Transport with congestion: optimization, equilibria and a fast marching gradient numerical algorithm

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1. Discrete models for traffic congestion
   - Nash - Wardrop equilibria
   - Convex optimization
2. Continuous models (joint work with G. Carlier and C. Jimenez)
   - Traffic intensity
   - Equilibria and optimal transport
   - Finite congestion
3. The dual problem (joint with F. Benmansour, G. Carlier, G. Peyré)
   - Duality
   - Discretization
   - Fast Marching Algorithm
4. A gradient algorithm
   - Fast Marching computation of the gradient
   - Convergences
   - Simulations

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The idea

Suppose two roads link two cities: a straight line highway and a longer country path.
If everybody chooses the former, it will become and congested and less performant of the latter. Hence everybody will change his mind and take the other one. And it will be even worse !
Is there an equilibrium?

The objects

- A finite graph with edges $e \in E$ and a set of sources S and destinations D,
- the set $C(s,d) = \{\sigma \text{ from } s \text{ to } d\}$ of possible paths from s to d,
- a demand input $\gamma(s,d)$ denoting the quantity of commuters from $s \in S$ to $d \in D$,
- an unknown repartition strategy (to be looked for) $q = (q_\sigma)_\sigma$ such that $\sum_{\sigma \in C(s,d)} q_\sigma = \gamma(s,d)$,
- a consequent traffic intensity (depending on q) $i_q = (i_q(e))_e$ given by $i_q(e) = \sum_{e \in \sigma} q_\sigma$,
- an increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(i_q(e))$ represents the congestioned cost of the edge e,
- the cost for each path σ, given by $c(\sigma) = \sum_{e \in \sigma} g(i_q(e)) \,\text{length}(e)$.

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Wardrop equilibria

The global strategy q represents the overall distribution of choices of commuters' paths. Imposing a **Nash equilibrium** condition (no single commuter wants to change his mind, provided all the others keep the same strategy) gives the following condition:

$$\sigma \in C(s,d),\ q_\sigma > 0 \Rightarrow c(\sigma) = \min\{c(\tilde{\sigma}) \ :\ \tilde{\sigma} \in C(s,d)\}.$$

This condition is well-known among geographical economists as Wardrop equilibrium.
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Variational principle

Optimizing an overall congestion cost means minimizing a quantity $\sum_e H(i_q(e))$ length(e) ($H : \mathbb{R}^+ \to \mathbb{R}^+$ being an increasing function: for instance with $H(t) = tg(t)$ we get the total cost for all commuters) among all possible strategies q.

Optimality conditions: **if q is optimal, then it is a Wardrop equilibrium** for $g = H'$.

To get a Wardrop equilibrium it is sufficient to solve a convex optimization problem (where H will be the primitive of g).

Unless $g(t) = t^p$, this problem **does not** amount to minimizing the total cost of all commuters!

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Measures formulation

In a domain $\Omega \subset \mathbb{R}^n$ the demand is represented by probabilities $\gamma \in \mathcal{P}(\Omega \times \Omega)$. We are given a set $\Gamma \subset \mathcal{P}(\Omega \times \Omega)$, the set of admissible demand couplings: usually $\Gamma = \{\overline{\gamma}\}$ or

$$\Gamma = \Pi(\mu, \nu) = \{\gamma \in \mathcal{P}(\Omega \times \Omega) \,:\, (\pi_X)_\sharp \gamma = \mu,\, (\pi_Y)_\sharp \gamma = \nu\}.$$

Let us also set

$$C = \{\text{Lipschitz paths } \sigma : [0,1] \to \Omega\}$$
$$C(s,d) = \{\sigma \in C : \sigma(0) = s,, \sigma(1) = d\}.$$

We look for a probability $Q \in \mathcal{P}(C)$ such that $(\pi_{0,1})_\sharp Q \in \Gamma$: it can be expressed as $Q = Q^{s,d} \otimes \gamma$ with $Q^{s,d} \in \mathcal{P}(C(s,d))$.
We want to define a traffic intensity $i_Q \in \mathcal{M}^+(\Omega)$ such that

$$i_Q(A) = \text{“how much ” the movement takes place in } A \ldots$$

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Traffic intensity and overall congestion

For $\phi \in C^0(\Omega)$ and $\sigma \in C$ set

$$L_\phi(\sigma) = \int_0^1 \phi(\sigma(t))|\sigma'(t)|dt.$$

Define i_Q by

$$< i_Q, \phi >= \int_C L_\phi(\sigma) Q(d\sigma).$$

Optimization: we minimize the convex functional

$$F(i_Q) = \begin{cases} \int H(i_q(x))dx & \text{if } i_q << \mathcal{L}^n, \\ +\infty & \text{otherwise} \end{cases}$$

among all admissible strategies Q, H being a convex, increasing and superlinear function. Typically $H(t) = t^p$.

