Optimal Dirichlet regions for mass transportation problems and for elliptic equations

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We want to study shape optimization problems of the form

$$\min\left\{F(\Sigma)\ :\ \Sigma\in\mathcal{A}\right\}$$

where F is a suitable shape functional and $\mathcal{A}$ is a class of admissible choices. In particular, we are interested in cases where Σ represents the Dirichlet region of an auxiliary variational problem that we write in the form

$$\min\left\{G(u)\ :\ u=0 \text{ on } \Sigma\right\} \tag{1}$$

whose solution will be indicated by u_Σ. The shape optimization problem will then be

$$\min\left\{F(\Sigma,u_\Sigma)\ :\ \Sigma\in\mathcal{A},\ u_\Sigma \text{ solves (1)}\right\}.$$

In this way the shape optimization problem can be seen as an optimal control problem where Σ is the control variable, u the state variable, and (1) is the state equation written in variational form.

We shall consider two main cases:

- the case when the auxiliary problem comes from the mass transportation theory;

- the case when the functional F is the elastic compliance and the state equation gives the equilibrium of an elastic structure.

The case of mass transportation problems

We consider a given compact set Ω of $\mathbf{R}^d$ (an urban region) and a probability measure f on Ω (the population distribution). We want to find Σ in some admissible class and the goal is to transport f on Σ in an optimal way.

To do that we consider all the probabilities g on Σ and the related Monge-Kantorovich cost (Wasserstein distance)

$$W(f,g) = \inf \int_{\Omega\times\Omega} |x-y| \, d\gamma$$

where the infimum is taken over all transport plans γ, i.e. probabilities on $\Omega \times \Omega$ with marginals f and g respectively.

The cost functional $F(\Sigma)$ is given by

$$F(\Sigma) = \inf\left\{W(f,g) \ : \ \operatorname{spt} g \subset \Sigma\right\}$$

and this turns out to coincide with

$$F(\Sigma) = \int_\Omega \operatorname{dist}(x,\Sigma)\, df(x)$$

whose state variable is the Kantorovich potential which solves the dual problem

$$\sup\left\{\int u\, df \ : \ u \in \mathrm{Lip}_1,\ u = 0 \text{ on } \Sigma\right\}$$

or equivalently the Monge-Kantorovich PDE

$$\begin{cases} -\operatorname{div}(\mu Du) = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \Sigma \\ u \in \mathrm{Lip}_1, \qquad |Du| = 1 & \text{on } \operatorname{spt}\mu \\ \mu(\Sigma) = 0. \end{cases}$$

Concerning the class of admissible controls we consider the following cases:

- $\mathcal{A} = \left\{\Sigma \ : \ \#\Sigma \le n\right\}$ called location problem;

- $\mathcal{A} = \left\{\Sigma \ : \ \Sigma \text{ connected}, \mathcal{H}^1(\Sigma) \le L\right\}$ called irrigation problem.

The location problem

We call optimal location problem the minimization problem

$$L_n = \min \Big\{ F(\Sigma) \;:\; \Sigma \subset \Omega,\, \#\Sigma \le n \Big\}.$$

It has been extensively studied, see for instance

Suzuki, Asami, Okabe: Math. Program. 1991
Suzuki, Drezner: Location Science 1996
Buttazzo, Oudet, Stepanov: Birkhäuser 2002
Bouchitté, Jimenez, Rajesh: CRAS 2002
Morgan, Bolton: Amer. Math. Monthly 2002
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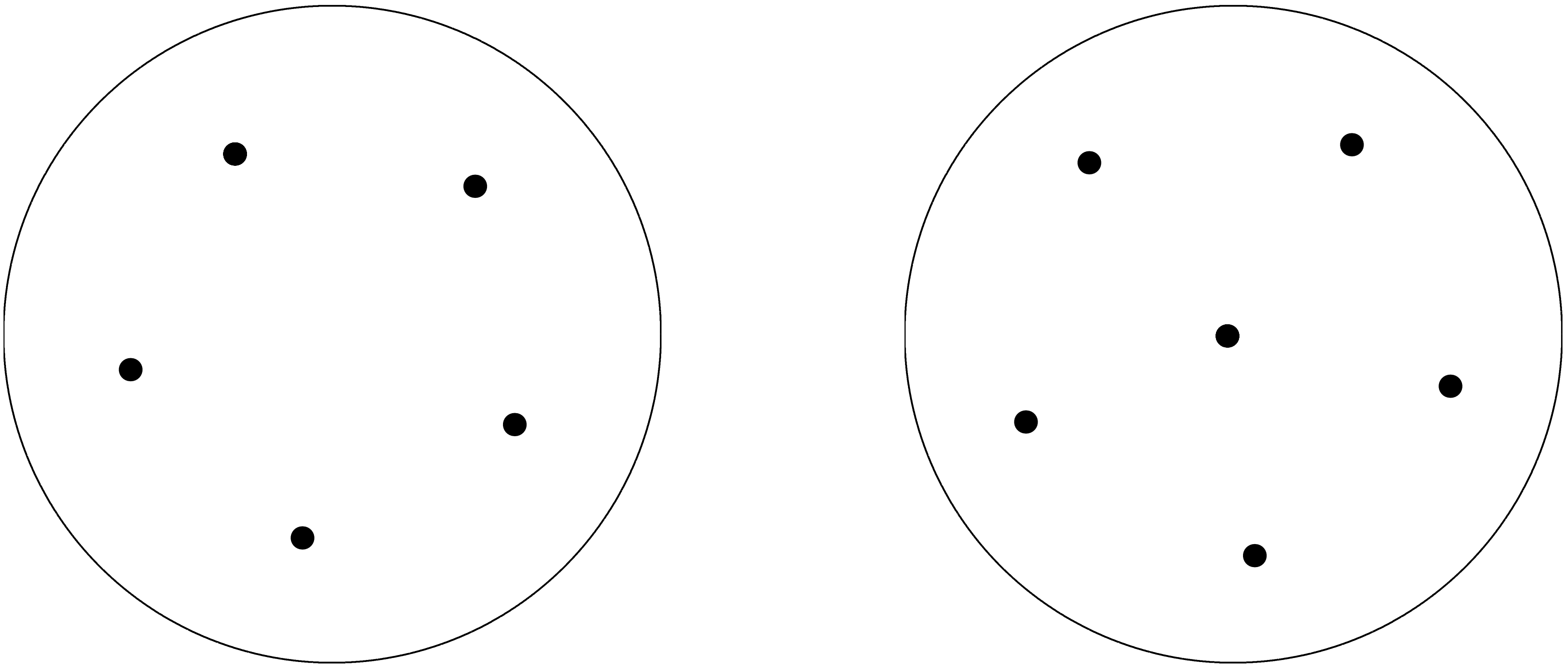


