Metrics in Spectral Analysis

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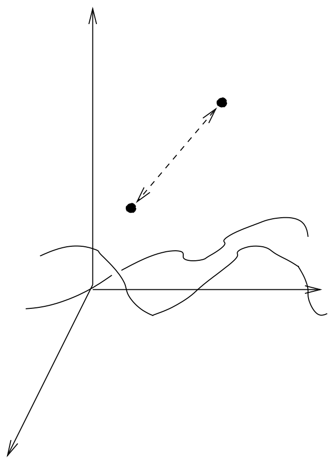
IPAM workshop April 2008

Notions of distance



Signals:

maximal separation (L_∞)
energy-like content (L_2)
integral of flow-rate (L_1)

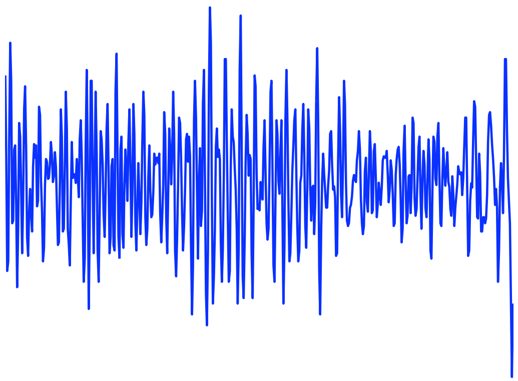
Input-output systems:

gains, operator norms
etc.

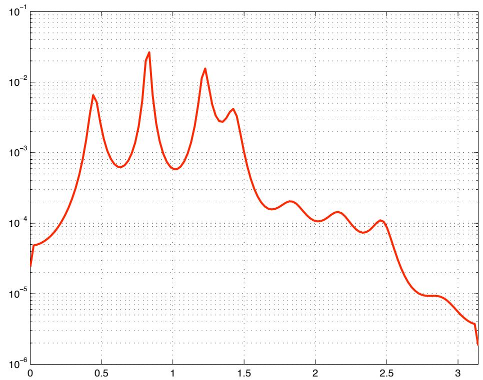
Power distributions:

...

Power spectra



$\Rightarrow$



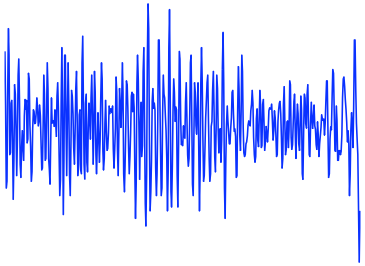
$$\ldots u_0, u_1, \ldots$$

$$u_k = \int e^{jk\theta} dX(\theta)$$

$$f(\theta)d\theta = E\{|dX(\theta)|^2\}$$

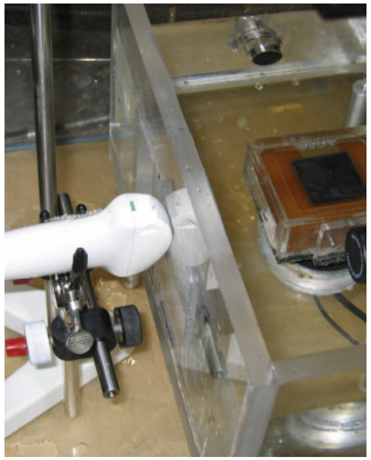
Periodogram, Blackman-Tukey, Levinson, Durbin, Burg, . . .

speech analysis



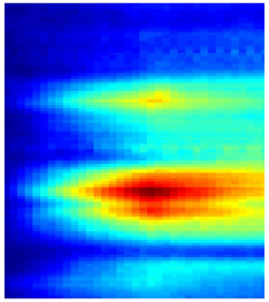
⇒

radar/sonar
medical diagnostics
system id
AFM
. . .



⇒

noninvasive temperature sensing

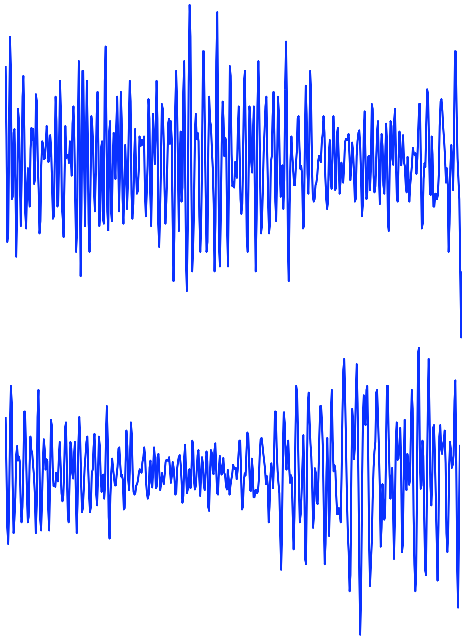


with E. Ebbini & A.N. Amini

In IEEE Trans. on Biomedical Engineering, 2005

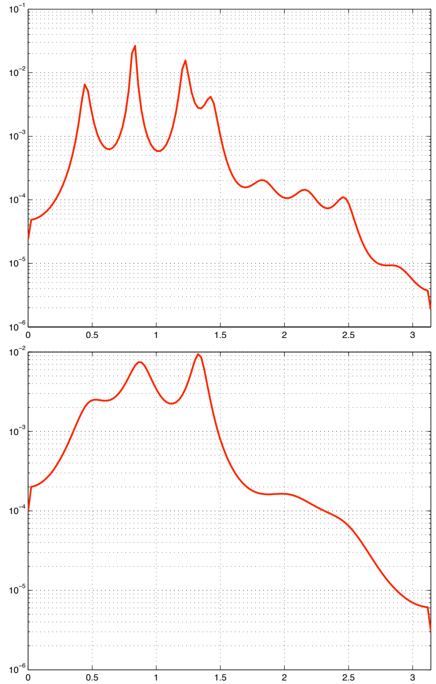
Signals vs. power densities

time-signals



$(u_1 - u_2)$ “error signal”

power distributions



$(f_1 - f_2)$ is not a “signal”

How can we compare power spectra?

Question:

what is a natural notion of distance between power spectral densities?

