Self-similar decay and generalizations via Optimal Transport and moment normalization. State of the art and open problems.

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Plan of the talk

Introduction Classical examples of self-similar decay.

Generalization of self–similarity Abstract description of the general strategy. Ingredients: Wasserstein Lipschitz–continuity + Estimate of solution moments + Fixed point theorem.

Nonlinear Diffusion Equations. 1 Intermediate behavior in the general non-homogeneous case via estimate of the temperature.

Nonlinear Diffusion Equations. 2 Further properties of the intermediate profile.

Scalar 1–d Conservation Law Via ∞–Wasserstein contraction.

Viscous conservation laws Via splitting method.

Open problems Behavior of the profile in case of asymptotical homogeneity. Possible applications to nonlocal transport equations.

List of references

[Tos 05 Toscani - JEE 2005] Basic idea: scaling solutions by their own temperature. Idea taken from cooling states in granular media.

[CDT 06 Carrillo, DF, Toscani - ARMA 2006] Toscani's idea applied to nonlinear diffusion equations.

[CV Carrillo, Vazquez - JEE 2007] Asymptotic complexity of the intermediate profile and characterization of the 'simple asymptotics'.

[CDL 07 Carrillo, DF, Lattanzio] 1–d scalar conservation law. Use of the ∞–Wasserstein distance.

[CDL2 07 Carrillo, DF, Lattanzio] Convection–diffusion equations via solution operator splitting.

[CDT] In progress.

Introduction. A classical example.

As an example of self-similar decay, we consider the Porous Medium Equation (PME) $\partial_t u = \Delta u^m$, with $m > 1$, posed on the whole space $\mathbb{R}^d$. (PME) features the so called L^1–L^∞ *smoothing effect*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C\left(\|u(0)\|_{L^1(\mathbb{R}^d)}\right) t^{-d/\lambda}, \qquad \lambda = d(m-1)+2,$$

for all initial datum $u_0 \in L^1_+(\mathbb{R}^d)$ and for all $t > 0$. In particular, $u(t)$ decays to zero uniformly for large times. An *intermediate* state for such a PDE is a (time–depending) profile which approximates (in a suitable norm) all the solutions $u(t)$ *better* than the zero state. In this case, the intermediate profile is the self–similar *Barenblatt* (or BZKP) solution

$$u^\infty(x,t) = t^{-d/\lambda}\left(C - k\left|xt^{-1/\lambda}\right|^2\right)_+^{\frac{1}{m-1}}, \quad k = \frac{(m-1)\lambda}{2m},$$

with $C > 0$ depending on the total mass.

Introduction. A classical example

This result goes back to [Kamin, Isr. J. Math. '73] and [Vazquez, TAMS '83, '84] (see also [Vazquez, JEE '03]). It relies on the classical *invariance* of the set of solutions to (PME) under the *scaling* $x \to x\varepsilon$, $t \to t\varepsilon^{\lambda}$, and it can be stated as follows: any solution $u(t)$ to (PME) with L^1_+ initial datum satisfies

$$\lim_{t\to+\infty} t^{\frac{q-1}{\lambda q}} \|u(t) - u^{\infty}(t)\|_{L^q(\mathbb{R}^d)} = 0, \qquad \text{for all} \ \ q \in [1, +\infty].$$

The above convergence rate (notice that no rate appears in the L^1 norm) cannot be improved unless one prescribes further assumptions in terms of the *second moment* of the initial datum. A well known result due to [Carrillo, Toscani - Ind. Univ. Math. J. '00], [Otto - CPDE '01], [Del Pino, Dolbeault - J. Math. P. Appl. '02] states the following: let $u_0 \in L^1_+$ such that $\int |x|^2 u_0(x)^2 dx < +\infty$, then

$$\|u(t) - u^{\infty}(t)\|_{L^1(\mathbb{R}^d)} \leq C t^{-\frac{2}{\lambda \max(2,m)}}. \tag{1}$$

Introduction. Time dependent scaling.

The main idea behind the latter result is to *re-scale* the solutions to (PME) in such a way that $u^{\infty}(t)$ turns into a *stationary* profile. A suitable choice of the scaling is

$$u(x,t) = R(t)^{-\frac{d}{\lambda}}\, v(y,\tau),$$
$$y = xR(t)^{-\frac{1}{\lambda}},\ \tau = \frac{1}{\lambda}\log R(t), \quad R(t) = (\lambda t + 1)\,. \tag{2}$$

The new independent variable $v(y,\tau)$ solves the nonlinear Fokker–Planck equation

$$\partial_\tau v = \operatorname{div}\left(yv + \nabla v^m\right) = \operatorname{div}\left(v\nabla\left(\frac{|y|^2}{2} + \frac{m}{m-1}v^{m-1}\right)\right). \tag{3}$$

The stationary solution $v^\infty(y) = \left(C - k|y|^2\right)_+^{\frac{1}{m-1}}$ corresponds to the Barenblatt solution via the scaling (2). v^∞ is the global minimum of the Entropy functional

$$E(v) = \frac{1}{m-1}\int v^{m-1}\,dy + \frac{1}{2}\int |y|^2 v\,dy.$$

A nontrivial study of the evolution in time of the *relative entropy* $E(v(\tau)) - E(v^\infty)$ and the use of a Csiszár–Kullback inequality provide exponential convergence to equilibrium in the L^1–norm, which yields (1) in the 'old' variable u.

