Probability evolution for complex multi-linear non-local interactions

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IPAM workshop , March 08

Aspects of Optimal Transport in Geometry and Calculus of Variations

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Motivations from statistical physics or interactive 'particle' systems

1. Rarefied ideal gases-elastic: ***conservative Boltzmann Transport eq.***

2. **Energy dissipative phenomena:** Gas of elastic or inelastic interacting systems in the presence of a thermostat with a fixed background temperature θ_b or Rapid granular flow dynamics: (**inelastic hard sphere interactions)**: homogeneous cooling states, randomly heated states, shear flows, shockwaves past wedges, etc.

3. **(Soft) condensed matter at nano scale:** Bose-Einstein condensates models, charge transport in solids: current/voltage transport modeling semiconductor.

4. **Emerging applications from stochastic dynamics** for multi-linear Maxwell type interactions : *Multiplicatively Interactive Stochastic Processes:* Pareto tails for wealth distribution, non-conservative dynamics: opinion dynamic models, particle swarms in population dynamics, etc

Goals:

- ***Understanding of analytical properties: large energy tails***

•long time asymptotics and characterization of asymptotics states: high energy tails and singularity formation

•A unified approach for Maxwell type interactions and generalizations.

A revision of the Boltzmann Transport Equation (BTE)

A general form for the space-homogenous BTE with external heating sources :

$$\frac{\partial f}{\partial t}(v,t) = Q_{\beta,\gamma,d}(f)(v,t) + \mathcal{G}(f)(v,t)$$

$$Q_{\beta,\gamma,d}(f)(v,t) = a(d,M)\int_{v_*,\sigma\in\mathbb{R}^d\times\sigma\in S^{d-1}} B_{\beta,\gamma,d}(|u|,\tfrac{u\cdot\sigma}{|u|})\left(J_\beta{}'f\,'f_* - f\,f_*\right)dv_*d\sigma$$

$$v' = v + \tfrac{\beta}{2}(|u|\sigma - u), \qquad v'_* = v_* - \tfrac{\beta}{2}(|u|\sigma - u) \text{ interaction law;}$$

$$u = v - v_* \text{ (relative velocity)}$$

$$B_{\beta,\gamma,d}(|u|,\sigma(\theta)) \qquad \text{(collisional kernel)}$$

$$\cos\theta = \frac{(u\cdot\sigma)}{|u|} \text{ cosine of scattering angle,}$$

$$\beta = \frac{1+e}{2}, \qquad e = \text{restitution coefficient}$$

$\beta = e = 1$ elastic interaction , $\beta < 1$ dissipative interaction

$$J_\beta = \frac{\partial(v,v_*)}{\partial('v,'v_*)} \text{ post-precollision Jacobian}$$

A revision of the Boltzmann Transport Equation (BTE)

Collisional kernel or transition probability of interactions is calculated using the intramolecular potential laws: Example for $d = 3$

$$V = r^{-s} \quad \text{with} \quad s \in (2, \infty)$$

$$B_{\beta,\gamma,d}(|u|, \sigma(\theta)) = b_{\beta,\gamma,d}(\sigma(\theta))\ |u|^{\gamma}\ ,$$

$$\cos\theta \ = \ \frac{(u \cdot \sigma)}{|u|} \quad \text{cos of scattering angle,}$$

with $\gamma = 0$ (Maxwell-Molecule models), $\gamma = 1$ (hard spheres),

We assume the growth condition $\quad 0 < b_{\beta,\gamma,d}(\sigma(\theta))\, \theta^{\alpha(d)} < K$

$\alpha > d - 1$ $\quad \int_{\sigma \in S^{d-1}} b_{\beta}(\sigma) d\sigma = 1$ **Grad cut-off condition**

In 3 dimensions:

$$\gamma = \frac{s-5}{s-1} \quad \text{and} \quad b_{\beta,\gamma,d}(\sigma(\theta)) \approx \theta^{-d+3-\nu} \text{ with } \quad \nu = \frac{2}{s-1}$$

- the Grad cut-off assumption is satisfied for variable hard potentials ($s \in (5, \infty)$)
- the α-growth condition on $b_{\beta,\gamma,d}$ is satisfied for $\alpha > 2$

Dynamically scaled solutions and stability

- Self-similar solution:

set $f(v,t) = V_0^{-n}(t)\,\tilde f(\tilde v, \tilde t)$, (ex: $\tilde v = \frac{v}{V_0(t)} = v\,(a+\mu t)$, $\tilde t(t) = \frac{1}{\mu}\ln(\frac{a+\mu t}{a})$),
then

$$\boxed{\frac{\partial \tilde f}{\partial \tilde t} = \kappa(t)\{Q^+(\tilde f, \tilde f) - \tilde f(\tilde f * |\tilde v|^\gamma)\} - \mu\,\mathrm{div}(\tilde v \tilde f)\,, \qquad \kappa > 0.}$$

- For **Maxwell type models**: $< Q^+(f,f), |v|^2 > \; = \; \mu\Theta(t)$ with μ the **energy production rate** $\Rightarrow$ linear ODE for the energy $\Rightarrow \Theta(t) = e^{\mu t}$

- An example $V_0^2 \; = \; \theta(t) \; = < f, |v|^2 >$ (i.e. V_0:=thermal speed)

- **Homogeneous cooling states:** energy dissipation $\mu < 0$

setting $f(v,t) \; = \; \Theta^{-n/2}(t)\,\tilde f(\frac{v}{\Theta^{1/2}}(t)) \; = \; e^{-\frac{n}{2}\mu t}\,\tilde f(v\,e^{-\frac{\mu t}{2}})$

$$\boxed{\mu\,\tilde v\,\frac{\partial}{\partial \tilde v}\tilde f = Q^+(\tilde f, \tilde f) - \tilde f\,.}$$

- Do they exist in the energy dissipative case?
- Are they attractors for solutions of the energy dissipative problem?
- what are their properties? $\Rightarrow$ They depend on the collision frequency

$$\frac{\partial f}{\partial t}(v,t) = \int_{\mathbb{R}^d \times S^{d-1}} [J_\beta f('v,t) f('v_*,t) - f(v,t) f(v_*,t)]\, |u|^\gamma \; b_{\gamma,d}\left(\frac{u\cdot\sigma}{|u|}\right) dv_* d\sigma + \theta \mathcal{G}(f)(v,t)$$

NESS satisfies : $\boxed{\int_{\mathbb{R}^d} f_\infty(v) \mathcal{M}_\gamma^{-1} dv}$

β	γ	$\mathcal{G}(f)$	$\mathcal{M}_\gamma$ = **NESS** tail asymptotics
$\beta = 1$	$0 \le \gamma \le 1\ (VHP)$	0	$C \exp(-r\,\lvert v\rvert^2)$
$\frac{1}{2} \le \beta < 1$	$0 \le \gamma \le 1\ (VHP)$	$\Delta_v f$	$C \exp(-r\lvert v\rvert^{\frac{\gamma+2}{2}})$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$\Delta_v f + \tau \nabla \cdot (v f)$	$C \exp(-r\,\lvert v\rvert^2)$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$v_2 \frac{\partial f}{\partial v_3}$	at least $C \exp(-r\,\lvert v\rvert^1)$
$\frac{1}{2} \le \beta < 1$	$0 < \gamma \le 1\ (VHP)$	$-\mu\ v \cdot \nabla f$	$C \exp(-\,r\,\lvert v\rvert^\gamma)$
$\frac{1}{2} \le \beta \le 1$	$\gamma = 0$ (MM)	$\theta_b Q(f, M_T) - \mu\, v \cdot \nabla f$	$C(c_1 + c_2\lvert v\rvert^{\mathrm{k}})^{-1}$

for $C = C_{(\gamma,\beta,\theta,d)}$ and $r = r_{(\gamma,\beta,\theta,d)}$. Also C, c_1, c_2 and $\mathbf{k}$ in the last case depend on $\beta, \theta, \theta_b, T, d$