Plan of the talk

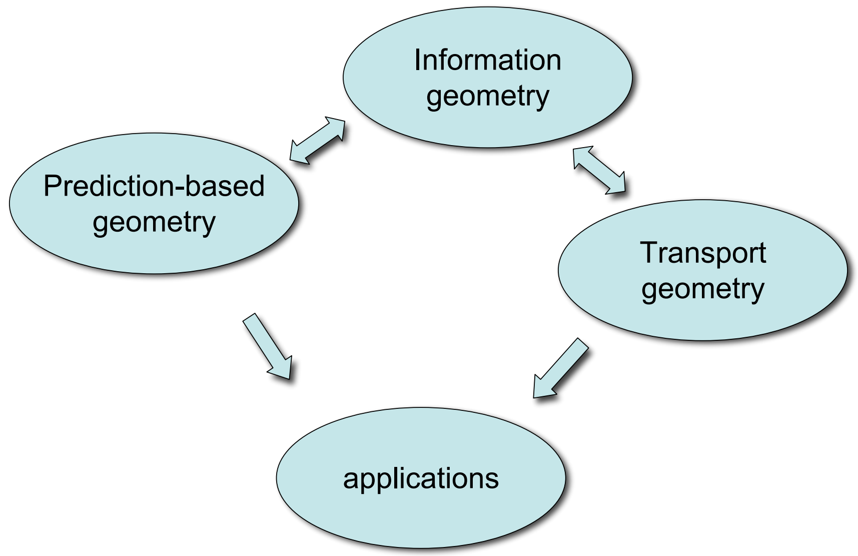
Metrics based on

prediction theory

parallels with information geometry

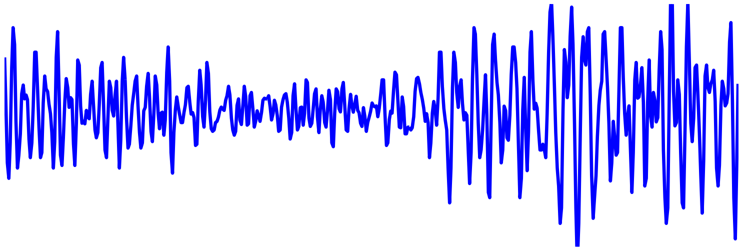
transportation theory

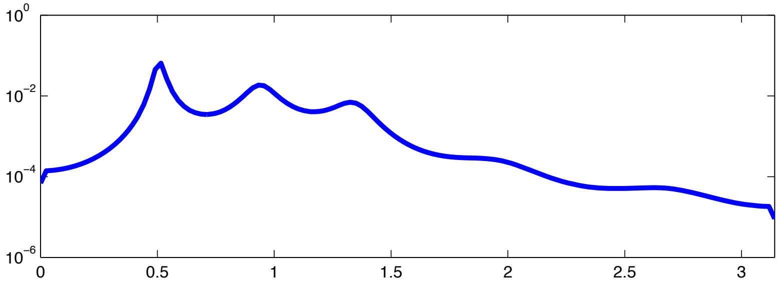
Thoughts + applications



Setting

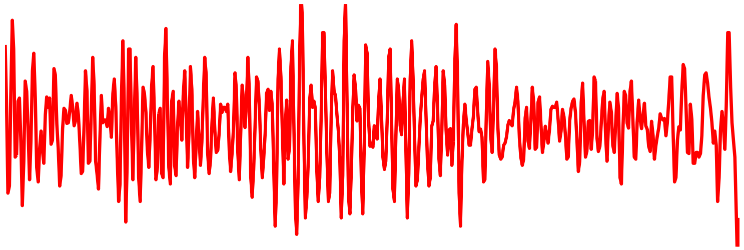
$\ldots u_{-1}, u_0, u_1, u_2, \ldots$

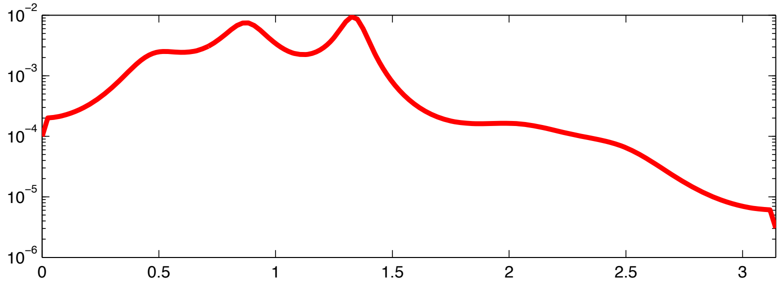




$f_1(\theta)$

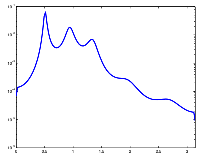
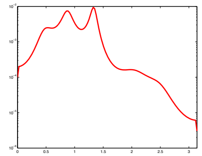
$\ldots u_{-1}, u_0, u_1, u_2, \ldots$





$f_2(\theta)$

We would like:

metric ( , )

$f_1(\theta)$, $f_2(\theta)$

candidates?

Kullback-Leibler, Bregman, Itakura-Saito, .. e.g., $\int (x - \log(x) - 1)|_{x=f_1/f_2}$

convex functionals

perceptual qualities

Linear prediction

One-step-ahead prediction: $u_{\text{present}} - \hat{u}_{\text{present}|\text{past}}$

with $\hat{u}_{\text{present}|\text{past}} := \sum_{\text{past}} \alpha_k u_k$

$$E\{|u_{\text{present}} - \hat{u}_{\text{present}|\text{past}}|^2\} = \text{variance of prediction error}$$

Szegö's theorem

One-step-ahead prediction:

$$\text{least error variance} \quad = \quad \exp\left\{\frac{1}{2\pi}\int \log f(\theta) d\theta\right\}$$

it is a geometric mean. . .

$$\exp\{\frac{1}{3}(\log f_1 + \log f_2 + \log f_3)\} = \sqrt[3]{f_1 f_2 f_3}$$

Degradation of prediction error variance

Use f_2 to design a predictor (assuming $u_{f_2,\text{time}}$).

Then compare how this performs on $u_{f_1,\text{time}}$ against the optimal based on f_1.

$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}} - \text{optimal variance}}{\text{optimal variance}} \geq 0$$

Degradation of prediction variance

$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}}}{\text{optimal variance}} = \frac{\text{arithmetic mean of } \left(\frac{f_1}{f_2}\right)}{\text{geometric mean of } \left(\frac{f_1}{f_2}\right)}$$

$$= \frac{\left(\frac{1}{2\pi}\int(\frac{f_1}{f_2})d\theta\right)}{\exp\left(\frac{1}{2\pi}\int\log(\frac{f_1}{f_2})d\theta\right)}$$

arithmetic over *geometric* mean (≥ 1)

Riemannian metric

$$f_1 = f,$$
$$f_2 = f + \Delta$$

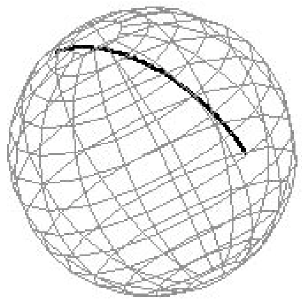
$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}} - \text{optimal variance}}{\text{optimal variance}} \simeq$$

$$\delta(f, f+\Delta) = \frac{1}{2\pi}\int \left(\frac{\Delta}{f}\right)^2 d\theta - \left(\frac{1}{2\pi}\int \left(\frac{\Delta}{f}\right) d\theta\right)^2$$

variance-like: (mean square) - (arithmetic-mean)^2

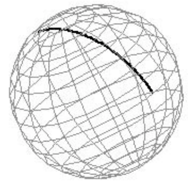
Geodesics

Paths $f_{\mathbf{r}}$ ($\mathbf{r} \in [0, 1]$) between f_0, f_1 of minimal length $\int_0^1 \sqrt{\delta\left(f_{\mathbf{r}}, f_{\mathbf{r}+d\mathbf{r}}\right)}$



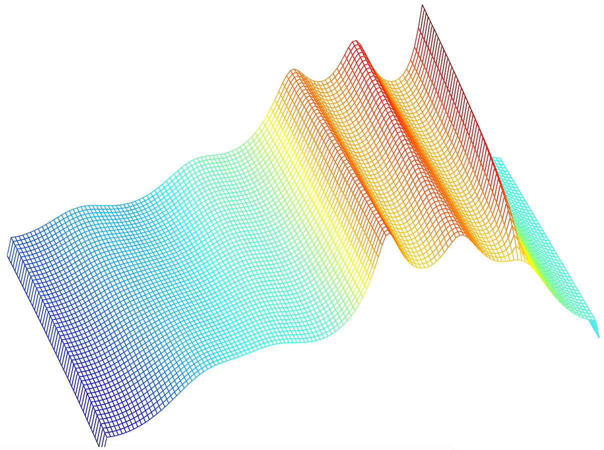
each point represents a different power spectral density

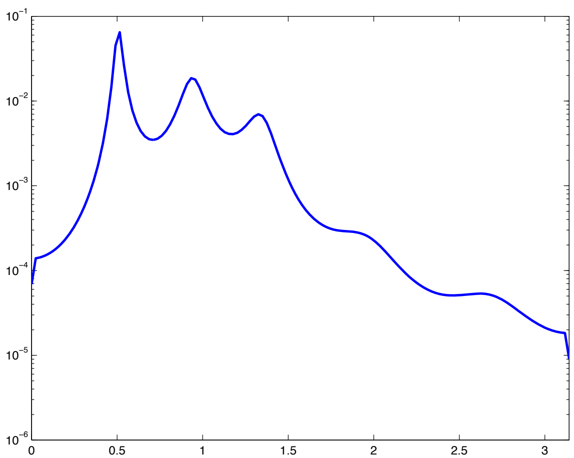
Geodesics

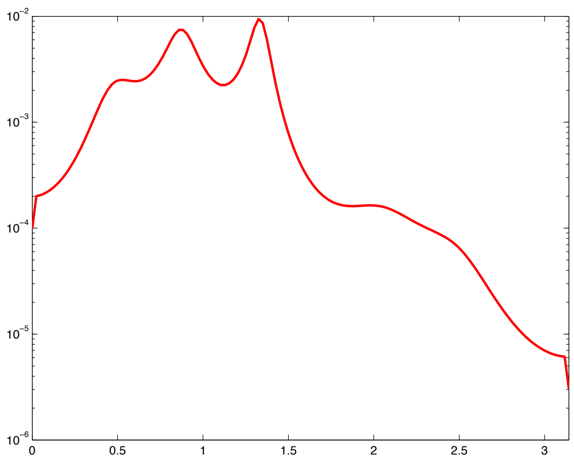


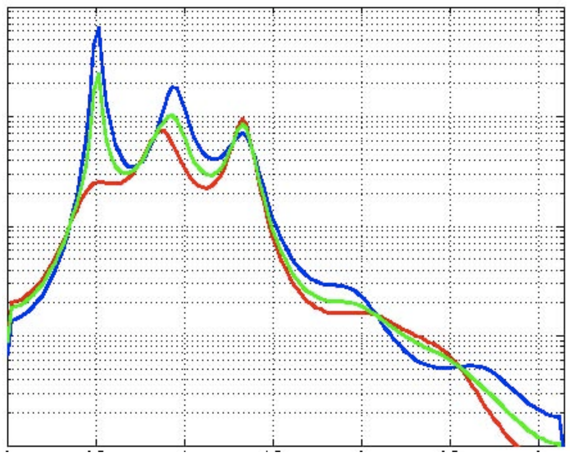
The geodesics are exponential families:

$$f_{\mathbf{r}} = f_0 \left(\frac{f_1}{f_0} \right)^{\mathbf{r}}, \quad \mathbf{r} \in [0, 1]$$