This method strongly relies on the *homogeneity* of the nonlinear function $u \mapsto \phi(u) = u^m$, which implies the *a-priori* existence of a self–similar solution (and, hence, of a candidate intermediate profile) and the existence of the time dependent scaling (2) which leads to (3).

Main question: How to do in case of a non-homogeneous ϕ?

Contraction in the 2–Wasserstein distance.

We recall the definition of 2–Wasserstein distance

$$d_2(u,v) := \inf \left\{ \int_{\mathbb{R}^d} |x - T(x)|^2 \, dx : \; T_\sharp u_1 = u_2 \right\}$$

between $u, v \in L^1_+(\mathbb{R}^d)$. It is known from [Otto - CPDE '01], [Carrillo, McCann, Villani - Rev. Mat. Iber. '03 and ARMA '05] and [Ambrosio, Gigli and Savaré - Birkhauser '05] that the functional $E(v)$ is 1–convex in the Wasserstein space of probability measures with finite second moment (1–*displacement* convex). Since the Fokker–Planck equation (3) can be formulated as the *gradient flow* of $E(v)$ with respect to the 2–Wasserstein distance, this implies the contraction estimate

$$d_2(v_1(\tau), v_2(\tau)) \leq e^{-\tau} d_2(v_1(0), v_2(0)),$$

for all $\tau \geq 0$ and for two solutions v_1, v_2 to (3).

Therefore, the solution semigroup S_τ of (3) is *strictly contractive* for all times $\tau > 0$ and therefore the unique stationary solution v^∞ can be interpreted as the unique fixed point of the semigroup S_τ. Moreover, the fixed point is the same for all $\tau > 0$. Once again, this is only possible due to the homogeneity of $\phi(u) = u^m$. In case of a general (not necessarily homogeneous) nonlinear diffusion equation $\partial_t = \Delta\phi(u)$ we cannot proceed in the same way because

- We don't have a scaling invariance which allows to write a Fokker–Planck type equation.
- We have no candidate intermediate profile.

The main scope of the present work is to reproduce an alternative scaling structure which allows to achieve the intermediate profile (at the re–scaled level) as a fixed point of a certain time depending map, *even in case of a lack of homogeneity*.

Introduction. A fundamental remark

A simple remark by Toscani motivates our strategy: we evaluate the *second moment* ('temperature' in kinetic language) of the Barenblatt solution

$$\theta_2[u^\infty(t)] := \int |x|^2 u^\infty(x,t)\,dx = t^{-d/\lambda} \int_{\mathbb{R}^d} |x|^2 \left(C - k\left|x t^{-1/\lambda}\right|^2\right)_+^{\frac{1}{m-1}}\,dx$$

$$= t^{\frac{2}{\lambda}} \int_{\mathbb{R}^d} |y|^2 (C - k|y|^2)_+^{\frac{1}{m-1}}\,dy = O(R(t)^{\frac{2}{\lambda}}) \quad \text{as} \;\; t \to +\infty.$$

This implies that, in the special case $u = u^\infty$, the scaling (2) is asymptotically (as $t \to +\infty$) equivalent to the following one

$$u^\infty(x,t) = \theta_2[u^\infty(t)]^{-\frac{d}{2}}\; v^\infty\left(x\;\theta_2[u^\infty(t)]^{-\frac{1}{2}}, \frac{1}{2}\log\theta_2[u^\infty(t)]\right)$$

Generalization of self similarity

Notations: Let $\mathbb{P}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$. For a $\mu \in \mathbb{P}(\mathbb{R}^d)$ and for $p \in [1, \infty)$ we denote the (possibly infinite) moment of order p of μ by

$$\theta_p[\mu] := \int_{\mathbb{R}^d} |x|^p \mu(x).$$

We introduce the space

$$\mathbb{P}_p(\mathbb{R}^d) := \left\{ \mu \in \mathbb{P}(\mathbb{R}^d) : \ \theta_p[\mu] < +\infty \right\}.$$

For two given $\mu_1, \mu_2 \in \mathbb{P}_p(\mathbb{R}^d)$, we recall the definition of p–Wasserstein distance between μ_1 and μ_2

$$d_p(\mu_1, \mu_2)^p = \inf \left\{ \int\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\gamma(x, y) : \quad \gamma \in \Gamma(\mu_1, \mu_2) \right\},$$

$\Gamma(\mu_1, \mu_2) = \{\gamma \in \mathbb{P}(\mathbb{R}^d \times \mathbb{R}^d) : \quad \pi^i_\sharp(\gamma) = \mu_i\}, \quad \pi_1, \pi_2$ projections.

