# Some limit problems for fractional order operators

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# 1. Fractional Laplacian

# Boundary reactions:

- Reaction in the interior

$$F_1(u) = \int_\Omega |\nabla u|^2 + \int_\Omega W(u)$$
$$\Updownarrow$$
$$\Delta u = W'(u) \quad \text{in } \Omega$$

- Reaction at the boundary

$$F_2(u) = \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} V(Tu)$$
$$\Updownarrow$$
$$\Delta u = 0 \text{ in } \Omega, \qquad \partial_\nu u = -V'(u) \text{ on } \partial\Omega$$

# Fractional Laplacian:

$$(-\Delta)^s f(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi) + \nabla f(x) \cdot (\xi - x)\chi_{\{|x-\xi|x<1\}}}{|x-\xi|^{n+2s}} d\xi, \qquad s \in (0,1)$$

Pseudo-differential operator: $\widehat{(-\Delta)^s f}(\xi) = |\xi|^{2s}\, \hat{f}(\xi)$

**Example:** If $T = (-\Delta)^{\frac{1}{2}}$, then $T^2 = -\Delta$

Poisson Kernel (upper half-space):

$$K_s(x,y) = c_{n,a} \frac{y^{1-a}}{\left(|x|^2 + |y|^2\right)^{\frac{n+1-a}{2}}}, \qquad s = \frac{1-a}{2}$$

# Extension problem for $(-\Delta)^s$:

**Theorem (Caffarelli-Silvestre):** For $s \in (0,1)$, $s = \frac{1-a}{2}$,

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ \\ u = f, & \text{at} \quad y = 0 \end{cases} \quad \Rightarrow \quad (-\Delta)^s f = -d_{n,s} \lim_{y \to 0} y^a \partial_y u$$

Moreover,

$$u = K \star_x f$$

**Idea:** Dirichlet-to-Neumann operator

## Theorem:

Caffarelli-Silvestre extension problem

$$\Updownarrow$$

scattering theory, Paneitz operator (conformal geometry),

- And moreover for $\gamma = m + \alpha$, $m \in \mathbb{N}$, $\gamma = \frac{1-a}{2}$

$$\begin{cases} \mathrm{div}(y^a \nabla u) = 0 & \text{in} \quad \mathbb{R}^{n+1}_+ \\ u = f, & \text{at} \quad y = 0 \end{cases} \qquad \Rightarrow$$

$$(-\Delta)^\gamma f = d_\gamma \lim_{y \to 0} y^a \partial_y \left( y^{-1} \left( \partial_y \left( ... y^{-1} \partial_y u \right) \right) \right)$$

# Mean curvature motion:

Numerical algorithm (Bence-Merriman-Osher): Let $C_0 \subset \mathbb{R}^n$ compact.

$$\text{Solve } \begin{cases} u_t - \Delta u = 0, & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = \chi_{C_0}, & \text{at } t = 0 \end{cases}$$

Fix $\epsilon > 0$ small. Consider $C_\epsilon := \{x \in \mathbb{R}^n : u(x,t) \geq 1/2\}$. Reinitiate.

**Theorem (Evans, etc):** Suitable scaling $\Rightarrow$ $\partial\Omega_t$ moves by mean curvature flow.

**Theorem: (Caffarelli-Souganidis):** Same result if we use Poisson kernel for fractional Laplacian
$s > 1/2 \rightarrow$ movement by mean curvature
$s = 1/2 \rightarrow$ movement by mean curvature (anomalous scaling)
$s < 1/2 \rightarrow$ movement by integral mean curvature

# 2. Phase transitions

# Notion of Γ-convergence:

Let $X$ metric space, $F_\epsilon : X \to [0, +\infty]$. $F_\epsilon \xrightarrow{\Gamma} F$ as $\epsilon \to 0$ if both:

- *Lower bound inequality:* $\forall u \in X$, $\forall (u_\epsilon)$ such that $u_\epsilon \to u$, then

$$\liminf_{\epsilon \to 0} F_\epsilon(u_\epsilon) \geq F(u)$$

- *Upper bound inequality:* $\forall u \in X$, $\exists (u_\epsilon)$ such that $u_\epsilon \to u$ and

$$\limsup_{\epsilon \to 0} F_\epsilon(u_\epsilon) = F(u)$$

**Aim:** Minimize $F$.
**Note:** If $(u_\epsilon)$ minimizes $F_\epsilon$, every cluster point of $(u_\epsilon)$ minimizes $F$.
**Strategy:** Let $\epsilon \to 0$, $(u_\epsilon)$ such that $F_\epsilon[u_\epsilon]$ bounded. To prove: $(u_\epsilon)$ precompact in $X$.