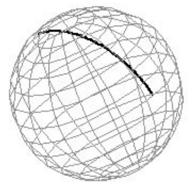






morphing

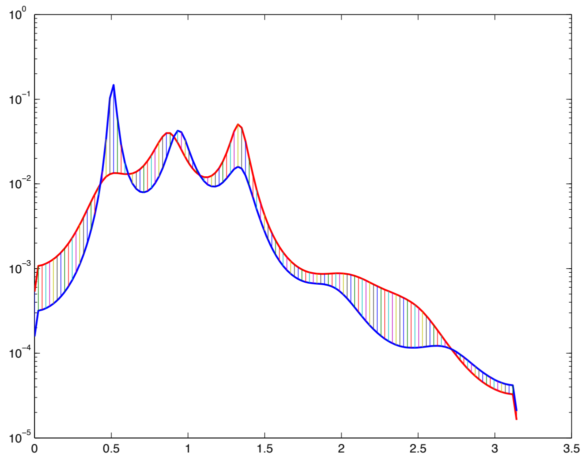
Geodesic distance: metric

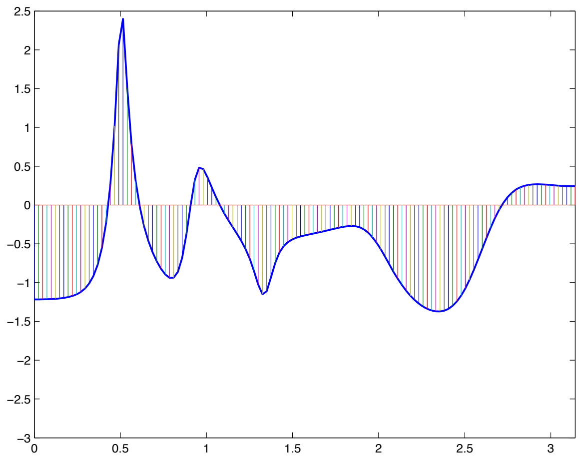


The path-length is

$$d(f_0, f_1) := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log\frac{f_1}{f_0}\right)^2 d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{f_1}{f_0}\right)d\theta\right)^2}$$

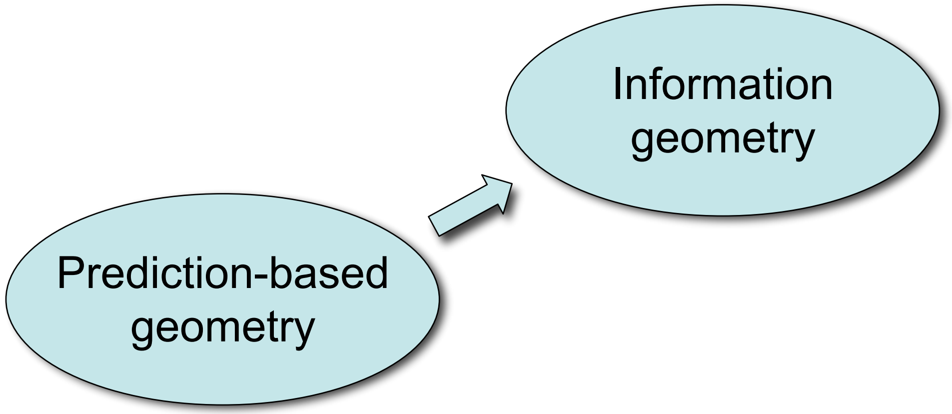
scale-insensitive, “shape” recognizer





Georgiou: IEEE Trans. on Signal Processing, Aug. 2007

$$\log\frac{f_1}{f_0} = \log(f_1) - \log(f_0)$$


Information geometry
Prediction-based geometry

Information geometry – *parallels*

$f \rightsquigarrow \mathbf{p}$: probability density

Expected "message-length increase":

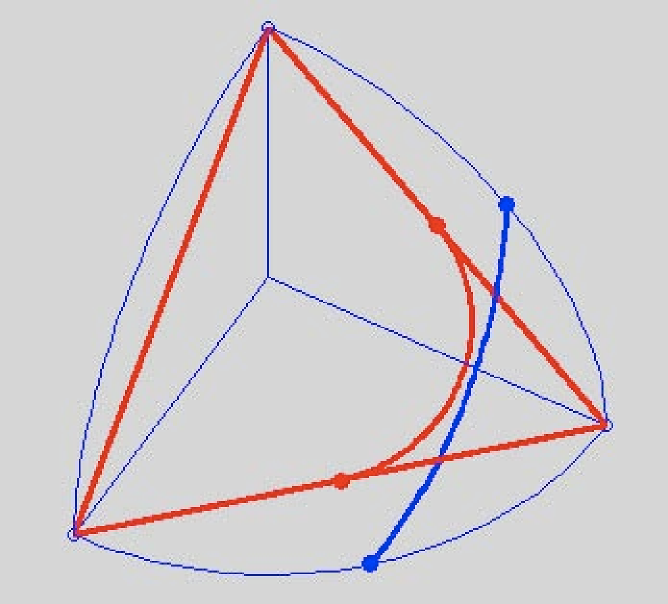
$$H(\mathbf{p}_1|\mathbf{p}_0) = \Big(-\sum \mathbf{p}_1 \log(\mathbf{p}_0)\Big) - \Big(-\sum \mathbf{p}_1 \log(\mathbf{p}_1)\Big)$$

R. Fisher
C. R. Rao
S. Kullback
R. Leibler

Fisher metric

$$H(\mathbf{p} + \Delta \mid \mathbf{p}) = \sum \frac{\Delta^2}{\mathbf{p}}$$

Information geometry – *parallels*



Geodesics: great circles

$$\mathbf{p} \mapsto \sqrt{\mathbf{p}} \in \mathrm{Sphere}$$

$$\begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{p(1)} \\ \sqrt{p(2)} \\ \sqrt{p(3)} \end{pmatrix}$$

Geodesic distance: Arclength
Battacharyya distance

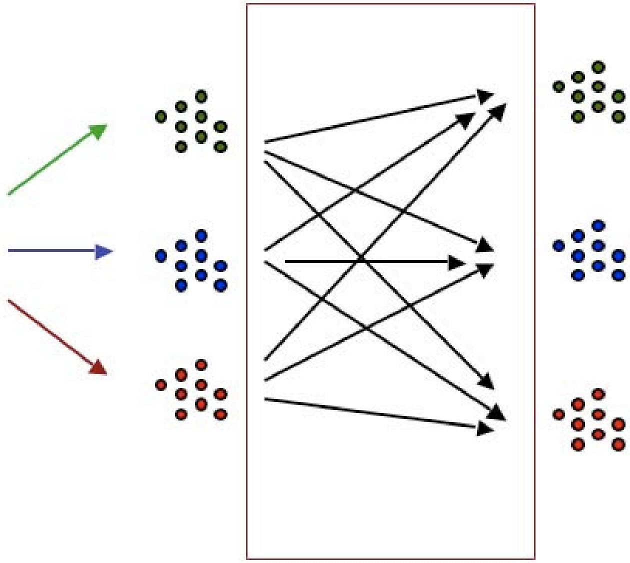
Information vs. prediction-based

$$\sum \frac{\Delta^2}{\mathbf{p}} \quad \text{vs.} \quad \int \left(\frac{\Delta}{\mathbf{f}}\right)^2 - \left(\int \frac{\Delta}{\mathbf{f}}\right)^2$$

$$\mathbf{p} \mapsto \sqrt{\mathbf{p}} \quad \text{vs.} \quad \mathbf{f} \mapsto \log \mathbf{f}$$

great circles vs. **logarithmic families**

Information geometry – *parallels*



Ability to differentiate decreases

$$\begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} \mapsto \mathrm{M} \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$$

Chentsov's theorem:

Stochastic maps are contractive

under *Fisher metric*

and

Fisher metric is the unique Riemannian metric with this property

Analogous properties for power spectra?

"Wish list"

a metric that behaves "naturally" under

addititive noise

$$f \mapsto f + f_{\text{noise}}$$

multiplicative noise

$$f \mapsto f \star f_{\text{noise}}$$

continuity of moments (second-order statistics)

$$f \mapsto \text{integrals of } f$$

Analogous properties for power spectra?