In the case $p = +\infty$ we have the following definition

$$d_\infty(\mu_1, \mu_2) = \lim_{p \to +\infty} d_p(\mu_1, \mu_2) \quad = \inf \left\{ \text{ess sup}_\gamma |x - y| : \quad \gamma \in \Gamma(\mu_1, \mu_2) \right\}.$$

We also introduce the space $\mathbb{P}_c(\mathbb{R}^d)$ of compactly supported probability measures on $\mathbb{R}^d$ and the *maximal transport moment* $\sigma[\mu]$ of $\mu \in \mathbb{P}_c(\mathbb{R}^d)$ as the quantity

$$\sigma[\mu] := d_\infty(\mu, \delta_0) = \sup\{|x|,\ x \in \text{supp}(\mu)\}.$$

Finally, we introduce the spaces

$$\mathcal{M}_p(\mathbb{R}^d) := \left\{ \mu \in \mathbb{P}_p(\mathbb{R}^d) :\ \theta_p[\mu] = 1 \right\}$$
$$\mathcal{M}_\infty(\mathbb{R}^d) := \left\{ \mu \in \mathbb{P}_c(\mathbb{R}^d) :\ \sigma[\mu] = 1 \right\}.$$

Generalization of self similarity. Abstract framework

Let $\mathcal{S}_t : \mathbb{P}_p(\mathbb{R}^d) \to \mathbb{P}_p(\mathbb{R}^d)$ be a continuous dynamical system on the p–Wasserstein space for a fixed $p \in [1, +\infty]$. Notice that we are not requiring $\mathcal{S}_t$ to be a semigroup. We prescribe the following basic assumptions on $\mathcal{S}_t$:

1. $\mathcal{S}_t$ features a (time depending) bound from below of the moment of order p, i. e. there exists a continuous map $[0, \infty) \ni t \mapsto \alpha(t) > 0$ such that
$$\theta_p[\mathcal{S}_t[\mu]] \geq \alpha(t)^p.$$
2. $\mathcal{S}_t$ is locally Liptschitz (in time) with respect to the p–Wasserstein distance, i. e. there exists a continuous map $[0, \infty) \ni t \mapsto \beta(t) > 0$ such that
$$d_p(\mathcal{S}_t[\mu], \mathcal{S}_t[\nu]) \leq \beta(t) d_p(\mu, \nu).$$
3. $\lim_{t \to +\infty} \dfrac{\beta(t)}{\alpha(t)} = 0$.

Remarks:

1. We remark that no assumptions are prescribed on the monotonicity of α and β. In particular, $\alpha(t)$ need not necessarily be diverging as $t \to +\infty$, which is the case in diffusion equations. The present framework also applies to those cases where $\theta_p[\mathcal{S}_t[\mu]] \to 0$, which means that solutions are concentrating to a Dirac's delta. In this case, we are generalizing the concept of *self–similar large time blow up.*

2. We emphasize the importance of the condition $\theta_p[\mathcal{S}_t[\mu]] \geq \alpha(t) > 0$, which makes the next definition of p–Toscani map well posed. However, in situations where it is possible to predict the exact time of collapse of the p–moment (i.e. the time where $\theta_p[\mathcal{S}_t[\mu]] = 0$), this condition could be removed. This case corresponds to investigate a *self–similar finite time blow up*.

Generalization of self similarity. The Toscani map

For a given $p \in [1, +\infty]$, we introduce the p–Toscani map

$$\mathcal{T}_t : \mathcal{M}_p(\mathbb{R}^d) \to \mathcal{M}_p(\mathbb{R}^d)$$

as follows: let $\mu \in \mathcal{M}_p(\mathbb{R}^d)$, let us denote $\mu(t) := \mathcal{S}_t[\mu]$. For a given test function $\varphi \in C_c(\mathbb{R}^d)$ we set

$$\int_{\mathbb{R}^d} \varphi(x) d\mathcal{T}_t[\mu](x) = \int_{\mathbb{R}^d} \varphi(x\theta_p[\mu(t)]^{-\frac{1}{p}}) d\mu(t)(x).$$

In case $\mu(t) = u(t)dx$, the above definition reads

$$\mathcal{T}_t[\mu](x) = \theta_p[u(t)]^{\frac{d}{p}} u\left(\theta_p[u(t)]^{\frac{1}{p}}\, x,\ t\right).$$