# Classical model of phase transitions:

Container: $\Omega \subset \mathbb{R}^3$, let $0 < V < vol(\Omega)$

Configuration: $u : \Omega \to \{0, 1\}$, $\int_\Omega u = V$, $u \in BV(\Omega, \{0, 1\})$

Interface: $S_u$ =set of discontinuities of $u$

Energy: $F(u) := \sigma H^2(S_u)$, $\sigma$=cst

Equilibrium: $\min F(u)$ among all possible configurations

# Cahn-Hilliard model:

Interface occurs in a thin layer

Configurations: $u : \Omega \to [0,1]$, $\int_\Omega u = V$, $u \in W^{1,2}(\Omega)$

Double well potential: $W(z) = 0 \Leftrightarrow z = 0, 1$

Energy: $F_\epsilon(u) = \epsilon \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon} \int_\Omega W(u)$

Equilibrium: $\min F_\epsilon$

# Connection between both:

$F_\epsilon(u) = \epsilon \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon} \int_\Omega W(u),\ u \in W^{1,2}(\Omega)$

$F(u) = \sigma H^2(S_u),\ u \in BV(\Omega, \{0,1\})$

$\sigma := 2 \int_0^1 \sqrt{W(t)} dt$

**Thm: (Modica-Mortola)**
$F_\epsilon \xrightarrow{\Gamma} F$ and compactness condition is satisfied.

**Cor:** If $u_\epsilon$ minimizes $F_\epsilon$, then $\exists$ subsequence $(u_\epsilon) \to u$, $u$ minimizer of $F$.

# Two phase fluids:

Reference: Alberti-Bouchitté-Seppecher

Idea: include boundary reactions

Double well potentials: $W(x) = 0 \Leftrightarrow x = \alpha, \beta, \quad V(x) = 0 \Leftrightarrow x = \alpha', \beta'$

**Functional:**

$$F_\epsilon(u) = \epsilon \int_\Omega |\nabla u|^2 + \frac{1}{\epsilon} \int_\Omega W(u) + \lambda_\epsilon \int_{\partial\Omega} V(Tu)$$

**Heuristics:** $u_\epsilon \to u_0$, and $Tu_\epsilon \to v$, but in general, $v \neq Tu_0$

## Capillarity model with line tension effect:

$$\phi(u,v) = \sigma\mathcal{H}^2(S_u) + \int_{\partial\Omega} |H(Tu) - H(v)| + c\mathcal{H}^1(Sv)$$
$$F(u) := \inf\{\phi(u,v) : v \in BV(\partial\Omega)\}$$

**Theorem:** (Alberti-Bouchitte-Seppecher)
Let $H$ primitive of $2\sqrt{W}$, $\sigma = |H(\beta) - H(\alpha)|$, $c = (\beta' - \alpha')K/\pi$, where

$$\lim \epsilon \log \lambda_\epsilon = K,$$

then $\quad F_\epsilon \xrightarrow{\Gamma} F \quad + \quad$ compactness

**Aim:** Study boundary reactions for the fractional Laplacian

**Theorem:** Fix $s > 1/2 \Leftrightarrow a < 0$. Let

$$F_\epsilon(u) = \epsilon^{1-a} \int_\Omega |\nabla u|^2 \text{dist}(x, \partial\Omega)^a + \frac{1}{\epsilon^{1-a}} \int_\Omega W(u) \text{dist}(x, \partial\Omega)^{-a} + \lambda_\epsilon \int_{\partial\Omega} V(Tu)$$

$$F[u] = \inf_v \Phi(u, v)$$

$$\Phi(u, v) := \sigma \mathcal{H}^2(S_u) + \int_{\partial\Omega} |H(Tu) - H(v)| + \kappa \mathcal{H}^1(Sv)$$

$$\sigma := H(\beta) - H(\alpha), \quad \kappa := C_s(\beta' - \alpha')^2,$$

$$H \text{ is a primitive of } 2\sqrt{W}, \quad \lambda_\epsilon \sim \epsilon^{\frac{1-a}{a}+\theta}, \quad \theta > 0$$

Then $F_\epsilon \xrightarrow{\Gamma} F$ + compactness

Proof:

$$F_\epsilon(u) = \epsilon^{1-a} \int_\Omega |\nabla u|^2 \text{dist}(x, \partial\Omega)^a + \frac{1}{\epsilon^{1-a}} \int_\Omega W(u) \text{dist}(x, \partial\Omega)^{-a} + \lambda_\epsilon \int_{\partial\Omega} V(Tu)$$

- 1-phase: $\lambda_\epsilon = 0$
- Slicing argument
- A singular perturbation of the norm $H^s = W^{s,2}$

$$\epsilon^{1-a} \iint_{E^2} \frac{|v(x') - v(x)|^2}{|x' - x|^{1+2s}} \, dx' dx + \lambda_\epsilon \int_E V(v) dx$$