Optimal locations of 5 and 6 points in a disk for $f = 1$

We recall here the main known facts.

- $L_n \approx n^{-1/d}$ as $n \to +\infty$;
- $n^{1/d} F_n \to C_d \int_\Omega \mu^{-1/d} f(x)\, dx$ as $n \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;

- $\mu_{opt} = K_d f^{d/(1+d)}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{d/(1+d)}$ and not as f (for instance as $f^{2/3}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by the centers of regular exagons.

- In dimension one we have $C_1 = 1/4$.
- In dimension two we have

$$C_2 = \int_E |x|\, dx = \frac{3 \log 3 + 4}{6\sqrt{2}\, 3^{3/4}} \approx 0.377$$

where E is the regular hexagon of unit area centered at the origin.

- If $d \geq 3$ the value of C_d is not known.
- If $d \geq 3$ the optimal asymptotical configuration of the points is not known.
- The numerical computation of optimal configurations is very heavy.

• If the choice of location points is made randomly, surprisingly the loss in average with respect to the optimum is not big and a similar estimate holds, i.e. there exists a constant R_d such that

$$E\Big(F(\Sigma_N\Big) \approx R_d N^{-1/d} \omega_d^{-1/d} \Big(\int_\Omega f^{d/(1+d)} \Big)^{(1+d)/d}$$

while

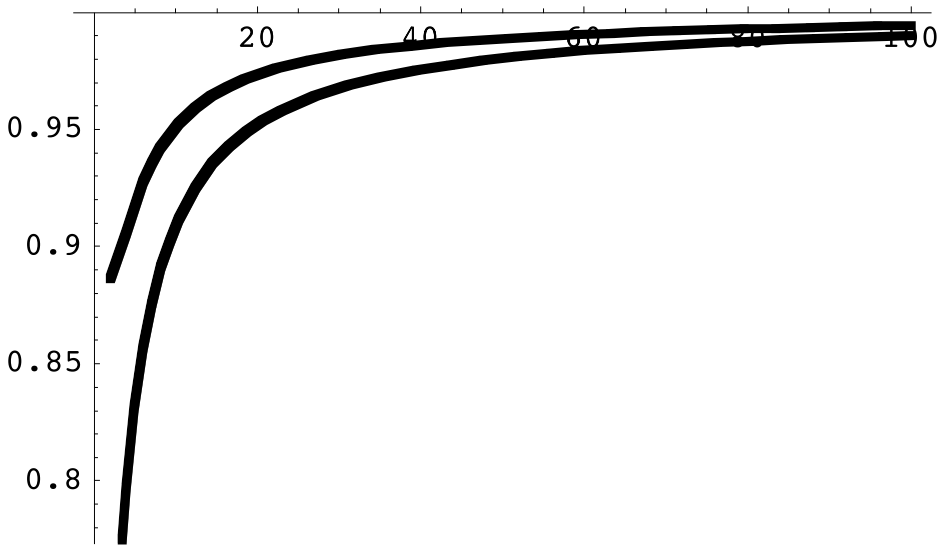
$$F(\Sigma_N^{opt}) \approx C_d N^{-1/d} \omega_d^{-1/d} \Big(\int_\Omega f^{d/(1+d)} \Big)^{(1+d)/d}$$

We have $R_d = \Gamma(1+1/d)$ so that

$C_1 = 0.5$ while $R_1 = 1$

$C_2 \simeq 0.669$ while $R_2 \simeq 0.886$

$\frac{d}{1+d} \leq C_d \leq \Gamma(1+1/d) = R_d$ for $d \geq 3$



Plot of $\frac{d}{1+d}$ and of $\Gamma(1+1/d)$ in terms of d

The irrigation problem

Taking again the cost functional

$$F(\Sigma) := \int_\Omega \mathrm{dist}(x, \Sigma)\, f(x)\, dx.$$

we consider the minimization problem

$$\min \Big\{ F(\Sigma) \ :\ \Sigma \text{ connected}, \mathcal{H}^1(\Sigma) \le \ell \Big\}$$

Connected onedimensional subsets Σ of Ω are called networks.