Reviewing elastic and inelastic properties

Due to symmetries of the collisional integral one can obtain (after interchanging the variables of integration)

Weak (Maxwell) form for the BTE

$$\boxed{\frac{\partial}{\partial t}\int_{I\!R^N} f\,\varphi\,dv = \int_{I\!R^d} Q(f,f)\varphi(v)\,dv = \frac{\kappa(t)}{2}\int_{I\!R^N}\int_{I\!R^N}\int_{S_+^{N-1}} f f_*\,(\varphi' + \varphi'_* - \varphi - \varphi_*)|u|^{\gamma}\,\tilde{b}(\sigma)\,d\sigma\,dv_*\,dv,}$$

Properties: It is easy to see, from the weak formulation:

conservation of mass ρ and momentum J: set $\varphi(v) = 1$ and $\varphi(v) = v$

Using local conservation of momentum on the test function: $\boxed{{}'v + {}'v_* = v + v_*}$

$$\frac{\partial}{\partial t}\int_{I\!R^N} f\,\{1, v_i\}dv = \kappa(t)\int_{I\!R^d} Q(f,f)(v)\{1, v_i\}\,dv = 0, \quad i = 1, 2, 3.$$

holds, both for the **Elastic** and **Inelastic** cases

Next, set $\varphi(v) = |v|^2 \Rightarrow$ It conserves energy for $\alpha = 1$ – **ELASTIC**:

Using local conservation of energy on the test function: $\boxed{|{}'v|^2 + |{}'v_*|^2 = |v|^2 + |v_*|^2}$

$$\boxed{\frac{\partial}{\partial t}\Theta(t) = \kappa(t)\int_{I\!R^d} Q(f,f)(v)\,|v|^2\,dv = 0}$$

Reviewing elastic and inelastic properties

However, it **dissipates energy for $\alpha < 1$ INELASTIC**:

Set $\varphi(v) = |v|^2$ and using local energy dissipation:

$$\boxed{|v|^2 + |v_*|^2 - |'v|^2 - |'v_*|^2 = -\frac{1-\alpha^2}{4}(1 - \nu \cdot \sigma)|v - v_*|^2}$$

One obtains and **energy dissipation inequality** by Jensen's ineq.

$$\frac{\partial}{\partial t}\Theta(t) = -c_N \frac{(1-\alpha^2)}{4}\kappa(t)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f f_* |v - v_*|^{2+\gamma}\, dv_*\, dv$$

$$\leq -c_N \frac{(1-\alpha^2)}{4}\kappa(t)\Theta(t)^{\frac{\gamma+2}{2}}$$

For a Maxwell type model: a linear equation for the kinetic energy

For $\gamma = 0$ one has the energy identity $\boxed{\Theta_t = -c_N \frac{(1-\alpha^2)}{4}\kappa(t)\Theta}$

Reviewing elastic properties

Recall Boltzmann H-Theorem for ELASTIC interactions:

$$\frac{\partial}{\partial t}\int f \log f \, dv = \kappa(t)\int_{I\!R^N} Q(f,f)\log f \, dv =$$

$$\frac{\kappa(t)}{2}\int_{I\!R^N}\int_{I\!R^N}\int_{S_+^{N-1}} \left(f f_* - f' f'_*\right)\log\frac{f' f'_*}{f f_*}|u|^{\gamma} b(\sigma)d\sigma\, dv\, dv_* \le 0$$

Time irreversibility is expressed in this inequality $\Longrightarrow$ **stability**

In addition:

The Boltzmann Theorem: *there are only N+2 collision invariants* $\Longleftrightarrow$

$$\int_{I\!R^N} Q(f,f)\log f\, dv = 0 \iff \log f(\cdot,v) = A + B\cdot v + C|v|^2 \iff$$

$f(\cdot,v) = M_{A,B,C}(v)$ Maxwellian (Gaussian in v-space) parameterized by A, B, C

related the first $N+2$ moments of the initial probability state of $f(0,v) = f_0(v)$

Reviewing elastic properties

Time Irreversibility and relation to Thermodynamics

- **Stability** $\lim_{t\to\infty}\|f(t,v)-M_{A,B,C}\|_{L^1_2}\to 0$ where $\{A,B,C\}\longleftrightarrow$ $\{\rho,u,w\}$, $\rho=\int f_0\,dv$, $\rho u=\int v f_0\,dv$ and $\rho w=\int |v|^2 f_0\,dv$

- **Macroscopic balance equations:** For the space inhomogeneous problem:

Under the ansatz of a Maxwellian state in v-space

$$\boxed{f(t,x,v)=M_{a,b,\mathbf{u}}=ae^{-(b|v-\mathbf{u}|^2)}}$$

where the dependance of (t,x) is only though the parameters $(a,b,\mathbf{u})$:

$\mathbf{u}=\dfrac{\mathbf{J}}{\rho}$ mean velocity and $\Theta=\rho\mathrm{w}=\dfrac{1}{2}\rho\mathbf{u}+\rho\,\mathbf{e}$ kinetic energy, $\mathbf{e}=$ internal en

choosing $a=\dfrac{3^{3/2}\rho}{(4\pi\,e)^{3/2}}$; $b=\dfrac{3}{4e}$

plus equilibrium constitutive relations : $P=\dfrac{2}{3}\rho\,e$ **pressure.**

Reviewing elastic properties

One obtains the Euler equations:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i}) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho\, \mathbf{u_i}) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i u_j} + p) = 0\,, \quad (j = 1, 2, 3)$$

$$\frac{\partial}{\partial t}(\rho(\frac{1}{2}|\mathbf{u}|^2 + e)) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i}(\frac{\mathbf{1}}{\mathbf{2}}|\mathbf{u}|^2 + \mathrm{e} + \frac{\mathrm{p}}{\rho})) = 0\,.$$

- **Hydrodynamic limits:** for ϵ-perturbations of Maxwellians plus constitutive relations $\Rightarrow \{A, B, C\}(t, x)$ the corresponding macroscopic system satisfy compressible Euler

or ϵ-Navier-Stokes equations with higher order partial derivatives terms proportional to an $O(\epsilon)$ deviations from Gaussian (Maxwellian) distributions.

Reviewing inelastic properties

Back to inelastic interactions:

INELASTIC Boltzmann collision term: No H-Theorem in the classical way! if $\alpha =$ constant < 1

$$\int_{I\!R^d} Q(f,f)\log f\,dv = \frac{1}{2}\int_{I\!R^d}\int_{I\!R^d}\int_{S_+^{d-1}} f f_* \left(\log\frac{f'f'_*}{ff_*} - \frac{f'f'_*}{ff_*} + 1\right)|u|^\gamma b(\sigma)\,d\sigma\,dv\,dv_* + \frac{1-\alpha^2}{2\alpha^2}\int_{I\!R^d}\int_{I\!R^d} f f_*\,|u|^\gamma\,dv\,dv_*.$$

Recall: it dissipates energy for $\alpha < 1$ by Jensen's inequality:

$$\boxed{\frac{\partial}{\partial t}\Theta(t) = -c_N\frac{(1-\alpha^2)}{4}\kappa(t)\int_{I\!R^d}\int_{I\!R^d} f f_* |v-v_*|^{2+\gamma}\,dv_*\,dv \le -c_N\frac{(1-\alpha^2)}{4}\kappa(t)\Theta(t)^{\frac{\gamma+2}{2}}}$$