Generalization of self similarity. The strategy

Let $\mu_1, \mu_2 \in \mathcal{M}_p(\mathbb{R}^d)$. We denote $\mu_i(t) := \mathcal{S}_t[\mu_i]$ for $i = 1, 2$ and we use a trivial scaling property of the p–Wasserstein distance to obtain

$$d_p(\mathcal{T}_t[\mu_1], \mathcal{T}_t[\mu_2])^p = \overline{\theta}_p(t)^{-1} d_p(\overline{\mu_1(t)}, \overline{\mu_2(t)})^p, \tag{4}$$

where $\overline{\theta}_p(t) := \min\{\theta_p[\mu_1(t)], \theta_p[\mu_2(t)]\}$ and $\overline{\mu_1(t)}, \overline{\mu_2(t)}$ are defined as follows: suppose for simplicity that $\theta_p[\mu_1(t)] \le \theta_p[\mu_2(t)]$. Then,

$$\overline{\mu_1(t)} := \mu_1(t),$$

$$\int \varphi(x) d\overline{\mu_2(t)}(x) = \int \varphi\left((\theta_p[\mu_1(t)]/\theta_p[\mu_2(t)])^{1/p}\ x \right) d\mu_2(t)(x)$$

for all $\varphi \in C_c(\mathbb{R}^d)$.

A technical lemma

Lemma 1. *Let* $p \in [1, +\infty]$. *Let* $\mu, \nu \in \mathbb{P}_p(\mathbb{R}^d)$ *if* $p < +\infty$ *(resp.* $\mu, \nu \in \mathbb{P}_c(\mathbb{R}^d)$ *if* $p = +\infty$*), such that* $\theta_p[\mu] = \theta_p[\nu]$ *(resp.* $\sigma[\mu] = \sigma[\nu]$ *if* $p = +\infty$*). For* $\alpha \geq 1$ *let* ν_α *be the probability measure defined by*

$$\int_{\mathbb{R}^d} \varphi(x) d\nu_\alpha(x) := \int_{\mathbb{R}^d} \varphi(\alpha x) d\nu(x), \qquad \varphi \in C_c(\mathbb{R}^d).$$

Then,

$$d_p(\mu, \nu) \leq 2 d_p(\mu, \nu_\alpha).$$

Proof. A simple computation yields

$$d_p^p(\mu, \nu_\alpha) = \inf \left\{ \int \int |x - \alpha y|^p d\gamma(x, y), \quad \gamma \in \Gamma(\mu, \nu) \right\},$$

where $\Gamma(\mu,\nu)$ is the set of all transport plans from between μ and ν. For a fixed $\gamma\in\Gamma(\mu,\nu)$ we denote $\psi_\gamma(\alpha) := \displaystyle\int\int |x-\alpha y|^p d\gamma(x,y)$ and compute

$$
\begin{aligned}
\psi_\gamma(1) &= \alpha^{-p}\|\alpha\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}\\
&\le \alpha^{-p}2^{p-1}\left(\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}+\|\pi_1-\alpha\pi_1\|^p_{L^p(d\gamma)}\right)\\
&= \alpha^{-p}2^{p-1}\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}+\alpha^{-p}2^{p-1}\left((\alpha-1)\theta_p[\mu]^{1/p}\right)^p\\
&= \alpha^{-p}2^{p-1}\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}+\alpha^{-p}2^{p-1}\left(\alpha\theta_p[\nu]^{1/p}-\theta_p[\mu]^{1/p}\right)^p\\
&= \alpha^{-p}2^{p-1}\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}+\alpha^{-p}2^{p-1}\left(\|\alpha\pi_2\|_{L^p(d\gamma)}-\|\pi_1\|_{L^p(d\gamma)}\right)^p\\
&\le \alpha^{-p}2^{p-1}\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}+\alpha^{-p}2^{p-1}(\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}\\
&= \alpha^{-p}2^{p}\|\pi_1-\alpha\pi_2\|^p_{L^p(d\gamma)}\le 2^p\psi_\gamma(\alpha). \qquad \square
\end{aligned}
$$

Remark: (the technical lemma for $p = 2$). In case $p = 2$, the statement in the above lemma can be improved to

$$d_2(\mu, \nu) \le d_2(\mu, \nu_\alpha), \qquad \text{for all } \ \alpha \ge 1.$$

This can be proven by a direct estimate of the first derivative with respect to α of the quantity $\psi_\gamma(\alpha) := \displaystyle\int\int |x - \alpha y|^2 d\gamma(x, y)$ for a $\gamma \in \Gamma_o(\mu, \nu)$:

$$\begin{aligned}
&\frac{d}{d\alpha}\psi_\gamma(\alpha) = -2\int\int (x - \alpha y) \cdot y d\gamma(x, y) = 2\alpha \int |y|^2 d\nu(y) \\
&\quad - 2\int\int x \cdot y d\gamma(x, y) = 2\alpha \int |y|^2 d\nu(y) + \int\int |x - y|^2 d\gamma(x, y) \\
&\quad - \int |x|^2 d\mu(x) - \int\int |y|^2 d\nu(y) = 2(\alpha - 1) \int |y|^2 d\nu(y) + d_2^2(\mu, \nu),
\end{aligned}$$

which shows that the minimum of $\psi_\gamma(\alpha)$ is achieved for $\alpha = 1 - \frac{d_2^2(\mu,\nu)}{2\int |x|^2 d\mu}$.