Theorem *For every $\ell > 0$ there exists an optimal network Σ_ℓ for the optimization problem above.*

Some necessary conditions of optimality on Σ_ℓ have been derived:

Buttazzo-Oudet-Stepanov 2002,
Buttazzo-Stepanov 2003,
Santambrogio-Tilli 2005
Mosconi-Tilli 2005
....

For instance the following facts have been proved:

- no closed loops;

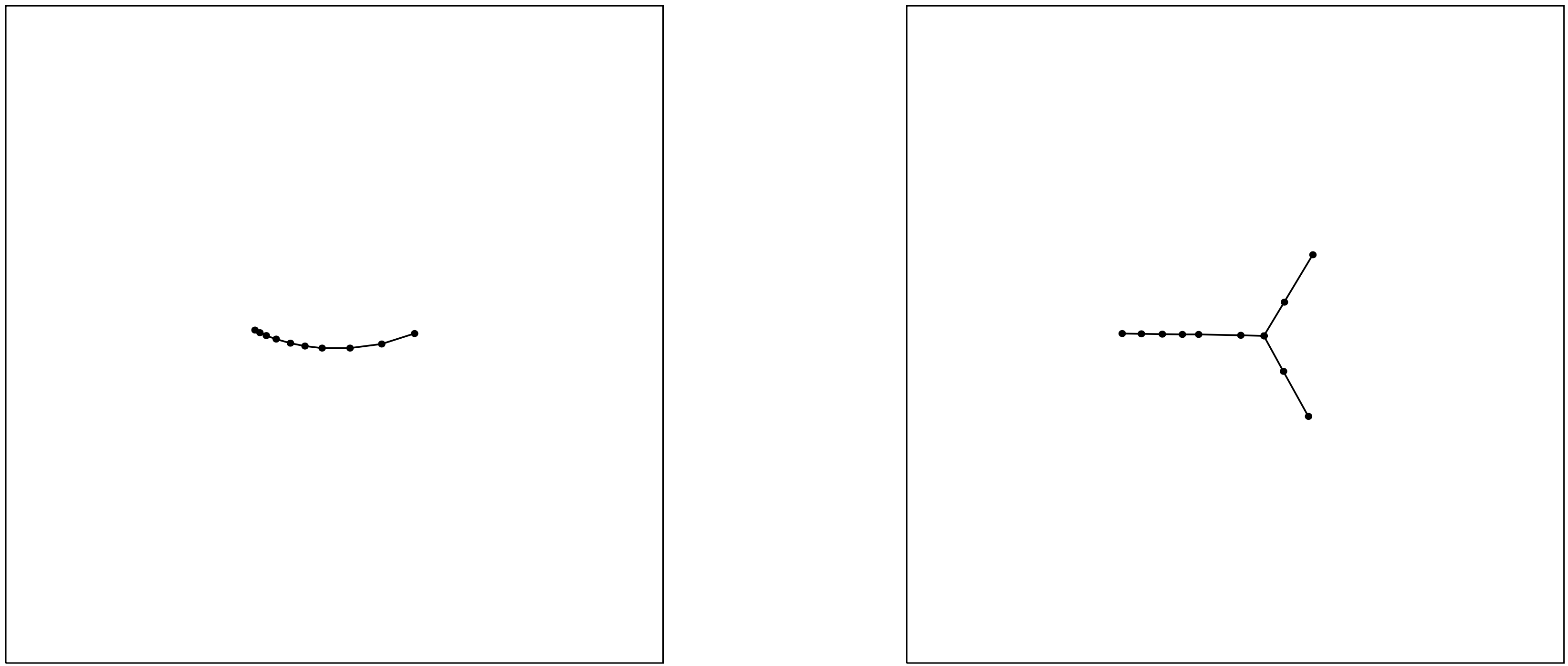
- at most triple point junctions;

- 120° at triple junctions;

