](x) + \alpha\} = 0$$

Solution Methods

$$u^{n+1} = u^n + \Delta t\left(F(D^2u^n) - f(x)\right)$$

Forward Euler.

Contraction in L^∞ if $\Delta t \le \frac{K(F)}{dx^2}$.

K Lipschitz constant of F.

(Fairly slow to converge)

Implicit Methods

Perron

u solves
$$\begin{cases} \min\ u(x) \\ \text{s.t.} \quad F(D^2u) - f \ge 0 \end{cases}$$

least super solution

This is a convex optimization problem:

$$\min \int 1 \cdot u(x)\,dx$$
$$\text{s.t.} \quad F[u] \ge 0 \qquad F \text{ convex.}$$

If F polyhedral, this is an ∞-dim^{l} LP

Linear Program.

Can discretize & solve using package. No CF Land.

E.g.

$$\min \int 1 \cdot u(x)\,dx$$

$$\text{s.j.} \quad L_1[u] \geq 0$$
$$L_2[u] \geq 0$$

if $L_1[u] \geq 0$, $L_2[u] \geq 0$ then $\min(L_1[u], L_2[u]) \geq 0$

More generally

$$F[u] = \min_{\alpha} L_\alpha[u]$$

Solve via

$$\begin{cases} \min \int 1 \cdot u(x)\,dx \\ \text{s.j.} \quad L_\alpha[u] \geq 0 \quad \alpha \in \Lambda \end{cases}$$

Linear Objective, Linear Constraints

(Not for MA which is quadratic)

Convergence Proof

Linear Case

$$L[u] = -u_{xx}$$

$$L^\varepsilon[u^\varepsilon] = -\frac{1}{\varepsilon^2}\left(u(x+\varepsilon) - 2u(x) + u(x-\varepsilon)\right)$$

$$L[u] = f \quad \Longrightarrow \quad (2)\ L^\varepsilon[u] = f + \delta^\varepsilon$$

trunc. error

$$(1)\ L^\varepsilon[u^\varepsilon] = f$$

$$\delta^\varepsilon = L^\varepsilon[u] - L[u] = O(\varepsilon^2)$$

$$z^\varepsilon = u - u^\varepsilon$$

subtract (1,2)

$$L^\varepsilon[z^\varepsilon] = \delta^\varepsilon \qquad \text{PDE for error}$$

Solve / Estimate

$$z^- < z < z^+$$

where by comparison

$$L^\varepsilon[z^+] = |\delta|_\infty$$

$$L^\varepsilon[z^-] = -|\delta|_\infty$$

Quadratic solutions in domain Radius R

$$z^\pm = \pm|\delta|\left(\frac{x^2 - R^2}{2}\right)$$

$$\Longrightarrow |z| \le \frac{\delta R^2}{2} = \frac{C R^2}{2}\varepsilon^2.$$

$\boxed{\checkmark}$

Convergence Proof

Nonlinear Case

$$F_M[u] = \max(L_1 u, L_2 u) \quad , \quad F_m[u] = \min(L_1 u, L_2 u)$$

finite diff.

$$F_M^\varepsilon[u] = \max(L_1^\varepsilon u, L_2^\varepsilon u) \quad , \quad F_m^\varepsilon[u] = \min(L_1^\varepsilon u, L_2^\varepsilon u)$$

where $L_1^\varepsilon, L_2^\varepsilon$ standard f.d. schemes for linear operators.

Truncation error

$$\delta^\varepsilon = F_M[u] - F_M^\varepsilon[u] = O(\varepsilon^2)$$

$$\parallel$$

$$\delta^\varepsilon = F_M^\varepsilon[u^\varepsilon] - F_M^\varepsilon[u] \qquad \text{since } F_M[u] = f, \; F_M^\varepsilon[u^\varepsilon] = f$$

$$z^\varepsilon = u - u^\varepsilon$$

$$\underbrace{F_M^\varepsilon[u^\varepsilon] - F_M^\varepsilon[u]}_{\delta^\varepsilon}$$

Nonlinear. How to estimate difference?

Claim $F_m[u^\varepsilon - u] \leq F_M^\varepsilon[u^\varepsilon] - F_M^\varepsilon[u] \leq F_M^\varepsilon(u^\varepsilon - u)$

Then have quadratic sub/super solutions as before,

$$|u^\varepsilon - u|_\infty \leq \frac{\delta R^2}{2} = \frac{CR^2}{2}\varepsilon^2$$

<u>Why Claim?</u>

$$X, Y \in \mathbb{R}^*$$

$$\min_i (X_i - Y_i) \leq \max_i (X_i) - \max_i (Y_i) \leq \max_i (X_i - Y_i)$$

<u>More generally</u>

(using Max norm)

<u>Lemma</u> Let $f: \mathbb{R}^* \to \mathbb{R}$ be Lipschitz cts with constant K. Suppose f is non-decreasing in each variable. Then

$$K \min_i (X_i - Y_i) \leq F(X) - F(Y) \leq K \max_i (X_i - Y_i)$$

<u>Proof</u> $F(X) - F(Y) \leq F(X \vee Y) - F(Y)$ since nondecreasing

$\leq K \| X \vee Y - Y \|_\infty$ since Lip

$\leq K \max_i (X_i - Y_i)$.

<u>Rmk</u> This is how convergence proof works for more general elliptic operators.

e.g. $\max_i L_i$ or $F = f(u_{v_1}, \ldots, u_{v_k v_k})$.

<u>Rmk</u> Can get better estimates that $\|\delta\|_\infty$ using ABP estimates for solution