$\Rightarrow$ Inelasticity brings loss of micro reversibility

$\Rightarrow$ but keeps time irreversibility !!: That is, there are stationary states and in some particular cases we can show stability to stationary and self-similar states (**Maxwell molecule equations of collisional type**)

$\Rightarrow$ However: Existence of NESS: Non Equilibrium Statistical States (stable stationary states are non-Gaussian **pdfs**)

Molecular models of Maxwell type

$$\boxed{f_t = Q^+(f,f)(t,v) - c_{\rho_0, b(cos(\theta)),N}\; f(v)}$$

Bobylev, '75-80, for the elastic, energy conservative case– For inelastic interactions: Bobylev,Carrillo, I.M.G. 00 Bobylev, Cercignani,Toscani, 03, Bobylev, Cercignani, I.M.G'06, for general non-conservative problem

For $\varphi(t,k) = \mathcal{F}_{v\to k}[f(t,v)], \quad \varphi(t,0) = \int f_0 dv = 1, \;\; \forall t > 0$

$$\frac{\partial \varphi}{\partial t} = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \{\varphi(t,k_-)\varphi(t,k_+) - \varphi(t,0)\varphi(t,k)\}\, b(\frac{k\cdot\sigma}{|k|})\, d\sigma = \Gamma(\varphi,\varphi) - c_N\varphi(t,k)$$

$$k_- = \beta\frac{(k-|k|n)}{2} = \beta|k|s, \qquad k_+ = k - k_- \qquad \beta = \frac{1+\alpha}{2}$$

$$s = \tfrac{1}{2}(1 - \tfrac{k\cdot\sigma}{|k|}) \;, \quad G(s) = \tfrac{|S^{N-2}|}{2^{N-2}}\, b(1-2s)\,(s-s^2)^{\frac{3-N}{2}} \quad , \quad \varphi(0,k) = \varphi_0(k) = \mathcal{F}(f_0)(k)\,;$$

$$\varphi(t,0) = 1, \qquad \nabla_k\varphi(t,0) = 0\;, \qquad \theta(t) = -\tfrac{1}{3}\Delta_k\varphi(t,0)\;.$$

For isotropic ($x = \frac{|k|^2}{2}$) or self similar solutions ($x = \frac{|k|^2}{2}e^{-\mu t}$) the Fourier transformed collisional form takes the simple form

$$\Gamma_\alpha(\varphi(x)) = \int_0^1 \{\varphi(\beta^2 s\, x)\varphi((1-\beta(2-\beta)s)\, x)\}\, G(s)\, ds = \int_0^1 \{\varphi(\,(\mathrm{a}_\beta(s)\, x)\varphi(\,(\mathrm{b}_\beta(s)\, x)\}\, G(s)\, ds$$

$$c_N = \tfrac{1}{2}\int_0^1 G(s)\, ds \qquad \text{and} \qquad 0 \le \mathrm{a}_\beta(s), \mathrm{b}_\beta(s) \le 1$$

A important applications:

Existence, asymptotic behavior - self-similar solutions and power like tails: From a unified point of energy dissipative Maxwell type models: λ_1 energy dissipation rate (Bobylev, I.M.G.JSP'05, Bobylev,Cercignani,I.G. '06)

$$\frac{\partial \varphi}{\partial t} = \int_0^1 ds\, G(s)\, \{\varphi(a(s)x)\, \varphi[(b(s)x] - \varphi(x)\, \varphi(0)\} + \theta \int_0^1 ds\, H(s)\, \{\varphi[c(s)x] - \varphi(x)\} =$$

$$= I_{a,b,\lambda_1}(\varphi,\varphi) + \theta\, I_{c,1,\lambda_1}\ ;$$

$\varphi_0(x) = 1 - x^p \quad p \leq 1$ initial state

$G(s), H(s)$ non-negative, integrable on $[0,1];\ 0 \leq a(s), b(s), c(s) \leq 1,\ s \in [0,1]$.

Examples:

- Classical elastic Maxwell gas with infinite initial energy:
  $a(s) = s,\ \ b(s) = 1 - s$, and $\boxed{\varphi_t = I_{a,b,0}(\varphi,\varphi)}$
- Gas of inelastic Maxwell particles with finite or infinite initial energy, with constant restitution coefficient $\beta = (1+\alpha)/2$:
  $a(s) = \beta^2 s,\ \ b(s) = 1 - \beta(2-\beta)s$ and $\boxed{\varphi_t = I_{a,b,\lambda_1}(\varphi,\varphi)}$
- Classical elastic or inelastic Maxwell gas with finite or infinite energy in the presence of an equilibrium background gas of particles with mass M, density n_1 and temperature T_1,
  $a(s) = \beta^2 s;\ \ b(s) = 1 - \beta(2-\beta)s$ for $\frac{1}{2} \leq \beta \leq 1;\ \ c(s) = 1 - 4M/(1+M)^2 s < 1;$
  and $\boxed{\varphi_t = I_{a,b,\lambda_{1\beta}}(\varphi,\varphi) + \theta I_{c,1,\lambda_1}(\varphi, e^{T_1 x}) \qquad \text{Energy non-conservative}}$

We will see that

1. *For more general systems multiplicatively interactive stochastic processes the lack of entropy functional* ***does not impairs*** *the understanding and realization of global existence (in the sense of positive Borel measures), long time behavior from spectral analysis and self-similar asymptotics.*

2. *"power tail formation for high energy tails" of self similar states is due to lack of total energy conservation,* ***independent*** *of the process being micro-reversible (elastic) or micro-irreversible (inelastic).* ***Self-similar solutions may be singular at zero****.*

3- *The long time asymptotic dynamics and decay rates are fully described by the* ***continuum spectrum associated to the linearization about singular measures*** *(when momentum is conserved).*

Existence, Self-similar asymptotics and Power-like Tails

(Bobylev, Cercignani, I.M.G.; To appear Comm. Math. Phys. 08)

For $\phi(k,t) = \mathcal{F}_{v\to k}[f(v,t)]$, *let* $\boxed{\Gamma(\phi) = \mathcal{F}_{v\to k}[Q^+]}$ *be the Fourier Transform of the contribution from the gain operator* $Q^+(f,f)$ *associated to a generalized BTE equation of Maxwell type.*

In the case of isotropic solutions $f(|v|^2,t) \longrightarrow \phi(|k|^2,t) = u(x,t)$.

$\boxed{\int f(v,t)|v|^2 dv = \Delta_k \phi(k,t)\,|_{k=0} = T(t) = u_x(0,t) \text{ is the kinetic energy}}$

The initial value problem:

For initial states $u(x,0) = u_o(x) = 1 + O(x^p) \in U$, $\|u_o\| = 1$, $p > 0$ with, *U the unit sphere in* $(C_B(\Re^d), \|\cdot\|_\infty)$, *take*

$\left\{ \begin{array}{l} p<1 \text{ infinity energy,} \\ P=>1 \text{ finite energy} \end{array} \right.$

$$u_t + u = \Gamma(u) = \sum_{n=1}^N \alpha_n \Gamma^{(n)}(u) \, , \qquad \sum_{n=1}^N \alpha_n = 1 \, , \ \alpha_n \geq 0 \, ,$$

$$\Gamma^{(n)}(u) = \int_0^\infty da_1 \ldots \int_0^\infty da_n \; A_n(a) \prod_{k=1}^n u(a_k x), \quad n = 1, \ldots, N \, ,$$

with $A_n(a) = A_n(a_1,\ldots,a_n) \geq 0$, have a compact support and $\int_0^\infty da A(a) = 1$.