"Wish list"

a metric $\delta(\cdot,\cdot)$:

addititive noise

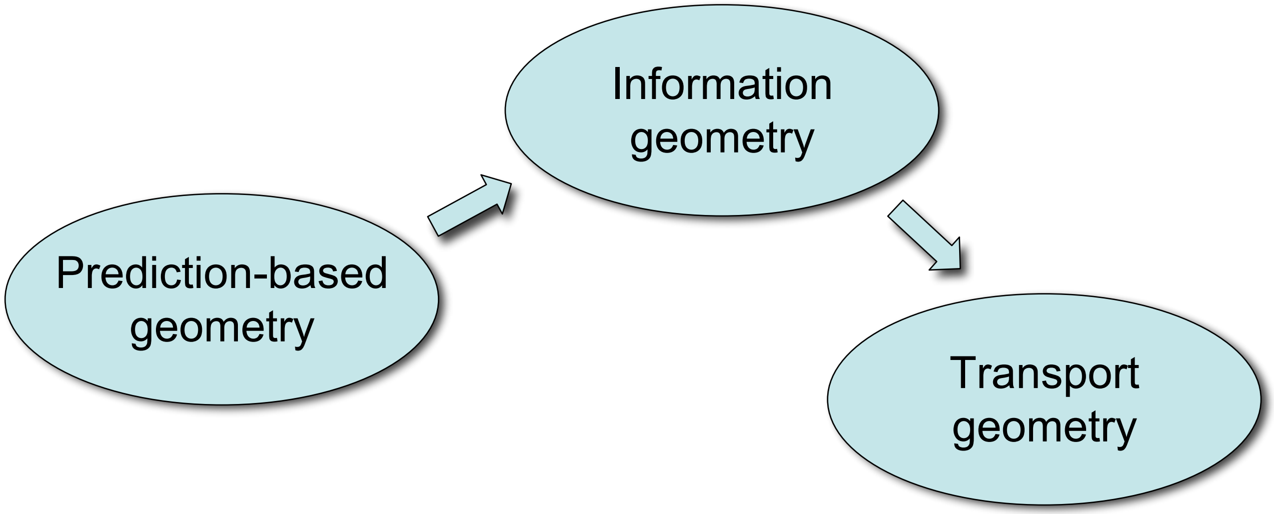
$$\delta(f_1 + f_{\rm noise}, f_2 + f_{\rm noise}) \leq \delta(f_1, f_2)$$

multiplicative noise –with $\int f_{\rm noise} \leq 1$

$$\delta(f_1 \star f_{\rm noise}, f_2 \star f_{\rm noise}) \leq \delta(f_1, f_2)$$

*continuity of moments (second-order statistics) —*weak* continuity

$$f_k \to f \quad \Leftrightarrow \quad \int {\rm G} f_k \to \int {\rm G} f, \quad \forall G \text{ continuous}$$


Information
geometry
Prediction-based
geometry
Transport
geometry

Transportation geometry

minimize cost of transferring mass

$$\int \mathrm{cost}(x \to y) \ \times \ \mathrm{mass}(dx, dy)$$

small twist: *unbalanced masses*

prior work by Benamou, Brenier: mixed "L^2-Wasserstein"

our work: mixed "Wasserstein-L_1" (a metric)

Mixed distances

Relaxation by Benamou:

$$\inf_{\int \phi = \int f_0} \{ d_{\text{Wasserstein}}(f_0, \phi)^2 + \kappa \cdot d_{L_2}(\phi, f_1)^2 \}$$

Here, (symmetric) relaxation:

"total variation" (f_0, ϕ_0) + transportation (ϕ_0, ϕ_1) + "total variation" (ϕ_1, f_1)

Total variation:

$$
\begin{aligned}
d_{\mathrm{TV}}(d\mu_0, d\mu_1) &= \int |\mu_0(dx) - \mu_1(dx)| \\
&= \min\{\int d\nu_0 + \int d\nu_1 \ : \ d\mu_0 + d\nu_0 = d\mu_1 + d\nu_1\}
\end{aligned}
$$

Transportation cost:

$$
T_c(d\mu_0, d\mu_1) := \min\left\{\int_{X\times X} c(x,y)d\pi(x,y) \ : \ d\pi \in \Pi(d\mu_0, d\mu_1)\right\}.
$$

Mixed metric

$$T_{c,\kappa}(d\mu_0, d\mu_1) \ := \inf_{\nu_0(X)=\nu_1(X)} \left(T_c(d\nu_0, d\nu_1) + \kappa \sum_{i=0}^{1} d_{\mathrm{TV}}(d\mu_i, d\nu_i) \right)$$

$$c(x,y) = |(x-y)_{\mathrm{mod} 2\pi}|^p, \qquad X = [0, 2\pi]$$

$$\delta_{p,\kappa}(d\mu_0, d\mu_1) \ := \ (T_{c,\kappa}(d\mu_0, d\mu_1))^{\min(1,\frac{1}{p})}$$

Mixed metric

$$X \to X \cup \{\infty\}$$

$$\mu \to \hat{\mu} \quad \text{where} \quad \begin{array}{l} \hat{\mu}_i(S) = \mu_i(S) \text{ for } S \subset X \\ \hat{\mu}_i(\infty) = M - \mu_i(X) \end{array}$$

$$c(x, y) \to \hat{c}(x, y) = \begin{cases} \min(c(x, y), 2\kappa) & \text{for } x, y \in X, \\ \kappa & \text{for } x \in X, y = \infty, \\ \kappa & \text{for } x = \infty, y \in X, \\ 0 & \text{for } x = \infty, y = \infty. \end{cases}$$

$$T_{c,\kappa}(d\mu_0, d\mu_1) = T_{\hat{c}}(d\hat{\mu}_0, d\hat{\mu}_1).$$

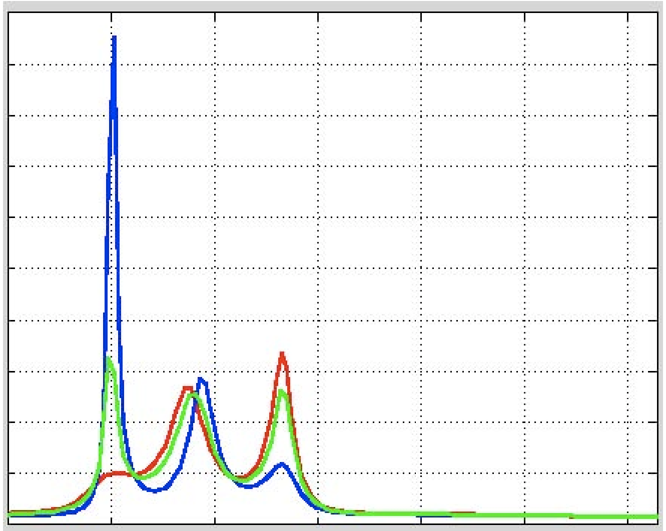
Mixed metric

If $\kappa > 0$ *and* $p \in (0, \infty)$,
then $\delta_{p,\kappa}(d\mu_0, d\mu_1)$ *is a metric & satisfies the "wish list", i.e.*