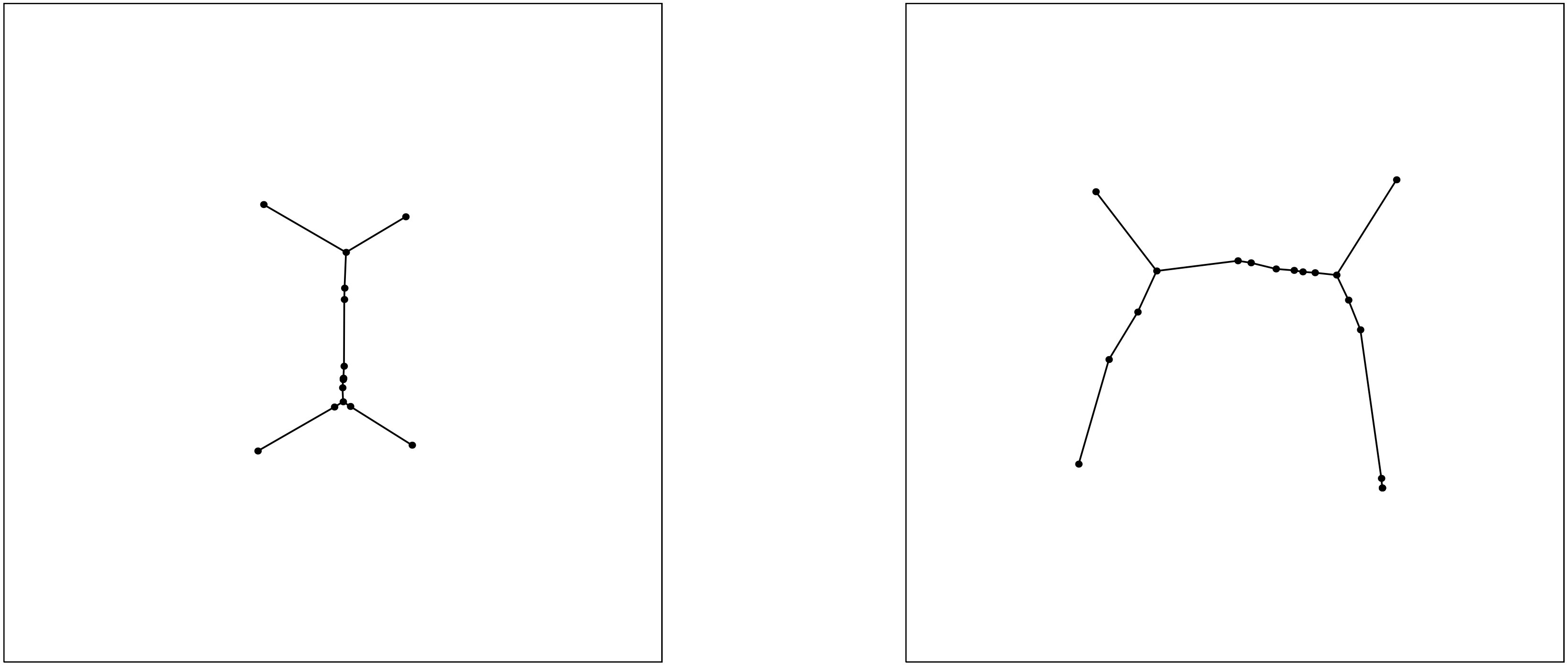
- no triple junctions for small ℓ;

- asymptotic behavior of Σ_ℓ as $\ell \to +\infty$ (Mosconi-Tilli JCA 2005);

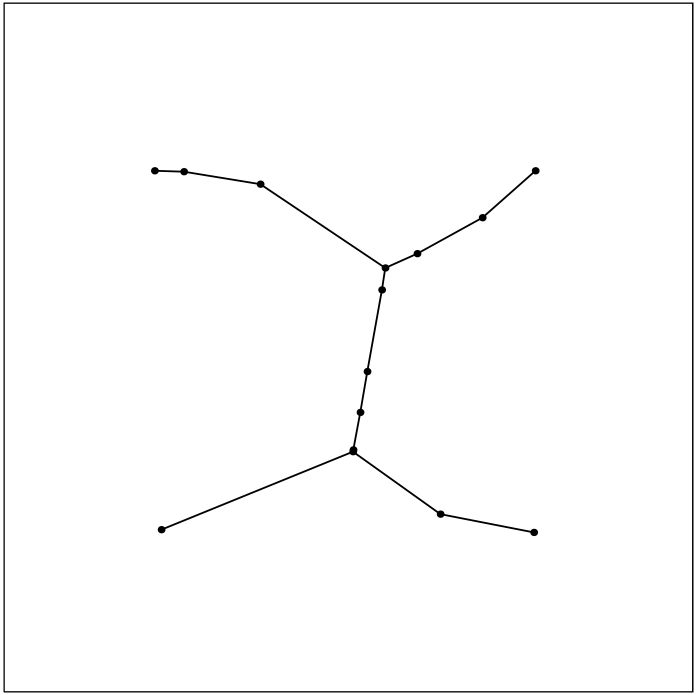
- regularity of Σ_ℓ is an open problem.