where $\Gamma(0) = 0$ and $\Gamma(1) = 1$ **are trivial solutions**

Theorem: The Γ-operator satisfies three fundamental properties

Theorem: The Γ-operator satisfies

- Preserves the unit sphere U in $(C_B(\Re^d), \|\cdot\|_\infty)$
- It has *L-Lipschitz condition*: there exists a linear bounded operator L from $(C_B(\Re^d), \|\cdot\|_\infty)$ into itself, such that, for $x = \frac{k^2}{2}$

$$\boxed{|\Gamma(u_1) - \Gamma(u_2)|(x,t) \leq L(|u_1 - u_2|(x,t)), \qquad \text{for} \|u_i\|_\infty \leq 1; i = 1, 2.}$$

- Invariance under dilations:
$e^{\tau\mathbf{D}}\Gamma(u) = \Gamma(e^{\tau\mathbf{D}}u)\ , \quad \mathbf{D} = x\frac{\partial}{\partial x}\ , \quad e^{\tau\mathbf{D}}u(x) = u(xe^{\tau}), \quad \tau \in \ .$

- L**-Lipschitz** condition on the operator Γ is a point-wise condition $\Rightarrow$ classical Lipschitz condition on $\mathcal{B}$.
- $\Gamma(u)$ is L-Lipschitz, where $L = L$ is the linearization of $\Gamma(u)(x,t) = \mathcal{F}_{v\to k}[Q(|v|,t)]$ about the state $u = 1$ is the linearization of Γ about $u = 1$

- relation to the contractive property of the Wasserstein distance between two probabilities:
  - **For Maxwell type of interactions that conserve momentum the 2^{nd}-Wasserstein distance from $W_2(f(v,t), \delta_{\langle v\ f\rangle(t)}) = \int f(v,t)|v|^2$ is the kinetic energy.**
  - **The eigenvalue of L for $u = x$ is the energy dissipation rate $\mu(1)$ so$\Theta' = -\mu(1)\Theta \ \Rightarrow$ for bounded initial energy, long time asymptotics and decay rates in Fourier space yield the same qualitatively properties in W_2 metrics, since this metric is equivalent to the usual weak convergence of measures plus convergence of second moments.**

Relates to the work of Toscani, Gabetta, Wennberg, Villani, Carlen, Carvalho, Carrillo, and many more (from '95 to date)

Self-similar asymptotics - spectral properties

Spectral Properties of L :

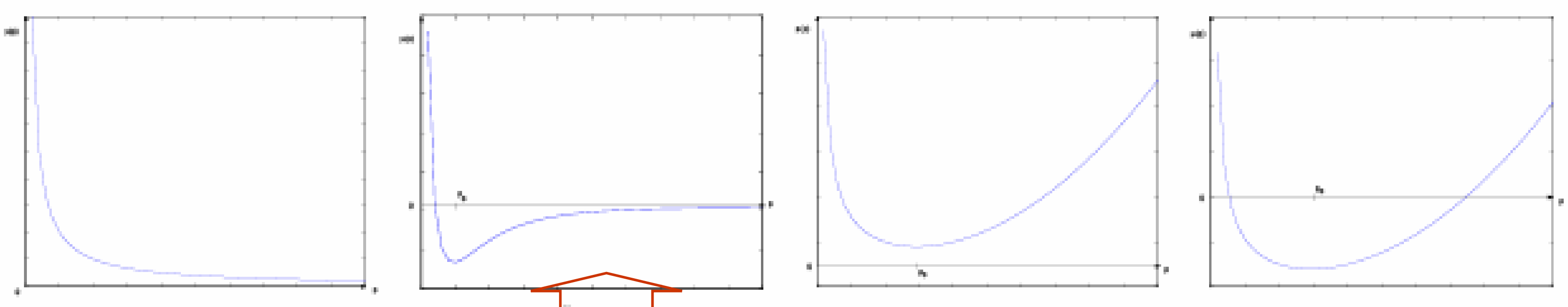
$Lu = \int_0^\infty K(a)u(a\,x)da$; $\qquad K(a) = \sum_{n=1}^{N} n\alpha_n K_n(a)$,

where $K_n(a) = \int_0^\infty da_2 \ldots \int_0^\infty da_n \, A_n(a_1, a_2, \ldots, a_n)$ and $\sum_{n=1}^{N} \alpha_n = 1$, and satisfies:

- x^p is the ei-function with ei-value $\lambda(p)$ of the linear operator L associated to Γ

$$Lx^p = \lambda(p)x^p, \qquad \lambda(p) = \int_0^\infty K(a)a^p da$$

- $\lambda(1) - 1$ is the energy dissipation rate.
- we call $\mu(p) = \frac{\lambda(p)-1}{p}$ **the spectral function associated to** Γ.
- $\mu(0+) = +\infty$ and $0 < p_0$, such that $\mu(p_0) = \min_{p>0} \mu(p)$ is the **unique minima** .



Boltzmann Spectrum

• Existence of solutions $\mathcal{B} = C_B(I\!R^+)$ (isotropic case) or in $\mathcal{B} = C_B(I\!R^N)$ (general case): **For finite or infinity initial second moment (kinetic energy)**

For $x = \frac{|k^2|}{2}$, with initial conditions $u_0 = \varphi_0(x) = 1 + O(x^p)$, as $x \to 0$, with $\|u_0\| = 1$,

there exists a unique solution $u(t, x)$ to

$$u_t + u = \Gamma(u), \quad u_{|t=0} = u_o(x) \quad \text{such that} \quad u(x,t) = 1 + O(x^p), \; 0 < p < p_0 \; for \; x \to 0$$

in addition, any two solutions to i.v.p. with same type of data satisfies:

1- $$|u_1(x,t) - u_2(x,t)| \leq Ce^{-t(\lambda(p)-1)} O(x^p)\,, \qquad \text{and}$$

2- $$|u_1(xe^{-\mu_* t}, t) - u_2(xe^{-\mu_* t}, t)| \leq Ce^{-tp(\mu_* - \mu(p))} O(x^p),$$

where the minimal constant for which the condition is satisfied

$$C_0 = \sup_{x \geq 0} \frac{|u_1(x,0) - u_2(x,0)|}{|x^p|} = \left\| \frac{u_1(x,0) - u_2(x,0)}{x^p} \right\|,$$

so it yields

$$\left\| \frac{u_1(x,t) - u_2(x,t)}{x^p} \right\| \leq e^{-t(1-\lambda(p))} \left\| \frac{u_1(x,0) - u_2(x,0)|}{x^p} \right\| \; \forall p > 0$$

These estimates are a consequence of the L-Lipschitz condition associated to Γ: they generalized Bobylev, Cercignani and Toscani,JSP'03 and later they have been interpeted as *"contractive distances"* (as originally introduced by Toscani, Gabetta, Wennberg, '96)

They imply, jointly with the property of the invariance under dilations for Γ, self-similar asymptotics and the existence of non-trivial dynamically stable laws.