We use the previous lemma in the following step:

$$d_p(\overline{\mu_1(t)}, \overline{\mu_2(t)})^p \leq 2 d_p \left(\mu_1(t), \mu_2(t)\right)^p .$$

Hence, the property 2 of the dynamical system $\mathcal{S}_t$ implies

$$d_p(\overline{\mu_1(t)}, \overline{\mu_2(t)})^p \leq 2 \beta(t)^p d_p \left(\mu_1, \mu_2\right)^p .$$

Finally, the above combined with (4) and with the property 1 of $\mathcal{S}_t$ implies

$$d_p(\mathcal{T}_t[\mu_1], \mathcal{T}_t[\mu_2]) \leq 2 \frac{\beta(t)}{\alpha(t)} d_p(\mu_1, \mu_2).$$

Property 3 of $\mathcal{S}_t$ implies that $2\frac{\beta(t)}{\alpha(t)} \leq c < 1$ for all $t \geq t^*$ for a t^* large enough, therefore the p–Toscani map $\mathcal{T}_t$ is a strict contraction on the complete metric space $(\mathcal{M}_p(\mathbb{R}^d), d_p)$.

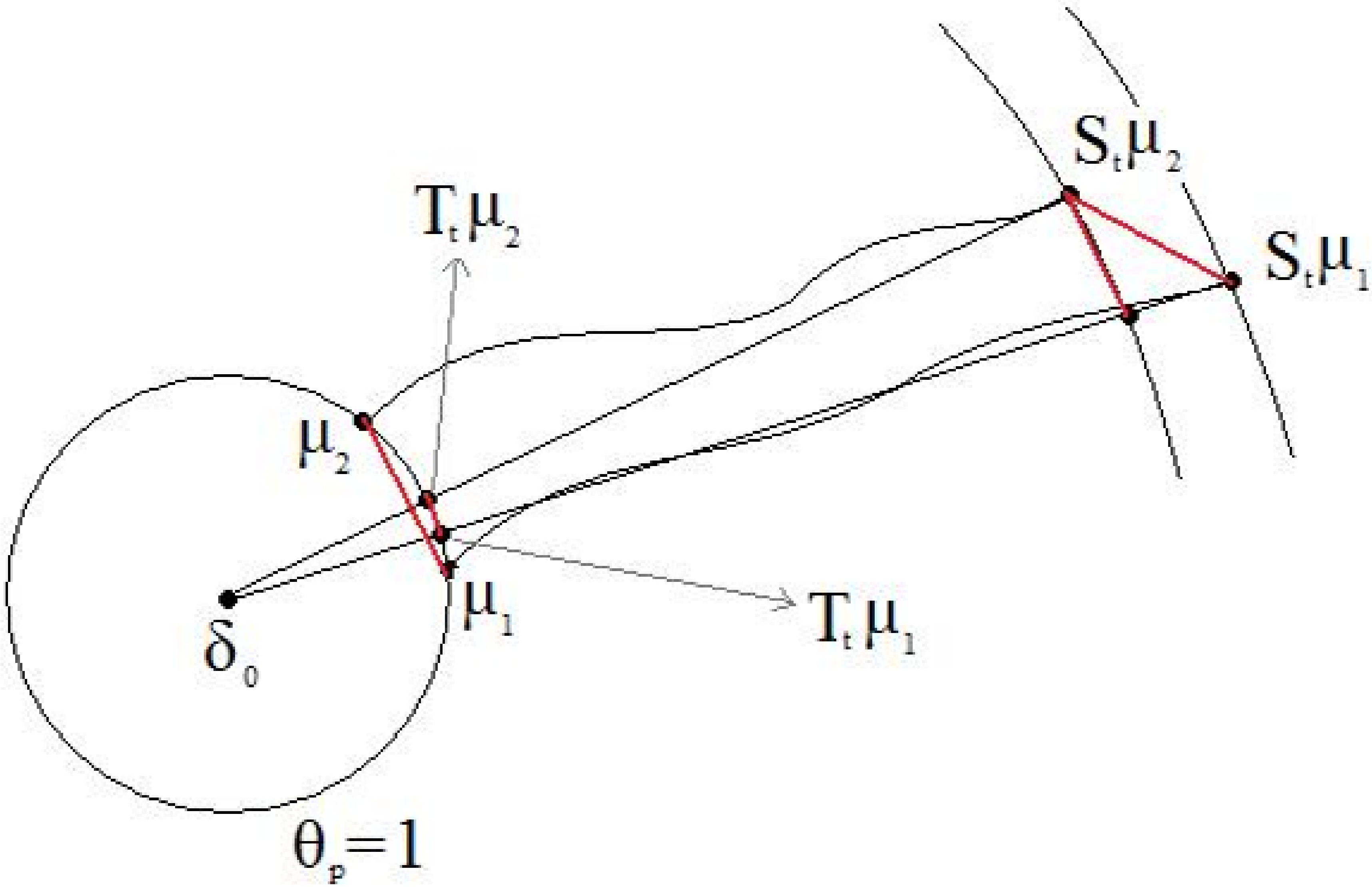

Figure 1: A geometric interpretation of the proof

The main theorem

We have proven the following theorem:

Theorem 1. *Let $\mathcal{S}_t$ be a dynamical system satisfying the above assumptions 1, 2 and 3 for a certain fixed $p \in [1, +\infty]$. Then there exist a time $t^* \geq 0$ and a unique one parameter family $\{\nu_t^\infty\}_{t \geq t^*} \subset \mathbb{P}_p(\mathbb{R}^d)$ such that*

$$d_p(\mathcal{T}_t[\mu(t)], \nu_t^\infty) \to 0 \qquad \text{as} \;\; t \to +\infty,$$

where $\mathcal{T}_t$ is the p–Toscani map defined above. Moreover, ν_t^∞ is the unique fixed point of $\mathcal{T}_t$ at each time $t > t^*$.

In the sequel we shall focus on specific cases where the assumptions on the dynamical system are satisfied and we shall study the fixed point family more in detail. Since all the models considered are mass preserving, we shall always assume that solutions have unit mass.

Remarks.