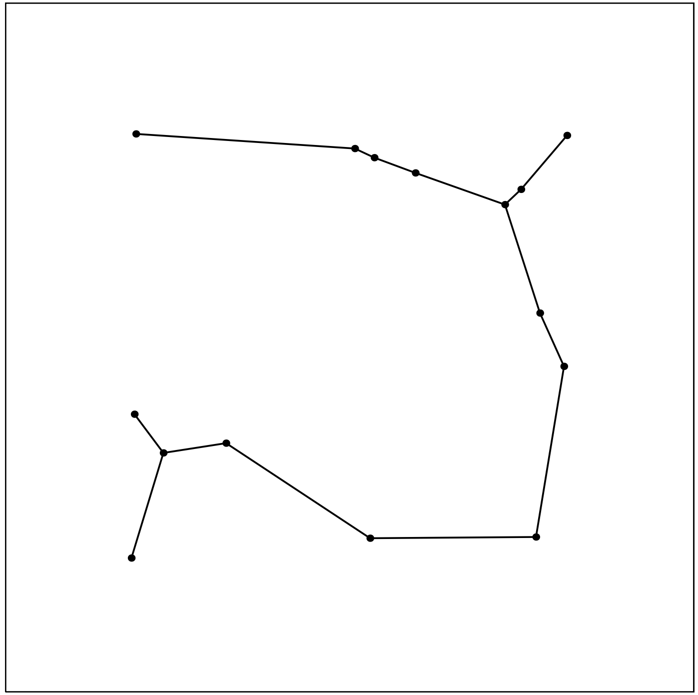


Optimal sets of length 0.5 and 1 in a unit square with $f = 1$.



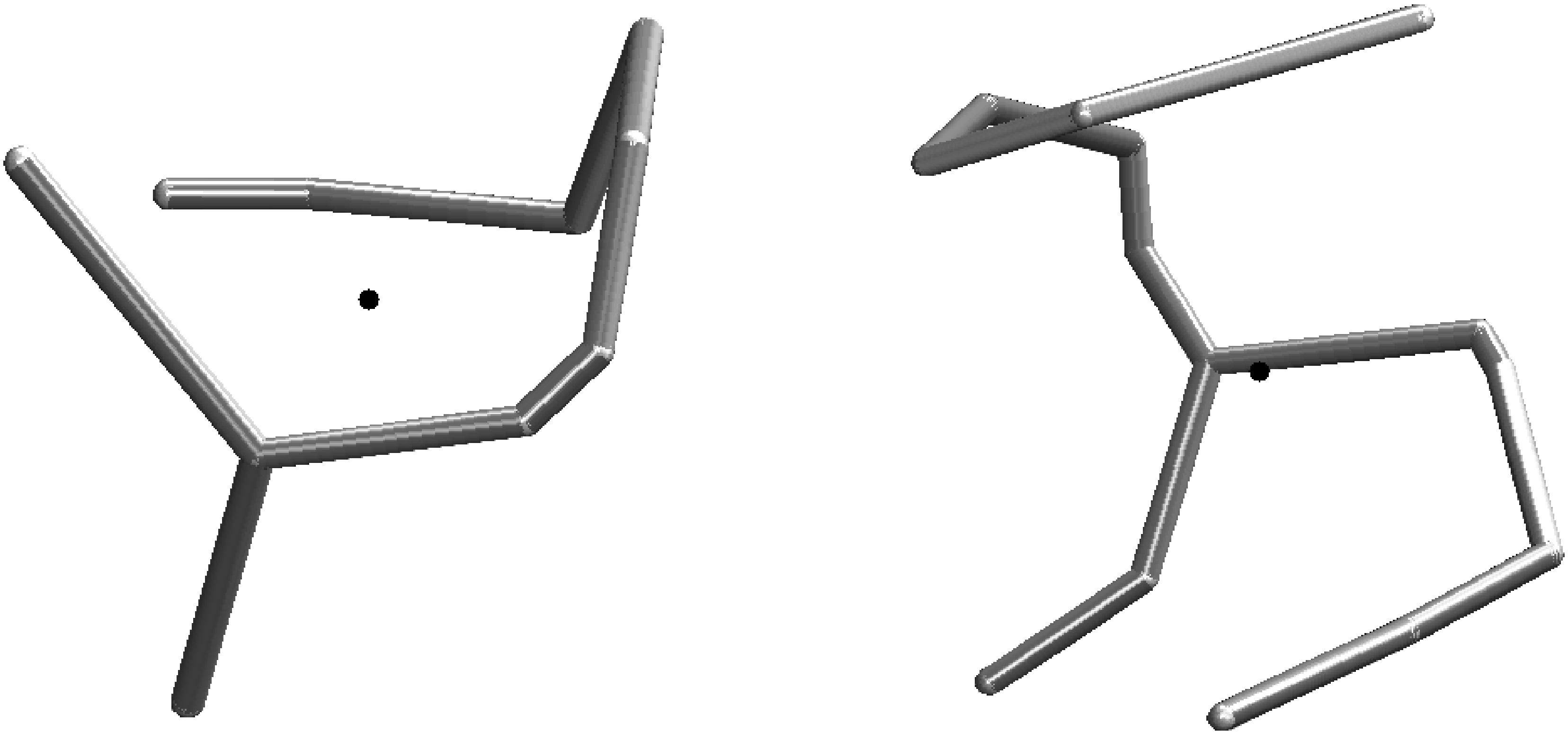
Optimal sets of length 1.5 and 2.5 in a unit square with $f = 1$.





Optimal sets of length 3 and 4 in a unit square with $f = 1$.

Optimal sets of length 1 and 2 in the unit ball of $\mathbf{R}^3$.



Optimal sets of length 3 and 4 in the unit ball of $\mathbf{R}^3$.