• **Existence of Self-Similar Solutions** For $x = \frac{|k^2|}{2}$ and $\boxed{\eta = x e^{\mu_* t}}$, with initial conditions on $\eta = x$: $\mu(p) x u_o' = \Gamma(u_o) + O(x^{p+\varepsilon})$, $u_o(x) = 1 + O(x^p)$ and $\|u_o\| \leq 1$

if $0 < p \leq 1 < p_0$ and $\mu_* = \mu(p)$ ($\Rightarrow$ one can take $u_o = e^{-x}$ to fulfill the conditions),

then, there exists a non-trivial self-similar solution $u(t,x) = \Psi(\eta)$ to

$$\boxed{\begin{array}{l} \mu_* \eta \Psi' + \Psi = \Gamma(\Psi), \qquad \text{with initial state} \qquad \Psi_{|\eta = x} = u_o(x) \quad \text{such that} \\ \Psi(\eta) = u_o(\eta) + O(\eta^{p+\varepsilon}) = 1 - \eta^p + O(\eta^{p+\varepsilon}) \quad \text{for } \eta \geq 0, \text{ and} \\ |\, u(x e^{-\mu_* t}, t) - \Psi(x) \,| \leq C e^{-t(p+\varepsilon)(\mu_* - \mu(p+\varepsilon))} O(x^{p+\varepsilon}) \quad \text{for } 0 < p < p + \varepsilon < p_0, \end{array}}$$

where $\Psi_{\mu_*}(x)$ satisfies:

$$\boxed{1 \geq \Psi_{\mu_*}(x) \geq e^{-x}, \qquad \lim_{x\to\infty} \Psi_{\mu_*}(x) = 0,}$$

and there exists a generalized non-negative function $R_{\mu_*}(\tau)$, $\tau \geq 0$, s.t.

$$\boxed{\Psi_{\mu_*}(x) = \int_0^\infty d\tau\, R_{\mu_*}(\tau)\, e^{-\tau x}, \qquad \int_0^\infty d\tau\, R_{\mu_*}(\tau) = \int_0^\infty d\tau\, R_{\mu_*}(\tau)\, \tau = 1 .}$$

Self-similar solutions - time asymptotics

Theorem *The following statements hold:*

[i]: *There exists a unique (in the class of probability measures) solution* $f(|v|,t)$ *with initial state* $f(|v|,0) = f_0(|v|) \geq 0$, $\int_{\mathbb{R}^3} f_0(|v|)dv = 1$ *such that, with* $x = \frac{|k|^2}{2}$
$u_0 = \mathcal{F}[f_0(|v|)] = 1 + O(x^p), x \to 0, 0 < p \leq 1.$

[ii]: *The solution* $f(|v|,t)$ *has self-similar asymptotics in the following sense:*
Take $p = 1$, $\quad \mu(1)xu_0' = \Gamma(u_0) + O(x^{1+\epsilon})$ *with* $\mu(1); \mu'(1) < 0$;
Then $\exists$ *a ! non-negative self-similar solution:*

$f_{ss}(|v|,t) = e^{-\frac{d}{2}\mu(1)t} F_1(|v|e^{-\frac{1}{2}\mu(1)t})$, $\mu(1)$ *- energy dissipation rate, and*

$$\boxed{|f(|v|e^{\frac{1}{2}\mu(1)t},t) - e^{-\frac{d}{2}\mu(1)t}F_1(|v|)| \leq C(\|f_0 - F_1(|v|)\|_{L_2^1})e^{-t(1+\varepsilon)(\mu_(1)-\mu(1+\varepsilon))}}$$

1- *For the statement with initial* $\frac{p}{2}$ *moments (i.e. in Fourier with order* $O(x^{p+\epsilon}); p > 1)$ *replace* $\mu(1)$, $\mu(1+\varepsilon)$ *by* $\mu(p), \mu(p+\varepsilon)$*, resp. (see Bobylev, Cercignani and I.M.G, Comm Math Phys'08 (arXiv.org 06))*
2- *This decay rate was computed first in Bobylev, Cercignani and Toscani, JSP'03, for the elastic collisions models converging to homogeneous cooling states example.*

[iii]: *However,*

$f(|v|e^{\frac{1}{2}\eta t},t) \to_{t\to\infty} e^{-\frac{d}{2}\eta t}\delta_0(|v|); \qquad \eta > \mu(1) \qquad$ *and*

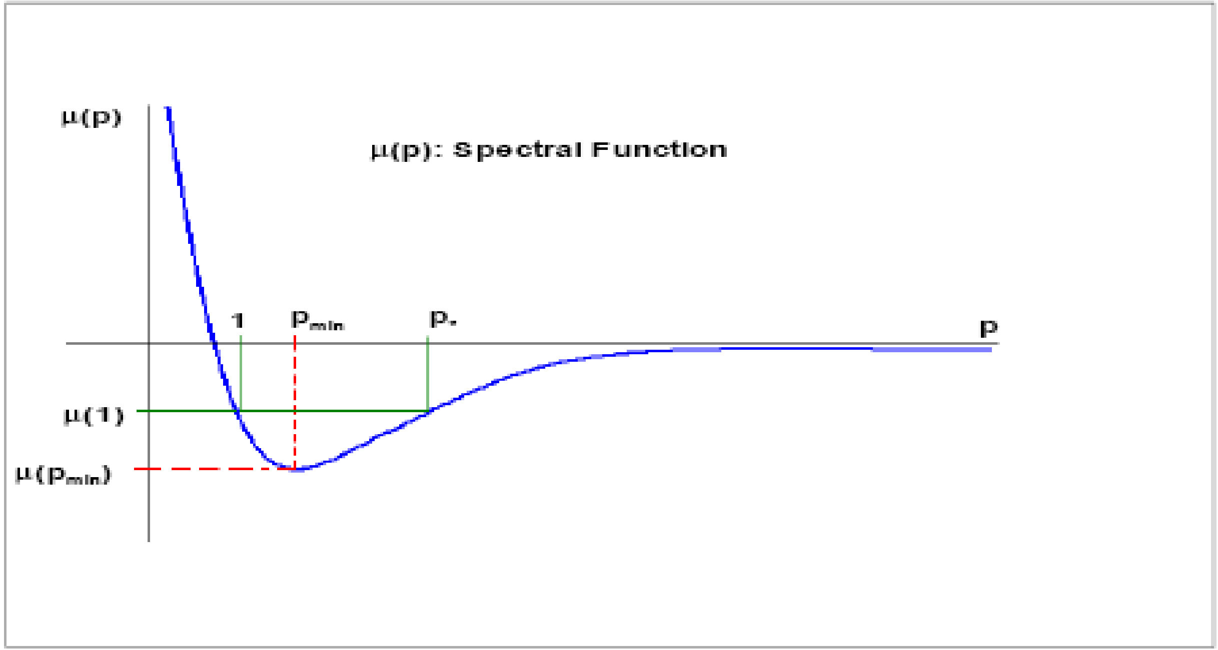
$f(|v|e^{\frac{1}{2}\eta t},t) \to_{t\to\infty} 0; \qquad \mu(p_{min}) < \mu(1+\delta) < \eta < \mu(1)$

Self-similar solutions and Power-like Tails

Theorem: *(Bobylev, Cercignani, I.M.G,06)* The self-similar asymptotic function $F_{\mu(p)}(|v|)$ does NOT have finite moments of all orders if the energy dissipates, i.e. $\mu(1) < 0$.

If $0 \leq p \leq 1$ then, $m_q = \int_{\mathbb{R}^3} F_{\mu(p)}(|v|)|v|^q dv \leq \infty; \quad 0 \leq q \leq p$

If $p = 1$ (finite initial energy) then, $m_q \leq \infty$ only for $0 \leq q \leq p_*$, where $p_* \geq 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$.



Example: Description of the Weakly Coupled Binary Mixture Problem *(Bobylev, I.M.G. JSP '06)*

Construction of **explicit solutions** to:

$$\frac{\partial f(v,t)}{\partial t} = \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)f('w,t) - f(v,t)f(w,t)]d\sigma dw$$
$$+ \quad \theta_b \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)M_T('w) - f(v,t)M_T(w)]d\sigma dw$$

with $M_T(v) = \frac{e^{\frac{-|v|^2}{(2T)}}}{(2\pi T)^{3/2}}$, $B(|u|,\mu) = C_\lambda = \frac{1}{4\pi}$, $\beta = 1.0$, θ_b - depending on the asymptotics and T being the background temperature.