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Optimality conditions

Let $\overline{Q} = \overline{Q}^{s,d} \otimes \overline{\gamma}$ be a minimizer and set $\overline{\xi} = H'(i_{\overline{Q}})$. For $\xi \geq 0$ set

$$c_\xi(s,d) = \inf_{\sigma \in C(s,d)} L_\xi(\sigma)$$

(it's the conformal Riemannian distance induced by ξ).

- $\overline{\gamma}$ minimizes $\int c_{\overline{\xi}} d\gamma$ among $\gamma \in \Gamma$,
- $\overline{Q}$−a.e. $L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(\sigma(0), \sigma(1))$.

Hence $\overline{\gamma}$ solves a Kantorovich **transport problem** and **almost any path is geodesic** (i.e. Wardrop equilibrium with respect to H').

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Finiteness of the minimum

In order to apply the theory, one needs $\min F(i_Q) < +\infty$.

- if $\Gamma = \Pi(\mu,\nu)$, then one can choose Q optimal for Monge Problem and apply De Pascale - Pratelli results to get L^p estimates on the **transport density**;
- if $\Gamma = \{\overline{\gamma}\}$ and $\mu = \nu = \mathcal{L}^n$ one can use incompressible fluid mechanics results and get Q concentrated on uniformly $L-$Lipschitz curves with $(\pi_t)_\sharp Q = \mu$, which implies $i_Q \in L^\infty$ (and, composing with diffeomorphisms, one arrives up to $\mu,\ \nu \in L^p$);
- if $\Gamma = \{\overline{\gamma}\}$ and μ and ν are discrete one can construct by hand a Q such that $i_Q(x) \approx |x - x_i|^{-1}$ near the atoms x_i, which implies $i_Q \in L^p$ for $p < n$.

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Applications

Where to use a continuous model instead of a network one?

In crowd and **pedestrian motion**, for instance.

As a **large scale limit** of car vehicle traffic models (look at the whole L.A. area: you may notice more congested area without necessarily seeing the one-dimensional structure of the road system).

G. Carlier, C. Jimenez , F. Santambrogio, Optimal transportation with traffic congestion and Wardrop equilibria, *SIAM Journal on Control and Opt.* **47** (3): 1330–1350, 2008.

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As a **large scale limit** of car vehicle traffic models (look at the whole L.A. area: you may notice more congested area without necessarily seeing the one-dimensional structure of the road system).

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Applications

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The dual problem

$$(P) = \min_{Q \text{ admissible}} \int_\Omega H(i_Q);$$

$$(D) = \max_{\xi \geq 0} - \int H^*(\xi) + \left(\min_{\gamma \in \Gamma} \int c_\xi \, d\gamma \right).$$

If $\Gamma = \Pi(\mu, \nu)$ then $\min_{\gamma \in \Gamma} \int c_\xi \, d\gamma = W_{c_\xi}(\mu, \nu)$ is the **value of a transport problem**. If $\Gamma = \{\overline{\gamma}\}$ then we obviously have $(D) = \max_{\xi \geq 0} - \int H^*(\xi) + \int c_\xi \, d\overline{\gamma}$.

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Duality and numerics

In the finite network case, the dual problem is usually analyzed instead of the primal one for numerical purposes (simpler constraints, smaller dimension...). Here it will be the same.

From an optimal ξ one can retrieve the corresponding traffic density by $H'(i_Q) = \xi$ (we need H to be strictly convex).

This dual problem involves computing geodesic distances according to the metric ξ, i.e. viscosity solutions of the Eikonal equation $|\nabla \mathcal{U}| = \xi$.