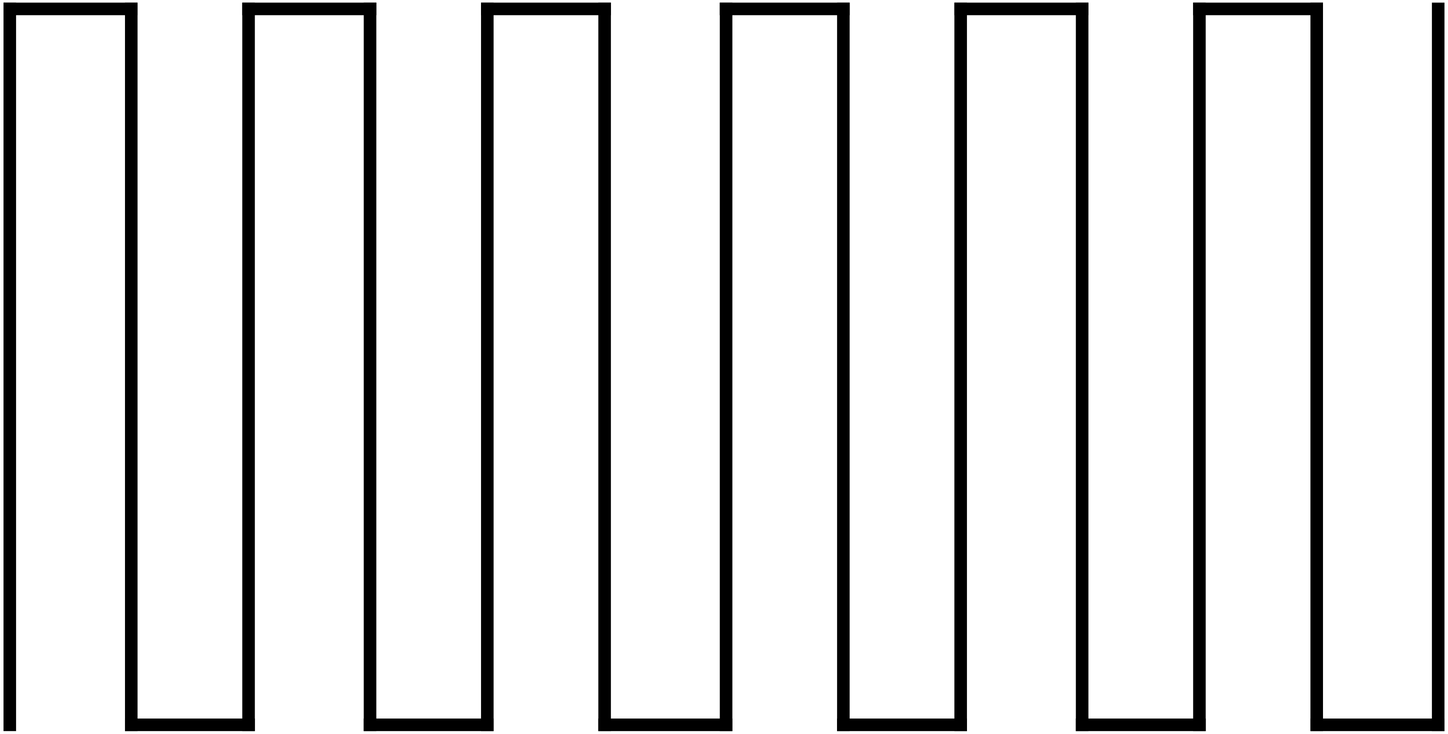
Analogously to what done for the location problem (with points) we can study the asymptotics as $\ell \to +\infty$ for the irrigation problem. This has been made by S.Mosconi and P.Tilli who proved the following facts.

- $L_\ell \approx \ell^{1/(1-d)}$ as $\ell \to +\infty$;

- $\ell^{1/(d-1)} F_\ell \to C_d \int_\Omega \mu^{1/(1-d)} f(x)\, dx$ as $\ell \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;

- $\mu_{opt} = K_d f^{(d-1)/d}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{(d-1)/d}$ and not as f (for instance as $f^{1/2}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by many parallel segments (at the same distance) connected by one segment.



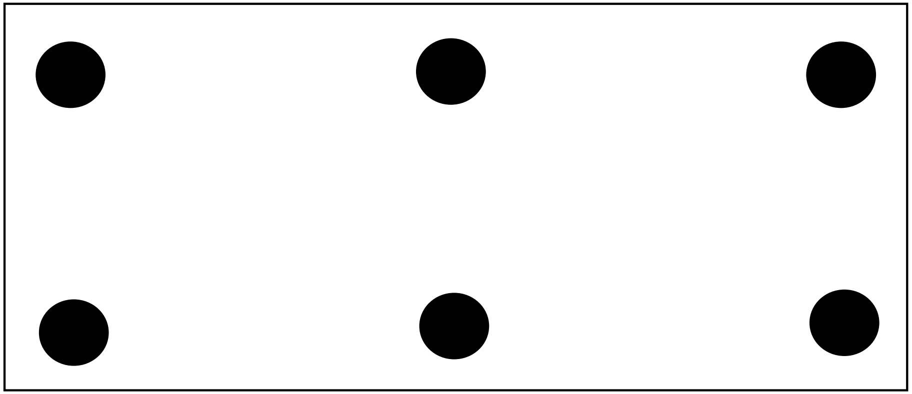
Asymptotic optimal irrigation network in dimension two.