- A system of two different particles with the same mass is considered. One set of particles is assumed to be at equilibrium i.e., with a Maxwellian distribution with temperature $T(t)$.
- Second set of particles is assumed to collide with themselves (first integral) and the background particles(Linear Boltzmann Collision Integral).

The collisions are assumed to be ***locally* elastic** i.e., $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$ but the above form leads to ***global*** energy dissipation i.e., $\int_{\mathbb{R}^3} |v|^2 f(v,t)dv \neq 0$.

Analytical and computational testing of the BTE with Thermostat: singular solutions

(with Bobylev, JSP 06**), and computational Spectral-Lagrangian solvers** (with S.H. Tarshkahbushanam, Jour.Comp.Phys. 08)

Self - Similar Asymptotics elastic BTE with thermostat

Taking Fourier Transform on the kinetic equation

$$\hat{f}_t = \hat{Q}(\hat{f},\hat{f}) + \theta_b \int_{\sigma\in S^2} b\left(\frac{k.\sigma}{|k|}\right)[\hat{f}(k_+)\hat{M}(k_-) - \hat{f}(k)\hat{M}(0)]d\sigma$$

$$k_\pm = \frac{1}{2}(k \pm |k|\omega)\,, \qquad \hat{f}(0)=1\,, \qquad \hat{M}(k) = e^{-\frac{T\,|k|^2}{2m}}$$

Set $\hat{f}(k,t) = \tilde{\hat{f}}(k,t)\exp\frac{-T\,|k|^2}{2} \Rightarrow$

$$\boxed{\tilde{\hat{f}}_t = \hat{Q}(\tilde{\hat{f}},\tilde{\hat{f}}) + \theta_b \int_{\sigma\in S^2} b\left(\frac{k.\sigma}{|k|}\right)[\tilde{\hat{f}}(k_+) - \tilde{\hat{f}}(k)]d\sigma}$$

which is equivalent to the untransformed equation with $T \equiv 0$.

- *set* $x = \frac{|k|^2}{2}$ *and look for similarity scaled solutions in* $axe^{\mu t} \Rightarrow$ *for* $\theta_b = \frac{4}{3}$, $\mu = \frac{2}{3}$ *and* $\hat{T} = T + s^2 e^{\frac{-2t}{3}}$; *a solution is*

$$\boxed{f_T^{ss}(|v|) = \frac{2}{\pi^5}^{\frac{1}{2}} \int_0^\infty \frac{1}{(1+s^2)^2}\,\frac{e^{-|v|^2/2\hat{T}}}{\hat{T}^{\frac{3}{2}}}ds}$$

Self - Similar Asymptotics elastic BTE with thermostat

- *For self similar asymptotics we study* $t \to \infty$ *so* $\hat{T} \to T$ *in* $f_T^{ss}(v,t)$ *(i.e. the particle distribution temperature approaches the background temperature as expected due to the linear coll. op.)*

- *Interesting NESS behavior can be observed if* $T \to 0$*: Set* $\hat{T} = s^2 e^{\frac{-2t}{3}}$ *so* $f_0^{ss}(|v|)$ *is explicit.*

- *Then* $f(|v|e^{-t/3}, t) \to_{t\to\infty} e^t f_0^{ss}(|v|)$ *where* $f_0^{ss}(|v|) = \frac{4}{\pi}\int_0^\infty \frac{e^{-|v|^2/(2s^2)}}{(2\pi s^2)(1+s^2)^2} ds$

- $\boxed{f_0^{ss}(|v|) = O(\frac{1}{|v|^6}) \ \text{ as } \ |v| \to \infty,}$ *and* **Power law tails for high energy**

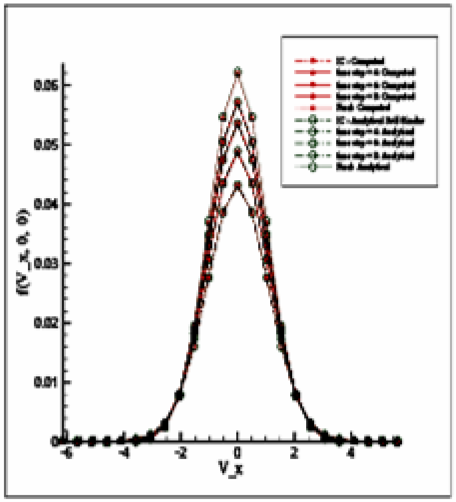
  $\boxed{f_0^{ss}(|v|) = O(\frac{1}{|v|^2}) \ \text{ as } |v| \ \to 0}$ **Infinitely many particles for zero energy**

Examples in soft condensed matter (Greenblatt and Lebowitz, *Physics A. 06)*

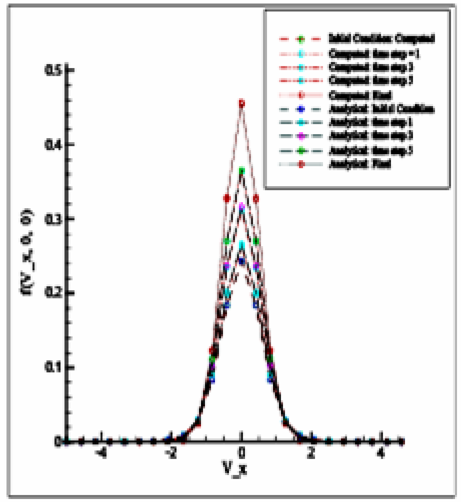
Testing: BTE with Thermostat

Spectral-Lagrangian solvers (with S.H. Tarshkahbushanam, JCP 08)

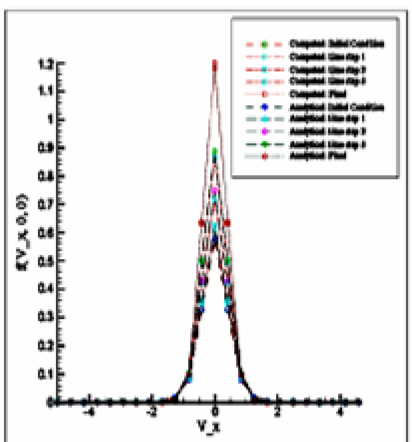
Computed Vs. Analytical Distribution:



$(N = 24, T = 1)$



$(N = 24, T = 0.25)$



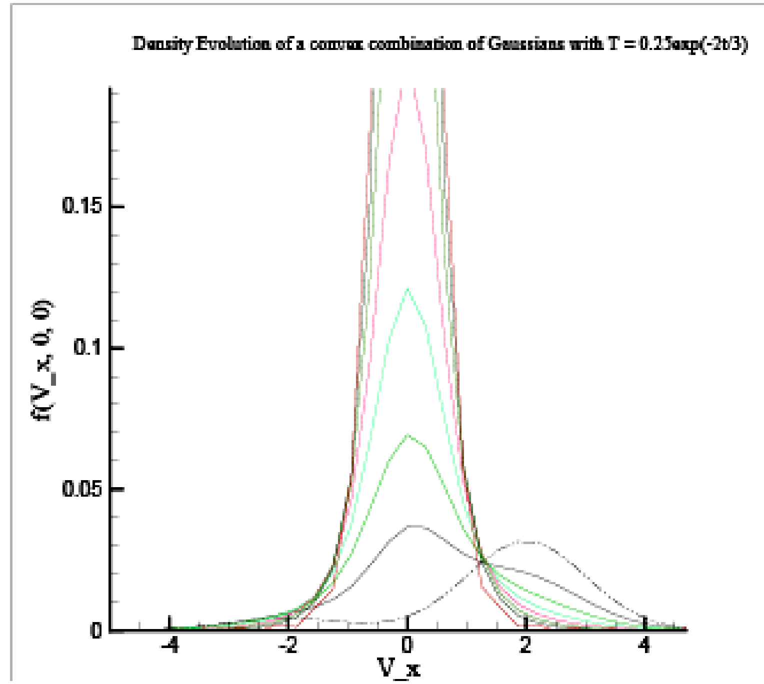
$(N = 24, T = 0.125)$

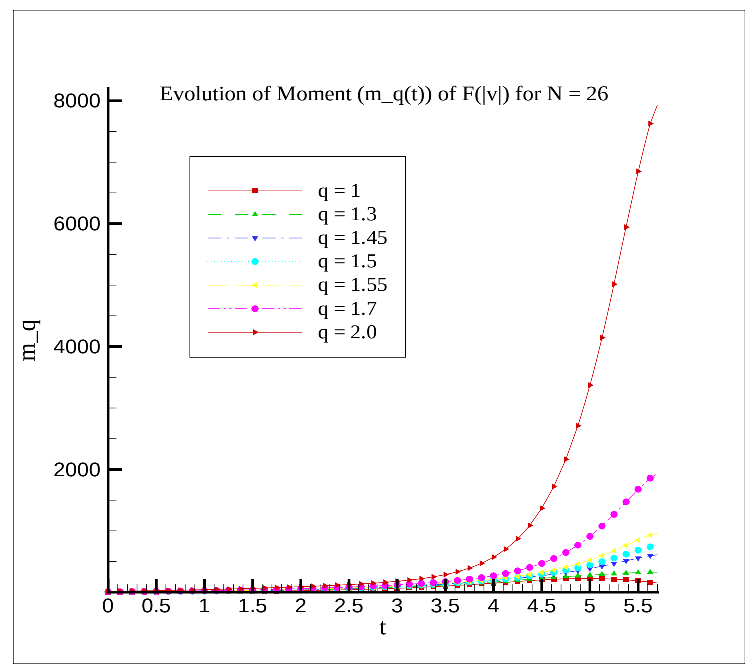
Maxwell Molecules model

Rescaling of spectral modes exponentially by the continuous spectrum with λ(1)=-2/3

Setting $\hat{T} = e^{-\frac{2}{3}t}(\frac{1}{4} + s^2)$

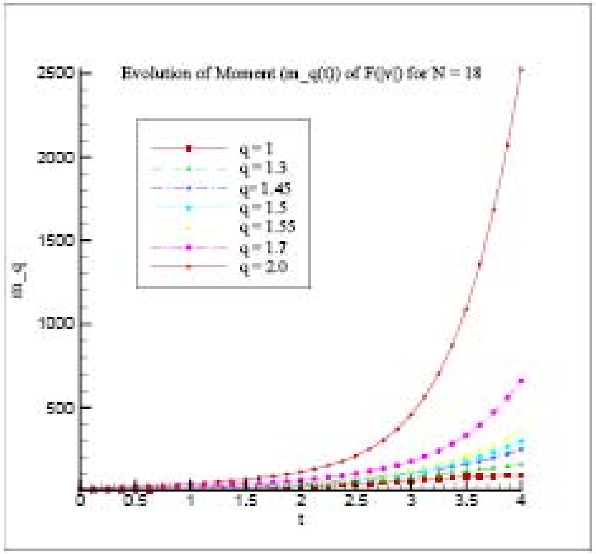
Evolution of the Particle Density $(N = 32, T = 0.25e^{-2t/3})$



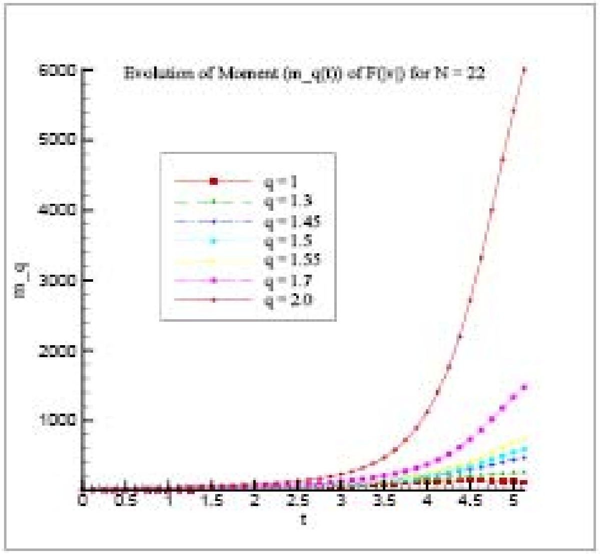


Moments calculations:

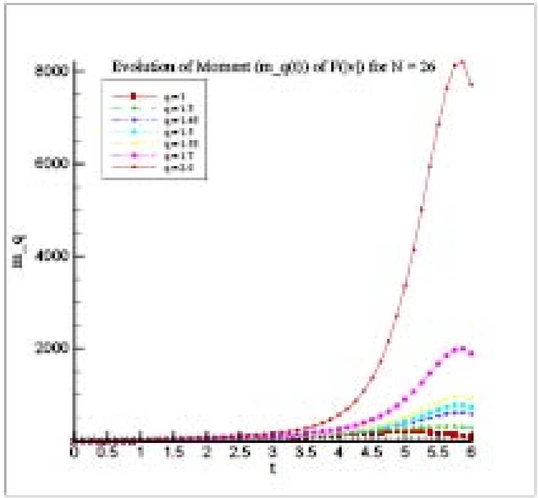
$$m_q(t) = \int_{\mathbb{R}^3} f(|v|,t)|v|^q dv \to \infty \text{ for } q > 1.5$$



(N = 18)



(N = 22)



(N = 26)

q-moment of $f(v,t)$ critical $q = 1.5$

Proof of 'power tails' by means of continuum spectrum and group transform methods

Back to the representation of the self-similar solution:

In addition: for $p_0 > 1$ and $p = 1$: the $R(\tau)$ satisfies (using the Laplace transform)

$$\boxed{-\mu(1)\frac{\partial}{\partial\tau}\tau R(\tau) + R(\tau) = Z(R) = \mathcal{L}^{-1}[\Gamma(w)]} \quad \Longleftrightarrow \quad \text{fractional moment equations}$$

$$\text{for} \quad Z(R) = \sum_{n=1}^{N} \alpha_n Z_n(R) \; , \quad \sum_{n=1}^{N} \alpha_n = 1 \; , \quad Z_n(R) = \int_{\mathbb{R}_+^n} da_1, \ldots, da_n \frac{A_n(a_1, \ldots, a_n)}{a_1 a_2 \ldots a_n} \prod_{k=1}^{n}{}^{*} R_k\left(\frac{\tau}{a_k}\right) \; ,$$

$$\prod_{k=1}^{n}{}^{*} R_k(\tau) = R_1 * R_2 * \ldots * R_n \; , \qquad R_1 * R_2 = \int_0^\tau d\tau' \, R_1(\tau') R_2(\tau - \tau') \; .$$

In addition, the corresponding *self-similar probability distribution admits an integral representation* **infinitely divisible** *distributions* $M_p(|v|)$ *with kernels* $R_p(\tau)$ for $p = \mu^{-1}(\mu_*)$ given by :

$$\boxed{F_p(|v|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^N} dv \, \Psi_{\mu(p)}\left(\frac{k^2}{2}\right) e^{ik\cdot v} = \int_0^\infty d\tau \, R_p(\tau) \, \tau^{-\frac{d}{2p}} M_p\left(|v| \tau^{-\frac{1}{2p}}\right)}$$

where

$$\boxed{M_p(|v|) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^N} dk \, e^{-|k|^{2p} + ik\cdot v} \; , \quad \text{is } \textbf{infinitely divisible} \text{ distribution}}$$

Note: $M_1(|v|)$ **is a classical Maxwellian distribution.**

$\Rightarrow$ This representation explains the connection of self-similar solutions of generalized Maxwell models with infinitely divisible distributions and stable laws.