1. The fixed value of the p–moment chosen for the Toscani map, unit in the above procedure, can be arbitrarily chosen to be $\theta > 0$. Asymptotic profiles are then obtained for solutions with initial data with that given value of the p–moment. These asymptotic profiles may depend on the value of the p moment.

2. In the special case of a dynamical system $\mathcal{S}_t$ with the translation invariance property $(\mathcal{S}_t[f])(x+h) = (\mathcal{S}_t[f(\cdot + h)](x)$ (which implies conservation of the center of mass), one can rephrase the whole procedure by considering the quantity $d_p(\cdot, \delta_{x_0})$ instead of θ_p in the assumption 2 on $\mathcal{S}_t$ for an arbitrary $x_0 \in \mathbb{R}^d$. In this case the asymptotic profile v_t^∞ will have center of mass x_0 and it be the x_0 translation of the previous one. This fact does not contradicts the uniqueness of the fixed point. Indeed, it proves *directly* that translated asymptotic profiles match asymptotically in the d_p with a faster rate than the growth rate of their p–moment.

3. We remark that the whole procedure works even in case the Lipschitz continuity property of the p–Wasserstein distance

$$d_p(\mathcal{S}_t[\mu], \mathcal{S}_t[\nu]) \leq \beta(t) d_p(\mu, \nu)$$

holds in a closed subspace $\mathcal{C}$ of $\mathbb{P}_p(\mathbb{R}^d)$. Indeed, the fixed point family would belong to $\mathcal{C}$ and the asymptotic result would hold only in that class.

Nonlinear diffusion equation

Let us consider the case of a nonlinear diffusion equation

$$\partial_t u = \Delta \phi(u), \tag{5}$$

posed on $x \in \mathbb{R}^d$, with the following assumptions on ϕ:

1. $\phi \in C[0, +\infty) \cap C^1(0, +\infty)$, $\phi(0) = 0$ and $\phi'(u) > 0$ for all $u > 0$.

2. $\exists C > 0$ and $m > \frac{d-2}{d}$ such that $\phi'(u) \geq C u^{m-1}$ for all $u > 0$.

3. $\dfrac{\phi(u)}{u^{1-1/d}}$ is nondecreasing on $u \in (0, \infty)$.