- New trace inequalities for weighted Sobolev spaces

$$v = Tu$$

**Result:** (A singular perturbation of the norm $H^s$).

Let $E \subset \mathbb{R}$ interval. Consider the functional

$$G_\epsilon(v, E) = \epsilon^{1-a} \iint_{E^2} \frac{|v(x') - v(x)|^2}{|x' - x|^{1+2s}} \, dx' dx + \lambda_\epsilon \int_E V(v) dx$$

such that

$$\lambda_\epsilon \sim \epsilon^{\frac{1-a}{a} + \theta}, \quad \theta > 0$$

and the functional

$$G(v, E) := C_0 \#(S_v)$$

Then

$$G_\epsilon \xrightarrow{\Gamma} G \quad + \quad \text{compactness}$$

# Trace inequalities:

**Spaces:**

$$W^{1,2}(\Omega, d^a) \quad \text{with norm} \quad \left(\int_\Omega |u|^2\, dist(\cdot, \partial\Omega)^a + \int_\Omega |\nabla u|^2 dist(\cdot, \partial\Omega)^a\right)^{\frac{1}{2}}$$

$$H^s(\partial\Omega) \quad \text{with norm} \quad \left(\int_{\partial\Omega} |v|^2 + \int\int_{\partial\Omega\times\partial\Omega} \frac{|v(x) - v(x')|^2}{|x - x'|^{1+2s}} dx dx'\right)^{\frac{1}{2}}$$

**Theorem (Nekvinda):** If $-1 < a < 1$, the class of traces on $\partial\Omega$ of $W^{1,2}(\Omega, d^a)$ equals the space $H^s(\partial\Omega)$, $s = \frac{1-a}{2}$.

**Theorem:** Let $\Omega := [-1,1] \times [0,1] \subset \mathbb{R}^2_+$. There exists $D_s$ such that

$$\iint_{[-1,1]^2} \frac{\left|Tu(x) - Tu(x')\right|^2}{|x - x'|^{1+2s}} \, dx dx' \leq D_s \int_\Omega |\nabla u|^2 y^a \, dx dy$$

and it is the best constant.

# 3. Dislocations

## Reaction-diffusion eq: $L = -(-\Delta)^{1/2}$

$$v_t = Lv - W'(v) + \sigma(t,x) \quad \text{in } \mathbb{R}$$

- $v$ independent of $t$: Peierls-Nabarro model describing dislocations

- Rescaling

$$\begin{cases} v_t^\varepsilon = \dfrac{1}{\varepsilon}\left\{Lv^\varepsilon - \dfrac{1}{\varepsilon}W'(v^\varepsilon) + \sigma(t,x)\right\} \\ v^\varepsilon(0,x) = v_0^\varepsilon(x) \quad \text{for} \quad x \in \mathbb{R}. \end{cases}$$

- **Aim:** Study limit dynamics as $\varepsilon \to 0$.

**Assumptions:**

$$W \in C^{2,\beta}, \quad W(v+1) = W(v), \quad W = 0 \text{ on } \mathbb{Z}, W > 0 \text{ on } \mathbb{R}\backslash\mathbb{Z}, \quad W''(0) > 0$$

$$\begin{cases} \sigma \in BUC\,([0,+\infty) \times \mathbb{R}) \quad |\sigma_x|_{L^\infty([0,+\infty)\times\mathbb{R})} \le K, \quad |\sigma_t|_{L^\infty([0,+\infty)\times\mathbb{R})} \le K, \\ |\sigma_x(t,x+h) - \sigma_x(t,x)| \le K|h|^\beta \quad \text{for all} \quad x, h \in \mathbb{R}, \quad t \in [0,+\infty) \end{cases}$$

For $x_1^0 < x_2^0 < \dots < x_N^0$, we set

$v_0^\varepsilon(x) = \frac{\varepsilon}{\alpha}\sigma(0,x) + \sum_{i=1}^N \phi\left(\frac{x - x_i^0}{\varepsilon}\right)$ with $\alpha = W''(0) > 0$

# Theorem:

ODE system of particles: $\{x_i(t)\}_{i=1}^N$

$$\begin{cases} \dfrac{dx_i}{dt} = \gamma \left( -\sigma(x_i, t) + \dfrac{1}{\pi} \displaystyle\sum_{j \neq i} \dfrac{1}{x_i - x_j} \right) \\ x_i(0) = x_i^0, \end{cases}$$

Then:

$\exists!$ viscosity solution $v^\varepsilon$

Let $v^0(t, x) = \sum_{i=1}^N H(x - x_i(t))$

As $\varepsilon \to 0$, $v^\varepsilon \to v^0$

**Ansatz:**

$$\tilde{v}^{\varepsilon}(t,x) = \varepsilon\tilde{\sigma} + \sum_{i=1}^{N} \phi\left(\frac{x - x_i(t)}{\varepsilon}\right)$$

**Layer solutions (Cabre-SolaMorales)**

$$\begin{cases} L\phi - W'(\phi) = 0 \quad \text{on} \quad \mathbb{R}, \\ \phi' > 0 \quad \text{and} \quad \phi(-\infty) = 0, \quad \phi(0) = \dfrac{1}{2}, \quad \phi(+\infty) = 1 \end{cases}$$

**Ansatz:**

$$\tilde{v}^{\varepsilon}(t,x) = \varepsilon\tilde{\sigma} + \sum_{i=1}^{N} \phi\left(\frac{x - x_i(t)}{\varepsilon}\right) + \mathit{corrector}$$

**Layer solutions (Cabre-SolaMorales)**

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**Thank you!**