distances do not increase

under additive noise
and multiplicative noise
with power ≤ 1

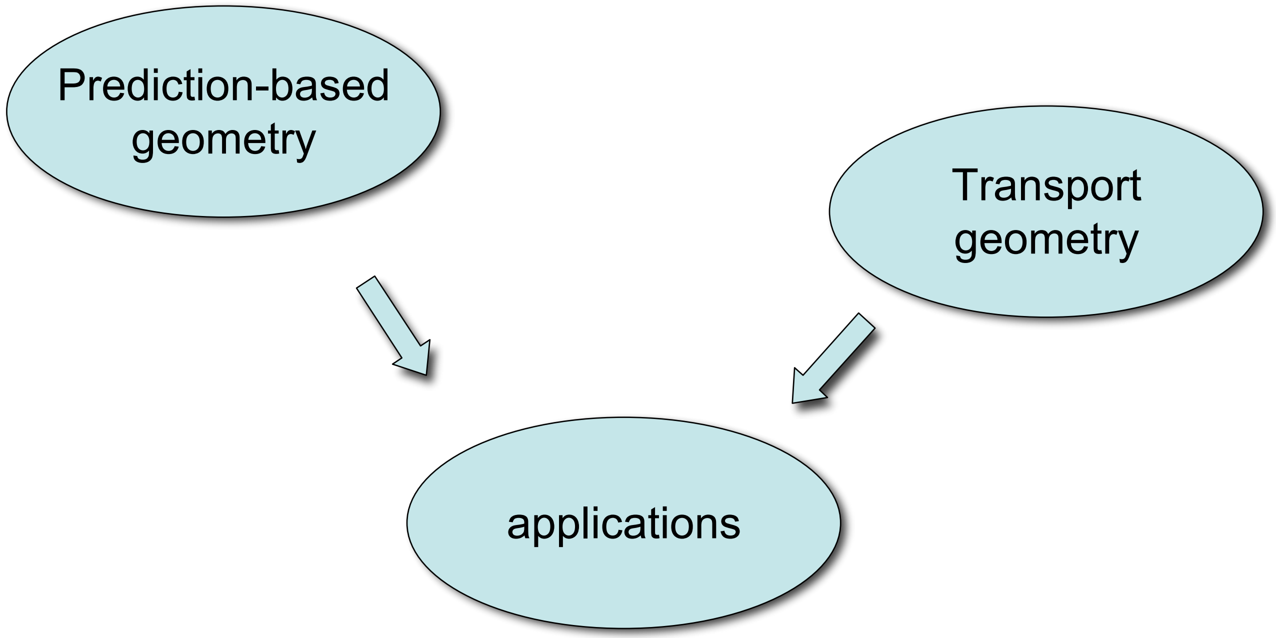
$+$ continuity of statistics



with J. Karlsson & M.S. Takyar

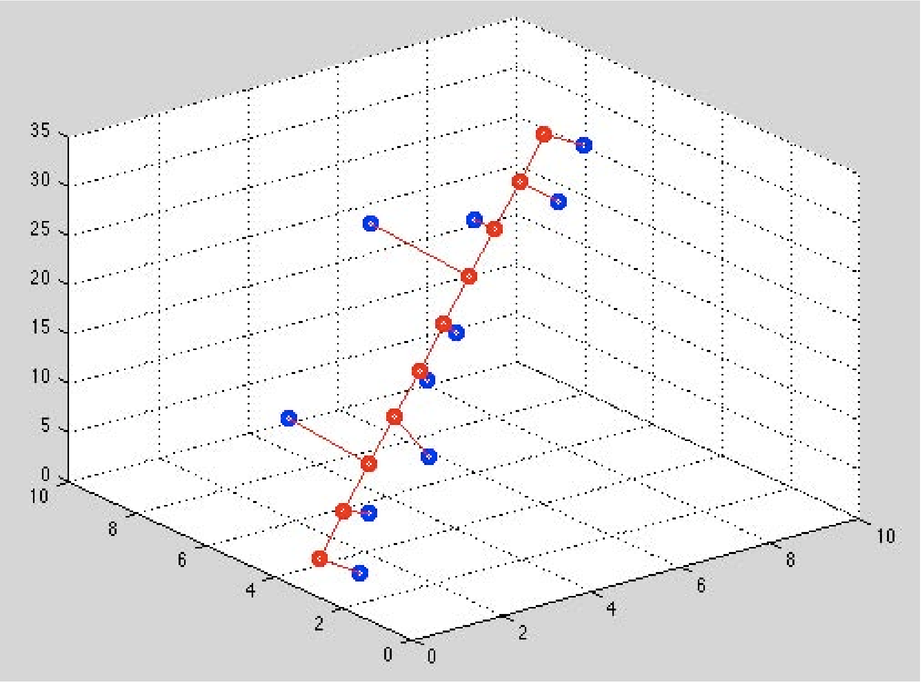
For $p = 1$, it holds that

$$\delta_{1,\kappa}(d\mu_0, d\mu_1) = \max_{\substack{\|g\| \leq \kappa \\ \|g\|_L \leq 1}} \int g(d\mu_0 - d\mu_1)$$


Prediction-based
geometry
Transport
geometry
applications

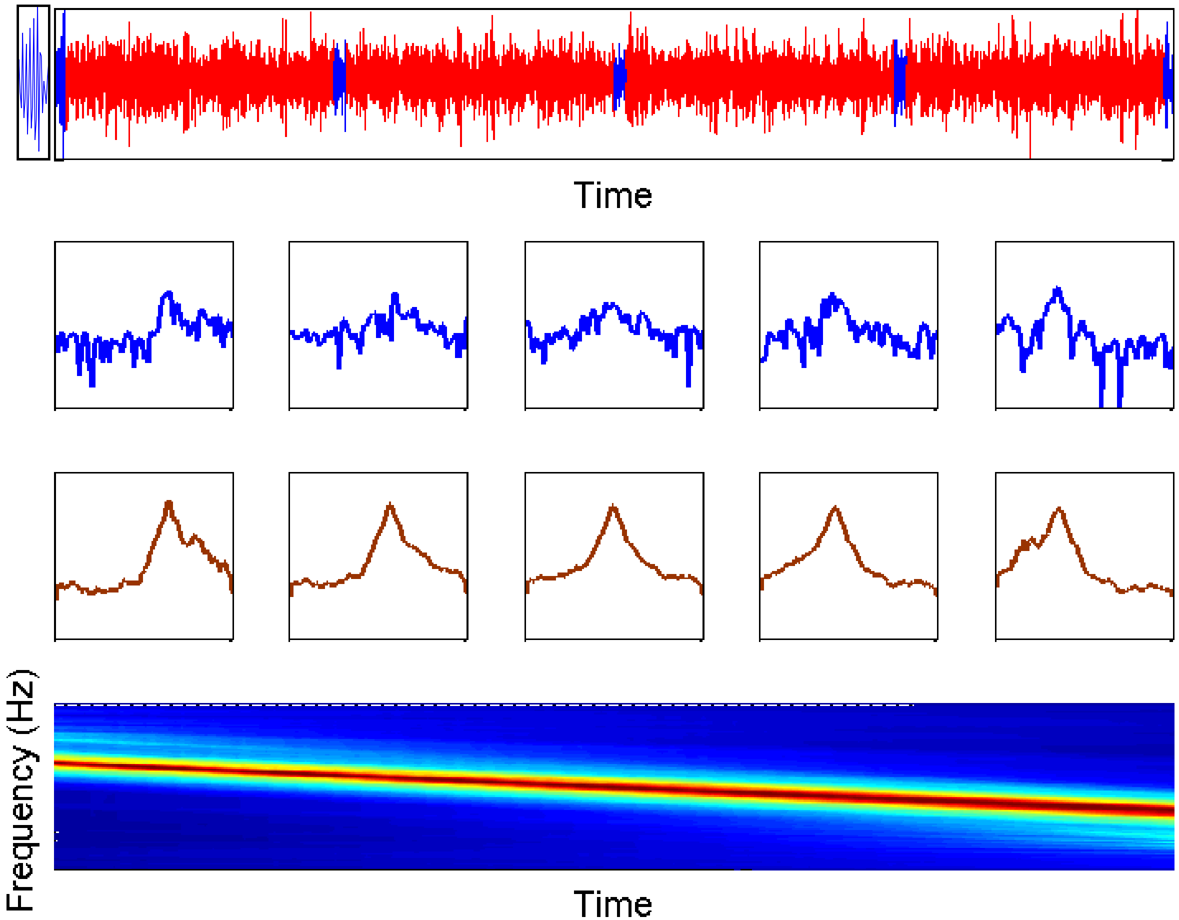
Fitting geodesics

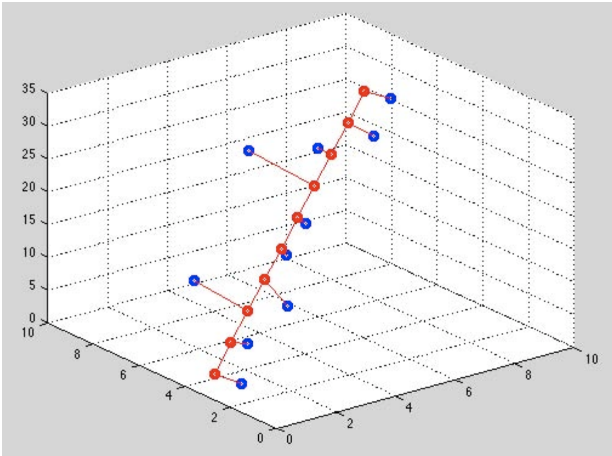
applications



Least squares: The theory of motion of heavenly bodies, Gauss, K.F.