Assumption 1 implies that the Cauchy problem for (5) is well posed in $L^1_+(\mathbb{R}^d)$ (cf. Benilán '76). Assumption 2 implies the L^1-L^∞ smoothing effect (cf. Verón '79)

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C\, t^{-\frac{d}{d(m-1)+2}} \|u_0\|_{L^1(\mathbb{R}^d)}. \tag{6}$$

Notice that assumption 2 is 'one–sided', in the sense that it does not require a power–like behavior of ϕ at zero. Assumption 3 implies that the entropy functional associated to equation (5) is displacement convex (cf. [McCann '97]) and thus, the flow map of (5) is a non-expansive contraction in time with respect to the Euclidean Wasserstein distance d_2 (cf. Otto '01, Agueh '2, Carrillo-McCann-Villani '04, Sturm '05). In formulas, if u_1, u_2 solve (5), then $d_2(u_1(t), u_2(t)) \le d_2(u_1(0), u_2(0))$. Therefore, in order to fit the abstract framework, we need *the temperature of the solution* $u(t)$ *to diverge to* $+\infty$ *for large times*.

This is ensured by the L^∞ decay of the solution in the following lemma.

Lemma 2. *Assume the nonlinearity ϕ satisfies assumptions 1, 2 and suppose that the temperature of all solutions $u(t)$ to (5) is finite. Then, all solutions $u(t)$ with to (5) satisfy*

$$\theta_2[u(t)] \geq \frac{1}{4} t^{\frac{2}{d(m-1)+2}}. \tag{7}$$

Proof. We have, for an arbitrary $R(t)$,

$$\begin{aligned}
\theta_2[u(t)] &= \int_{\mathbb{R}^d} \frac{|x|^2}{2} u(x,t)\,dx = \int_{|x|\geq R(t)} \frac{|x|^2}{2} u(x,t)\,dx + \int_{|x|\leq R(t)} \frac{|x|^2}{2} u(x,t)\,dx \\
&\geq \frac{R(t)^2}{2} \int_{|x|\geq R(t)} u(x,t)\,dx = \frac{R(t)^2}{2} \left[1 - \int_{|x|\leq R(t)} u(x,t)\,dx\right] \\
&\geq \frac{R(t)^2}{2} \left[1 - C_d \|u(t)\|_{L^\infty} R(t)^d\right],
\end{aligned}$$

where C_d is the volume of the unit sphere in $\mathbb{R}^d$. Taking into account the smoothing effect, we have

$$\|u(t)\|_{L^\infty} \leq C_0 t^{-\frac{d}{d(m-1)+2}}, \tag{8}$$

where the constant C_0 depends on the mass of the initial datum, and thus, by choosing

$$R(t) = \frac{1}{2C_0} t^{\frac{1}{d(m-1)+2}},$$

we obtain the desired below estimate (7). $\square$

As a consequence of this lemma and of the non-expansive contraction of the 2–Wasserstein distance, one obtains the following theorem stated in [CDT, ARMA '06]. The fact that the fixed point family here is absolutely continuous w.r.t. Lebesgue measures depends on a L^∞ regularizing effect occurring for measure valued initial data.

Theorem 2. [Asymptotic profile for general nonlinear diffusions] *Given ϕ verifying the above hypotheses, there exists $t_* > 0$ and a one parameter curve of probability densities v_t^∞, with unit temperature defined for $t \geq t_*$ such that, for any solution of (5) with initial data $(1+|x|^2)u_0 \in L^1_+(\mathbb{R}^d)$ of unit mass and temperature,*

$$d_2\left(\theta_2[u(t)]^{d/2}u(\theta_2[u(t)]^{1/2}\,\cdot\,,t), v_t^\infty\right) \longrightarrow 0 \quad \textit{as} \ \ t \to \infty.$$

Moreover, the asymptotic profile v_t^∞ is characterized as the unique fixed point of the 2–Toscani map associated to the flow map of (5)

Remarks:

- R. McCann gave a geometric interpretation of this phenomenon for nonlinear diffusion in 2002 at the Pims thematic programme on aymptotic geometric analysis.

- Let us remark here that the 2–Toscani map is a *projection* onto the unit second moment manifold $\mathcal{M}_2(\mathbb{R}^d)$, as it was proven in [Carlen, Gangbo - Annals of Math. 2003].

Nonlinear diffusion. The asymptotic profile

In the homogeneous case $\phi(u) = u^m$, the family v_t^∞ does not depend on t and coincides with the Barenblatt self-similar profile at the time in which it has temperature 1. This result is a consequence of the main theorem in [Toscani '05] and the uniqueness of the fixed points v_t^∞. Three natural question then arise:

1. Can one characterize the set of nonlinearities for which v_t^∞ is constant in time?
2. In case v_t^∞ is not constant, how does v_t^∞ behave as $t \to +\infty$? Does it converge to a limit point?
3. Is it possible to prove further regularity of v_t^∞?

All there issues have been addressed in [Carrillo, Vazquez - JEE 2007].

Nonlinear diffusion. The asymptotic profile

Main results in [Carrillo, Vazquez - JEE 2007]:

1. The asymptotic profile v_t^∞ belongs to $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and is a radially symmetric non increasing function. Moreover, if the nonlinear diffusion function $\phi(u)$ is C^∞ for $u > 0$, the fixed point v_t^∞ is locally C^∞ wherever it is positive.