2 - Properties for moments equations: $-\mu(1)\frac{\partial}{\partial\tau}\tau R(\tau)+R(\tau)=\mathcal{L}^{-1}[\Gamma(w)]$

Set $\boxed{m_s=\int_0^\infty d\tau\, R(\tau)\tau^s \ ,\ \ s>0}$; with $m_0=m_1=1$. $m_s>0$ for all $s>1$.

Then multiply by τ^s and integrate to obtain *(see Bobylev, Cercignani, I.M.G, CMP'08) for the definition of* $\mathbf{I}_n(s)$)

$\boxed{s\,[\mu(1)-\mu(s)]m_s=\sum_{n=2}^{N}\alpha_n I_n(s) \text{ for } s>1}$, with $\mu(1)=$ energy dissipation rate

Now, one can show that $\boxed{0\le\sum_{n=2}^{N}\alpha_n I_n(s)\le C_N m_{s-1}}$, then

while $\boxed{\mu(1)-\mu(s)>0}$ then $\boxed{0<m_s\le\frac{C_N}{s\,[\mu(1)-\mu(s)]}\ m_{s-1}\ \text{ is finite}}$,

otherwise, if $\boxed{\mu(1)-\mu(s)<0}$ then m_s must be unbounded.

$\Rightarrow$ the following **Theorem** holds:

[i] If the equation $\boxed{\mu(s)=\mu(1)}$ has the only solution $s=1$, then $\boxed{m_s<\infty \text{ for any } s>0}$.

[ii] If $\boxed{\mu(s)=\mu(1)}$ has two solutions $s=1$ and $s=s_*>1$, then $\boxed{m_s<\infty \text{ for } s<s_*}$ and $\boxed{m_s=\infty \text{ for } s>s_*}$.

[iii] $m_{s_*}<\infty$ only if $I_n(s_*)=0$ in the above equation, for all $n=2\ldots N$.

⇒ The boundedness properties of the moments m_s of R_p implies the boundedness of moments for the self-similar solutions constructed by Fourier or Laplace transform methods: with $v \geq 0$, $0 < p \leq 1$:

$$F_p(|v|) = \int_0^\infty d\tau \, R_p(\tau)\tau^{-\frac{d}{2p}} M_p(|v|\tau^{-\frac{1}{2p}}) \; , \qquad \text{then}$$

$$m_{2s}(F_p) = m_{2s}(M_p) m_{s/p}(R_p) \quad \text{(for Fourier Transform),}$$

and

$$\Phi_p(v) = \int_0^\infty d\tau \, R_p(\tau)\tau^{-\frac{1}{p}} N_p(v\tau^{-\frac{1}{p}}) \; , \; \text{then}$$

$$m_s(\Phi_p) = m_s(N_p) m_{s/p}(R_p) \quad \text{(for Laplace Transform) .}$$

⇒ the following **Theorem**:

1- If $0 < p < 1$, then $m_{2s}(F_p)$ and $m_s(\Phi_p)$ are finite if and only if $0 < s < p$.

2- For $p = 1$ the result holds for $m_s = m_{2s}(F_1)$ and for $m_s = m_s(\Phi_1)$.

⇒ $F(|v|)$ can not have all (even) moments bounded ≡ power tails.

Typical Spectral function μ(p) for Maxwell type models

- For $p_0 > 1$ and $0 < p < (p + \epsilon) < p_0$ ⇒ **Self similar asymptotics for:**

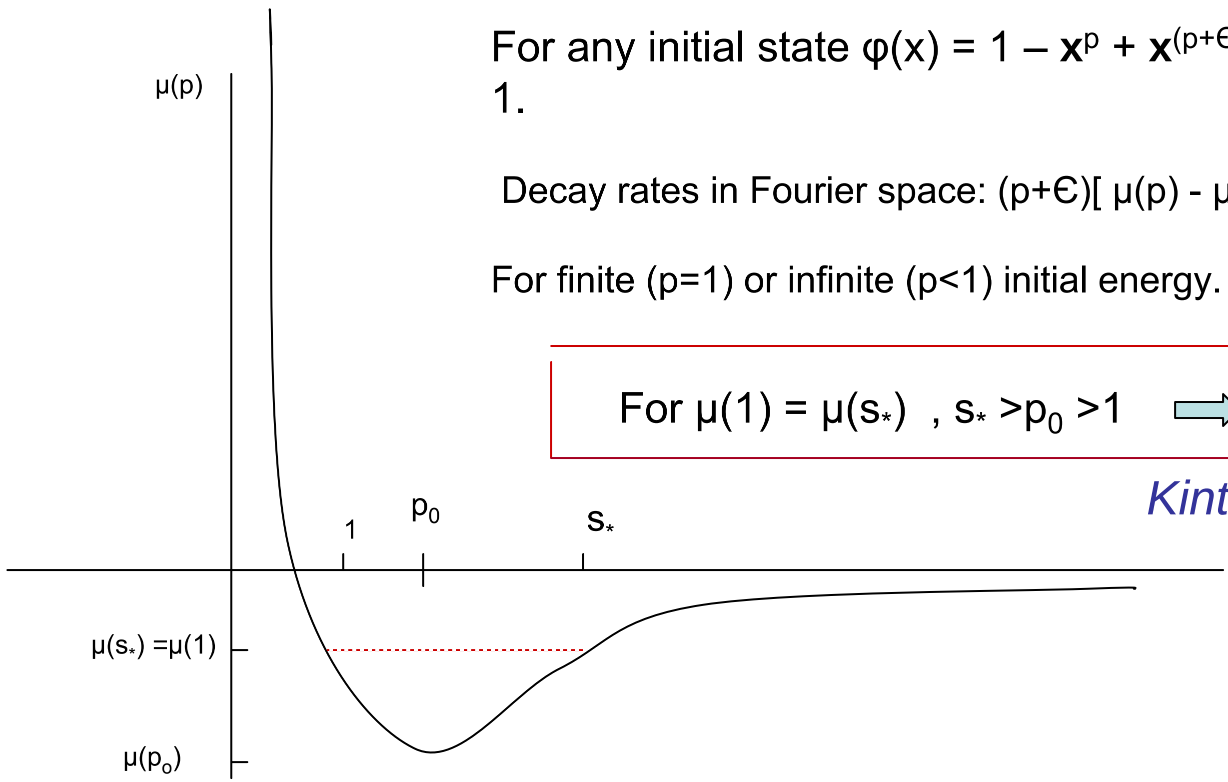
For any initial state $\varphi(x) = 1 - \mathbf{x}^p + \mathbf{x}^{(p+\epsilon)}$, $p \leq 1$.

Decay rates in Fourier space: $(p+\epsilon)[\,\mu(p) - \mu(p+\epsilon)\,]$

For finite (p=1) or infinite (p<1) initial energy.

For $\mu(1) = \mu(s_*)$, $s_* > p_0 > 1$ ⇒ Power tails

Kintchine type CLT



•For $p_0 < 1$ and $p=1$ ⇒ **No self-similar asymptotics with finite energy**

Thank you very much for your attention!

References (www.ma.utexas.edu/users/gamba/research and references therein)