Tracking with geodesics

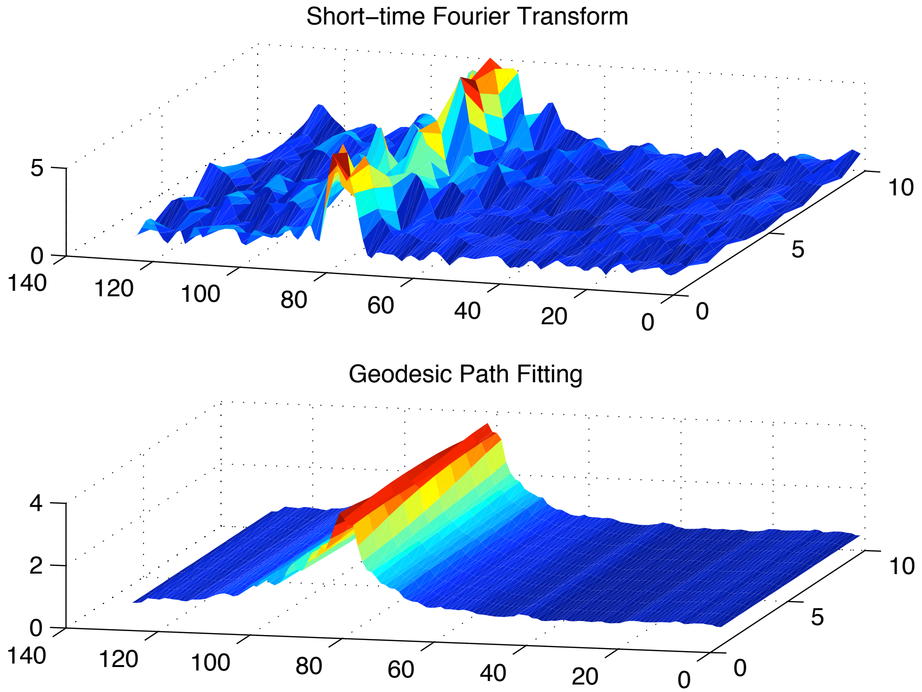


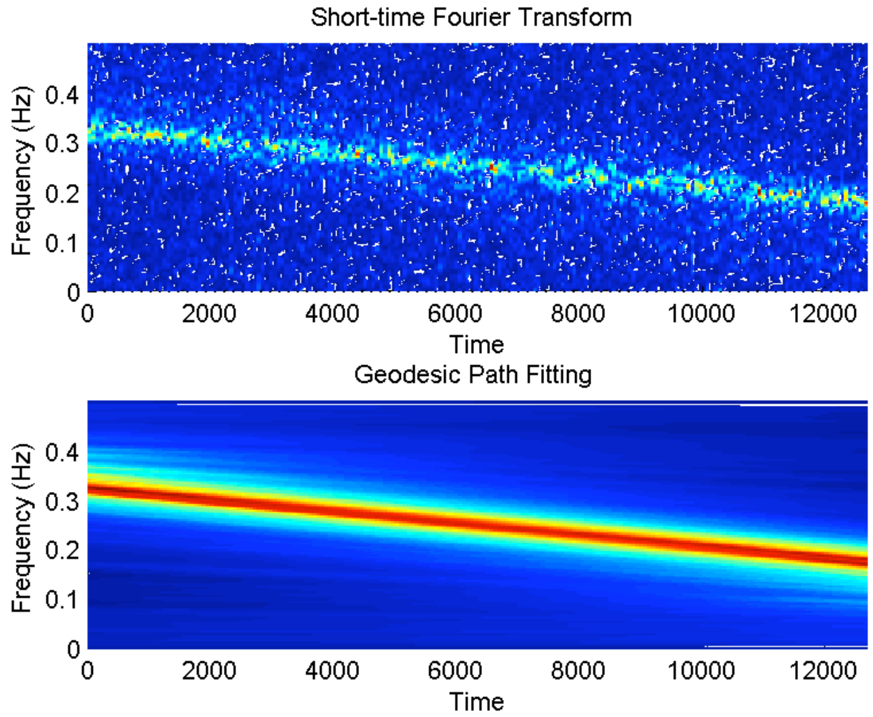


with X. Jiang & T. Luo

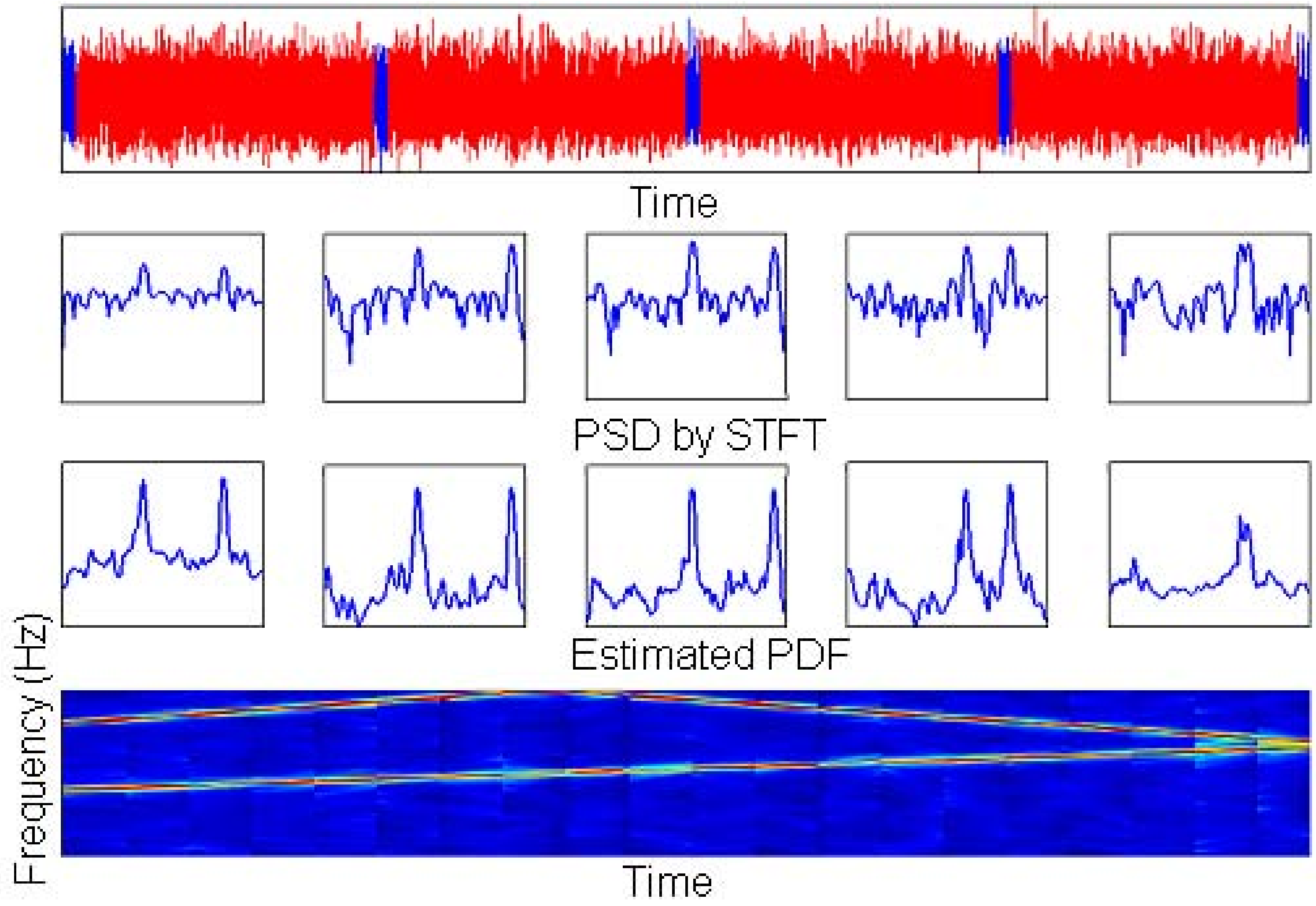
numerics ∼ convex quadratic program

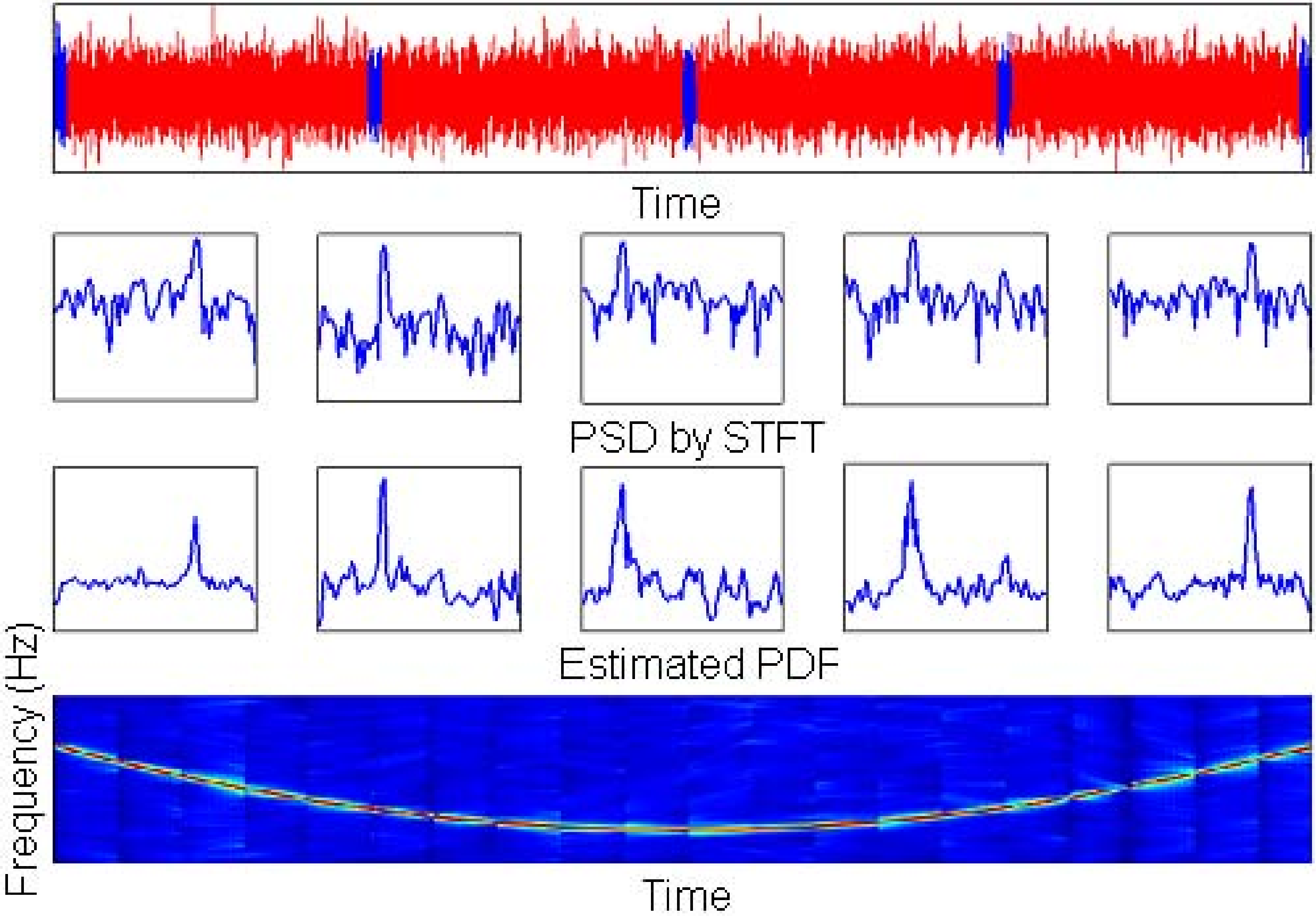
Tracking with geodesics





with Xianhua Jiang