The case of elastic compliance

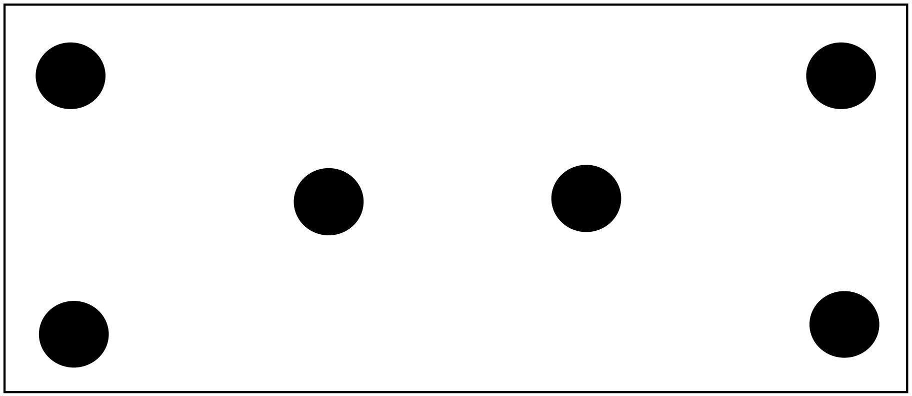
The goal is to study the configurations that provide the minimal compliance of a structure. We want to find the optimal region where to clamp a structure in order to obtain the highest rigidity.

The class of admissible choices may be, as in the case of mass transportation, a set of points or a one-dimensional connected set.

Think for instance to the problem of locating in an optimal way (for the elastic compliance) the six legs of a table, as below.



An admissible configuration for the six legs.



Another admissible configuration.

The precise definition of the cost functional can be given by introducing the elastic compliance

$$\mathcal{C}(\Sigma) = \int_{\Omega} f(x) u_{\Sigma}(x) \, dx$$

where Ω is the entire elastic membrane, Σ the region (we are looking for) where the membrane is fixed to zero, f is the exterior load, and u_{Σ} is the vertical displacement that solves the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{in } \Sigma \cup \partial\Omega \end{cases}$$

The optimization problem is then

$$\min\Big\{\mathcal{C}(\Sigma)\ :\ \Sigma \text{ admissible}\Big\}$$

where again the set of admissible configurations is given by any array of a fixed number n of balls with total volume V prescribed.

As before, the goal is to study the optimal configurations and to make an asymptotic analysis of the density of optimal locations.

Theorem. *For every $V > 0$ there exists a convex function g_V such that the sequence of functional $(F_n)_n$ above Γ-converges, for the weak* topology on $\mathcal{P}(\overline{\Omega})$, to the functional*

$$F(\mu) = \int_\Omega f^2(x)\, g_V(\mu^a)\, dx$$

where μ^a denotes the absolutely continuous part of μ.

The Euler-Lagrange equation of the limit functional F is very simple: μ is absolutely continuous and for a suitable constant c

$$g_V'(\mu) = \frac{c}{f^2(x)}\,.$$

Open problems

- Exagonal tiling for $f = 1$?

- Non-circular regions Σ, where also the orientation should appear in the limit.

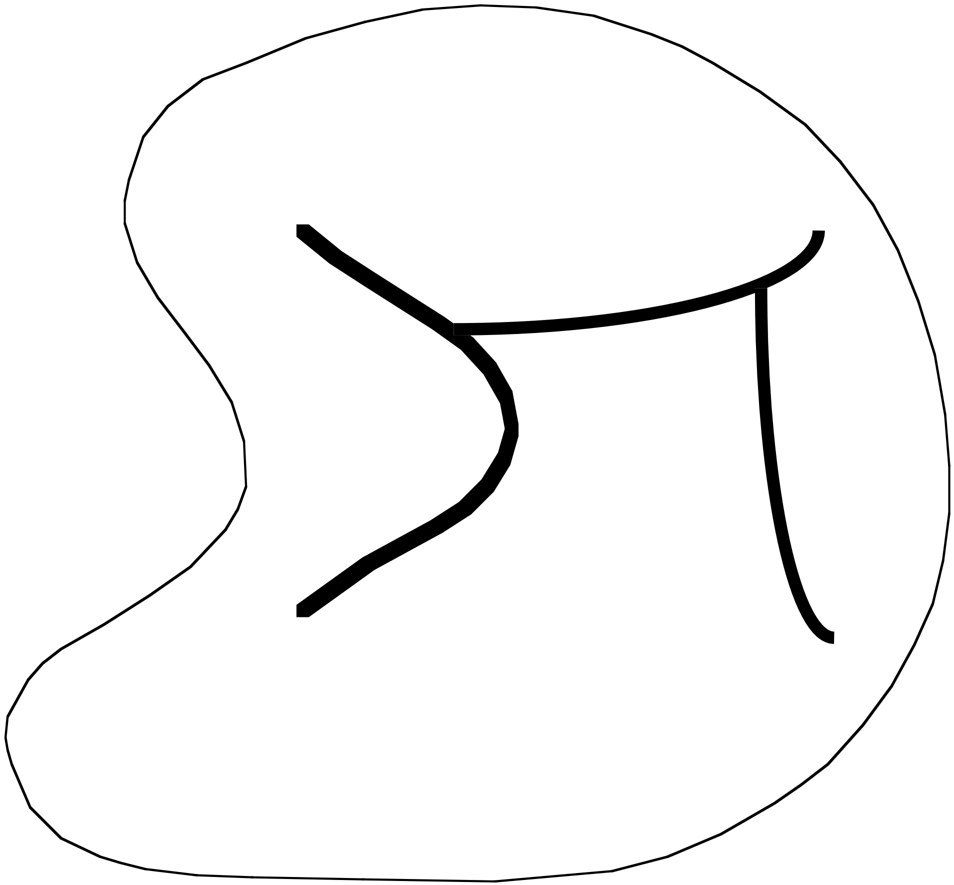
- Computation of the limit function g_V.