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Fast marching algorithm for distances

Take $\xi \geq 0$ defined on a square grid of size h. Set $\mathcal{U}(x_0) = 0$. Look for

$$(D_x\mathcal{U})^2_{i,j} + (D_y\mathcal{U})^2_{i,j} = h^2(\xi_{i,j})^2,$$

where we denote

$$(D_x\mathcal{U})_{i,j} := \max\{(\mathcal{U}_{i,j} - \mathcal{U}_{i-1,j}), (\mathcal{U}_{i,j} - \mathcal{U}_{i+1,j}), 0\}/h,$$
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The **Fast Marching Method (FMM)** is a numerical method introduced by Sethian for efficiently solving this system (whose solution converges to $c_\xi(\cdot, x_0)$). The numerical complexity of the FMM is $O(N \log(N))$ operations for a grid with N points.

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Grid discretization of the dual problem

$$(DD) = \quad \min J(\xi) = \sum_i H^*(\xi_i) - \sum_{j,k} \gamma(j,k) \mathcal{U}_{x_j;\xi}(x_k)$$

$\mathcal{U}_{x;\xi}(y)$: FMM solution of $|\nabla U| = \xi$ with $\mathcal{U}(x) = 0$, computed at y.
This problem is convex ($\xi \mapsto \mathcal{U}_{x,\xi}(y)$ is concave) and we can solve it by a gradient method.
Hence, we have to derive the $\mathcal{U}$-term w.r.t. ξ.

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FMM Gradient computation

Compute $\mathcal{U}_{x_0,\xi}(y)$ and let ξ vary. The value at y depends on the value of two parents y_1 and y_2 through

$$(\mathcal{U}(y) - \mathcal{U}(y_1))^2 + (\mathcal{U}(y) - \mathcal{U}(y_2))^2 = h^2\xi^2(y)$$

or on the value of one parent only through

$$\mathcal{U}(y) - \mathcal{U}(y_1) = h\xi(y).$$

If ξ varies one gets either

$$\delta\mathcal{U}(y) = \frac{2h^2\xi(y)\delta\xi(y) + \delta\mathcal{U}(y_1)(\mathcal{U}(y) - \mathcal{U}(y_1)) + \delta\mathcal{U}(y_2)(\mathcal{U}(y) - \mathcal{U}(y_2))}{2\mathcal{U}(y) - \mathcal{U}(y_1) - \mathcal{U}(y_2)}$$

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Convergences - 1

We apply a usual **subgradient algorithm** (why subgradient? in general $\mathcal{U}$ is not differentiable w.r.t. ξ, it happens where there are ex-aequo among the parents):

$$\xi^{(1)} = 1; \quad \xi^{(k+1)} = \max\{0, \xi^{(k)} - \rho_k w^{(k)}\}$$

$$\text{where} \quad (w^{(k)})_{i,j} = (H^*)'(\xi^{(k)}_{i,j}) + (v^{(k)})_{i,j} \in \partial J(\xi^{(k)})$$

where $v^{(k)}$ is a vector in the subdifferential of the $\mathcal{U}-$part at the previous point $\xi^{(k)}$ and ρ_k is a suitable sequence of steps.
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Convergences - 2

What about convergence when the discretization gets finer and finer? We prove Γ-**convergence** of the $h-$discretized functional to the original dual one (à la De Giorgi). This implies convergence of the minima and of the minimizers.

Main tool for the $\Gamma - \liminf$: if $\xi_h \rightharpoonup \xi$ in L^{p^*} (ξ_h being identified to a cell-wise constant function), and $\mathcal{U}_h$ are the corresponding solutions of the Eikonal equation, with $\mathcal{U}_h(x_0) = 0$, on the grid (according to FMM), then

$$\limsup_{h \to 0} \mathcal{U}_h(y) \leq \mathcal{U}_\xi(y),$$

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Other applications

Other problems involving geodesic distances under unknown metrics may be approached through this gradient algorithm.

Military applications (Buttazzo, Davini, Fragalà and Macia)

$$\max \quad c_\xi(x_0, x_1) \qquad\qquad a \le \xi \le b, \ \int \xi \le M;$$

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Travel time tomography (Cavalca and Lailly, Leung and Qian)

$$\min \int |\nabla \xi|^p + \sum_k \left(c_\xi(x_k, y_k) - d_k\right)^2.$$

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