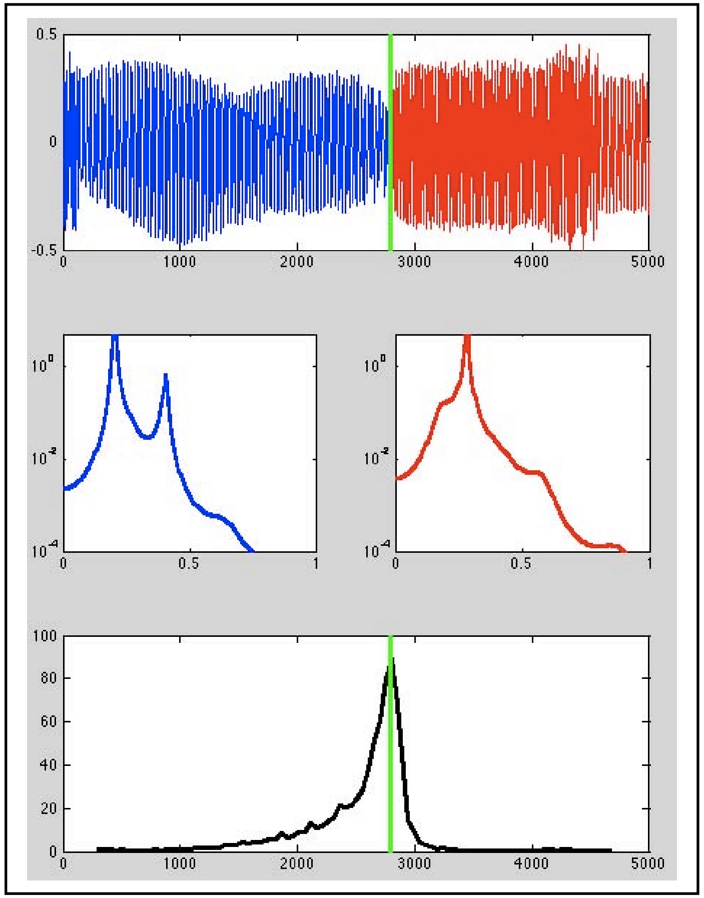

Time
PSD by STFT
Estimated PDF
Frequency (Hz)
Time


Time
PSD by STFT
Estimated PDF
Frequency (Hz)
Time

Voice & sounds

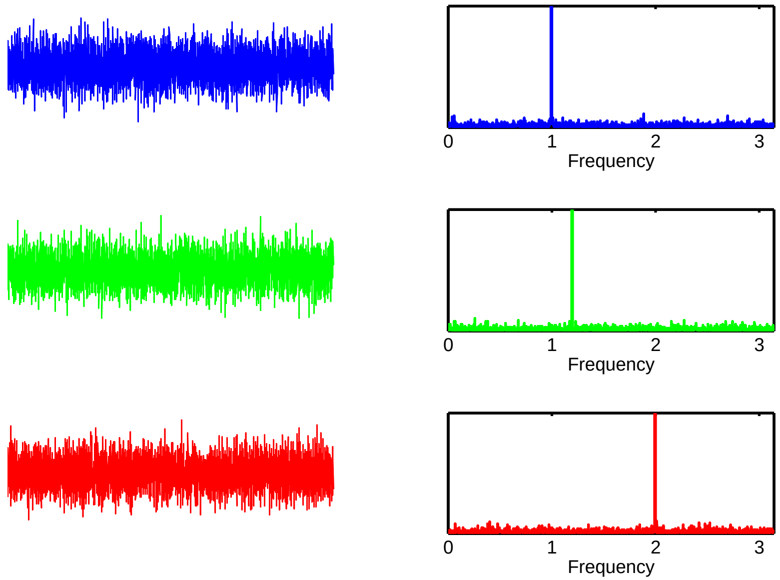
John Weissmuller's MGM Tarzan Yell

http://www.complxmind.com

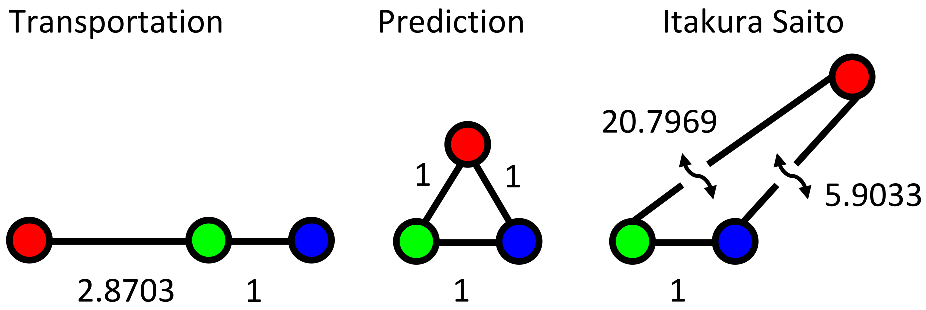


Voice, sounds, radar, etc.