- Quasistatic evolution, when the points are added one by one, without modifying the ones that are already located.

Optimal compliance networks

We consider the problem of finding the best location of a Dirichlet region Σ for a two-dimensional membrane Ω subjected to a given vertical force f. The admissible Σ belong to the class of all closed connected subsets of Ω with $\mathcal{H}^1(\Sigma) \leq L$.

The existence of an optimal configuration Σ_L for the optimization problem described above is well known; for instance it can be seen as a consequence of the Sverák compactness result.



An admissible compliance network.

As in the previous situations we are interested in the asymptotic behaviour of Σ_L as $L \to +\infty$; more precisely our goal is to obtain the limit distribution (density of lenght per unit area) of Σ_L as a limit probability measure that minimize the Γ-limit functional of the suitably rescaled compliances.

To do this it is convenient to associate to every Σ the probability measure

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and to define the rescaled compliance functional $F_L : \mathcal{P}(\overline{\Omega}) \to [0, +\infty]$

$$F_L(\mu) = \begin{cases} L^2 \int_\Omega f u_\Sigma \, dx & \text{if } \mu = \mu_\Sigma,\ \mathcal{H}^1(\Sigma) \le L \\ +\infty & \text{otherwise} \end{cases}$$

where u_Σ is the solution of the state equation with Dirichlet condition on Σ. The scaling factor L^2 is the right one in order to avoid the functionals to degenerate to the trivial limit functional which vanishes everywhere.

Theorem. *The family of functionals* (F_L) *above* Γ*-converges, as* $L \to +\infty$ *with respect to the weak* topology on* $\mathcal{P}(\overline{\Omega})$*, to the functional*

$$F(\mu) = C \int_\Omega \frac{f^2}{\mu_a^2}\, dx$$

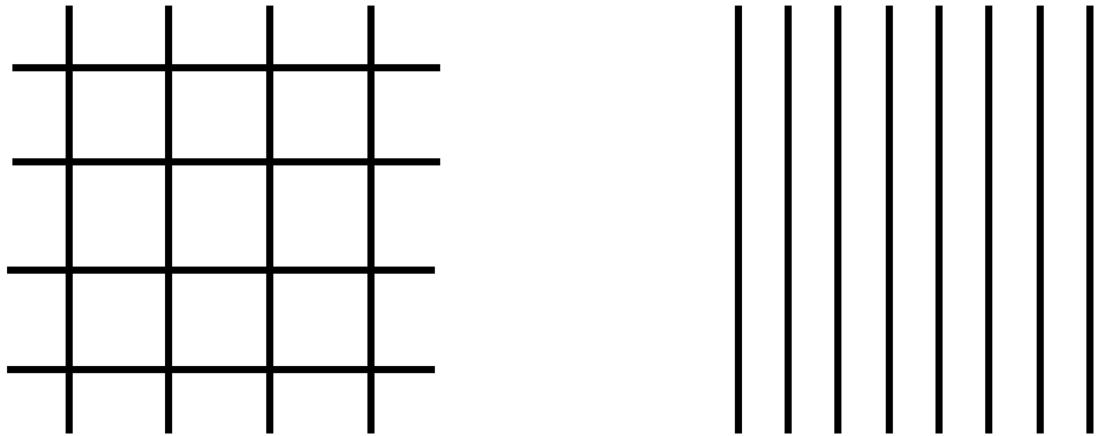
where C *is a constant.*

In particular, the optimal compliance networks Σ_L are such that μ_{Σ_L} converge weakly* to the minimizer of the limit functional, given by

$$\mu = c f^{2/3}\, dx.$$

Computing the constant C is a delicate issue. If $Y = (0,1)^2$, taking $f = 1$, it comes from the formula

$$C = \inf \left\{ \liminf_{L \to +\infty} L^2 \int_Y u_{\Sigma_L} \, dx \ : \ \Sigma_L \text{ admissible} \right\}.$$



A grid is less performant than a comb structure, that we conjecture to be the optimal one.

Open problems

- Optimal periodic network for $f = 1$? This would give the value of the constant C.

- Numerical computation of the optimal networks Σ_L.

- Quasistatic evolution, when the length increases with the time and Σ_L also increases with respect to the inclusion (irreversibility).

- Same analysis with $-\Delta_p$, and limit behaviour as $p \to +\infty$, to see if the geometric problem of average distance can be recovered.

References

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