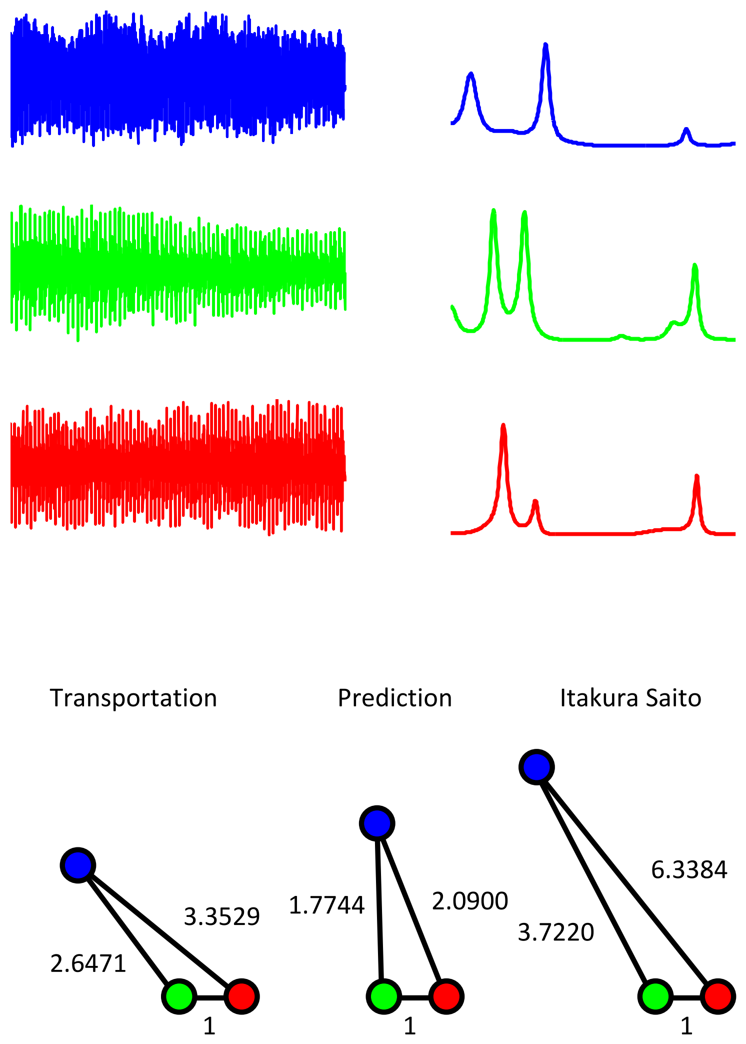
$$y_k = \cos(k\theta + \phi) + w_k$$



discrimination qualities

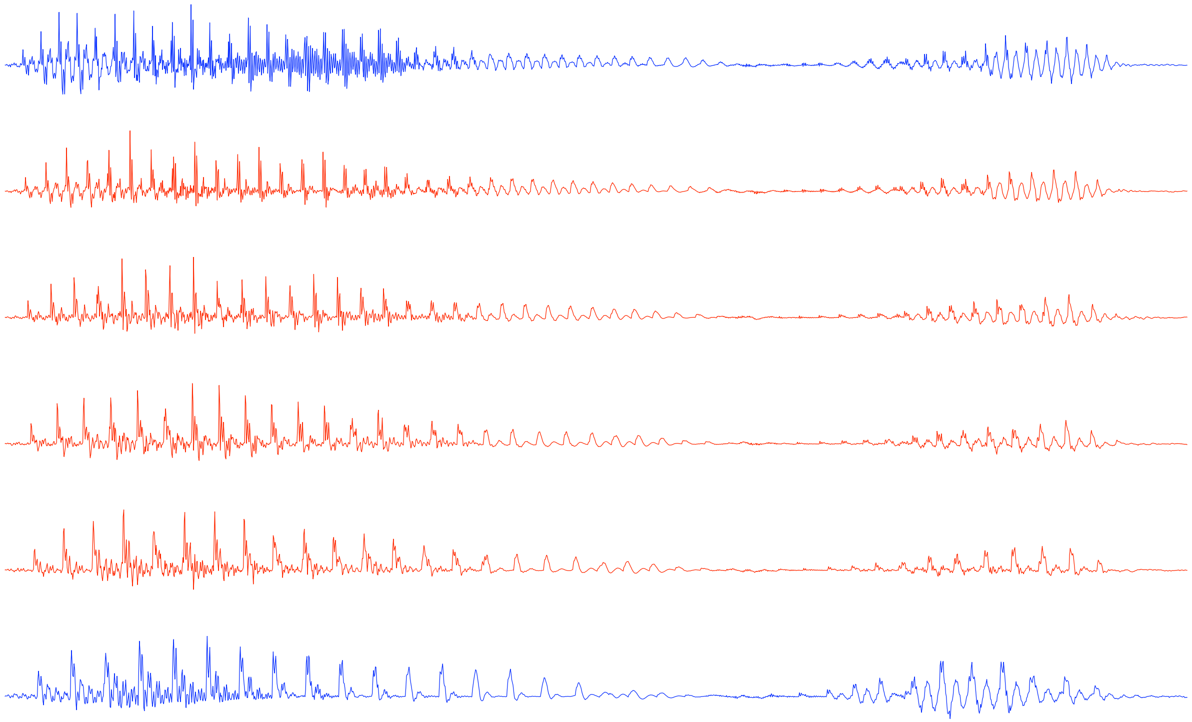


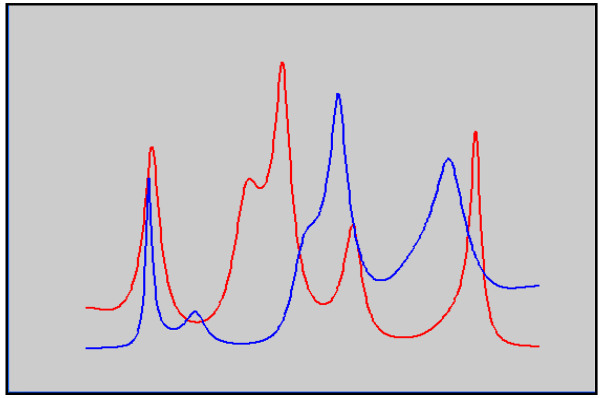
Voice & sounds: phoneme recognition

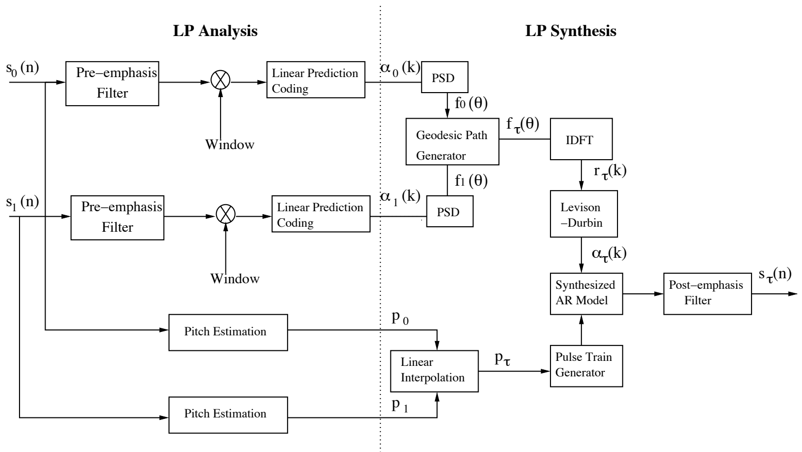


sound "ah"
blue: female
green: male
red: male

Voice & sounds: morphing of speech







Voice & sounds: morphing of speech

advantages:

no fade-in/fade-out

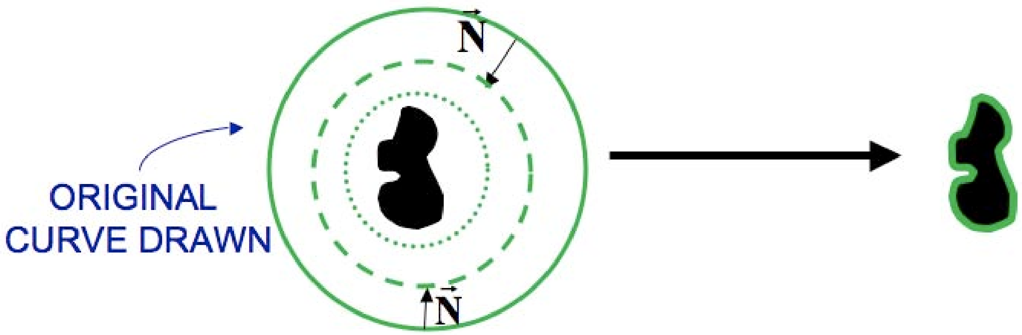
possible drawbacks:

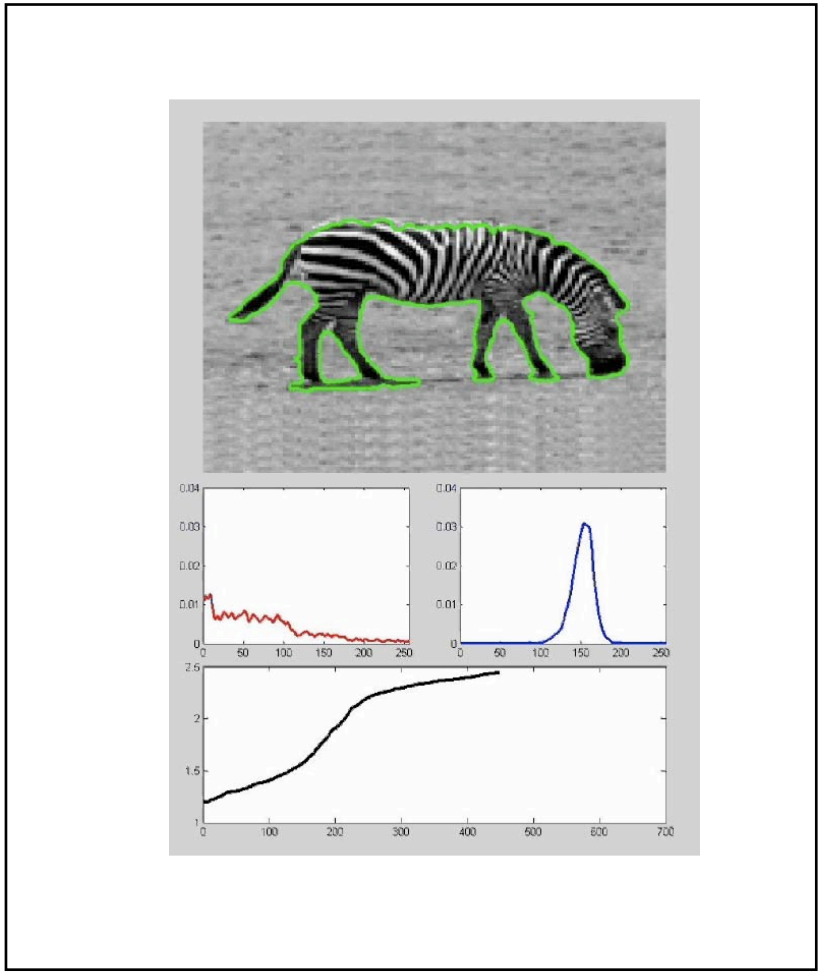
occasionally “artificial sounding” frigatives

Images

Geometric active contours

$$\frac{\partial}{\partial t}\text{Curve} = \nabla_{\text{Curve}} \ \text{metric}(f_{\text{inside}}, f_{\text{outside}})$$





with Romeil Sandhu and Allen Tannenbaum

Concluding thoughts

Metrics
in spectral analysis

- operational significance
- “respect” natural transformations

Thank you for your attention

thanks to

Xianhua Jiang   Johan Karlsson   Romeil Sandhu   Mir Shahrouz Takyar

& Allen Tannenbaum