2. There exists a nonlinearity $\phi(u)$ such that the adherence points of its asymptotic profile v_t^∞ contains more than one element. This example is constructed by taking a nonlinearity with a behavior at $u = 0$ which oscillates between two power–like diffusions (e.g. a linear and a quadratic one). The asymptotic behavior of v_t^∞ 'can be made arbitrarily complex' (chaotic behavior).

3. If the 2–Toscani map $\mathcal{T}_t$ for a nonlinear diffusion equation $u_t = \Delta\phi(u)$ is constant in time for some open time interval I, then the function ϕ is a power function on the range of the family of the constant fixed point v^∞. This means that $\phi(u) = cu^m$ for some m and $c > 0$ and for all $u \in [0, A]$ for some $A > 0$.

Remark: We expect that the family of fixed points v_t^∞ is asymptotically stable as $t \to +\infty$ in case of $\phi(u)$ such that $\lim_{u\to 0} \frac{\phi(u)}{u^m} = l$ for certain $m, l > 0$. This fact is suggested by the L^1 convergence result in [CDT, ARMA '06] and by previous results in [De Pablo, Vazquez - Ann. Math. Pura Appl. 1991] and [Biler, Dolbeault, Esteban - Appl. Math. Lett. 2006], which show that the behavior of $\phi(u)$ near $u = 0$ determines the intermediate asymptotic behavior.

A nonlocal equation for the scaled solution.

The scaled variable $v(y,\tau)$ defined by the 2–Toscani map via the scaling

$$u(x,t) = \theta_2[u(t)]^{-d/2}\, v\left(\theta_2[u(t)]^{-1/2}x,\ \frac{1}{2}\log\theta_2[u(t)]\right)$$

satisfies the following nonlocal/nonlinear Fokker–Planck type equation

$$\partial_\tau v = \mathrm{div}(yv) + \Delta\left(\frac{\phi(\theta_2[u(t)]^{-d/2}v)}{d\int_{\mathbb{R}^d}\phi(\theta_2[u(t)]^{-d/2}v)dy}\right). \tag{9}$$

Suppose ϕ is asymptotically homogeneous, i. e.

$$\lim_{u\searrow 0}\frac{\phi(u)}{u^m} = \alpha > 0, \qquad \text{for some } \ m > \max\left\{1-\frac{1}{d}, \frac{d}{d+2}\right\}. \tag{10}$$

Then, due to $\theta_2[u(t)] \to +\infty$ as $t \to +\infty$, it is reasonable to expect the large time asymptotics of (9) to be governed by the leading order equation

$$\partial_\tau v = \mathrm{div}(yv) + \Delta\left(\frac{v^m}{d\int_{\mathbb{R}^d} v^m dy}\right).$$

The unique stationary solution v^∞ of the above equation is the candidate limit of the family of fixed points v_t^∞.

Remark (A possible limit for non asymptotically homogeneous ϕ's): Consider the example $\phi(u) = -u^2 \log u + \frac{u^2}{2} + u^3$, which satisfies all the above assumptions. The leading term of $\phi(u)$ at $u \to 0$ is $-u^2 \log u$, which is not power–like. However, the equation (9) still suggests a possible limit for v_t^∞, since we can still detect a leading order term. The expected limiting equation would be in this case

$$\partial_\tau v = \mathrm{div}(yv) + \Delta\left(\frac{v^2}{d\int_{\mathbb{R}^d} v^2 dy}\right).$$

These issues are part of a work in progress.

Nonlinear scalar conservation laws

We now focus on another classical example of dissipative PDE, namely the nonlinear scalar 1–d conservation law

$$\partial_t u + \partial_x f(u) = 0 \tag{11}$$

with initial condition $u(x, 0) = \bar{u}(x)$. We assume

- f convex, $f(0) = f'(0) = 0$ w. l. o. g.
- $\bar{u} \in L^\infty(\mathbb{R})$, $\bar{u} \geq 0$, $\bar{u}$ with compact support
- w. l. o. g. $\displaystyle\int_{\mathbb{R}} \bar{u}(x)dx = 1$

Unlike the case of nonlinear diffusion equations, here a Liptschitz continuity result of a p–Wasserstein distance (even in one space dimension) is not immediate. A previous result by Bolley, Brenier and Loeper (JHDE 2005) proved $d_p(u_x(t), v_x(t)) \leq d_p(u_x(0), v_x(0)), \quad p \geq 1$, for nondecreasing solutions u, v whose distributional derivative is a probability measure. In fact, such result could be also used in order to prove asymptotic stability of diffusive waves for nondecreasing solutions (see [CDL, JDE '07]) by a similar strategy. We shall omit the details of this result in the present context.

As we shall see, the only distance which easily allows for a stability result of the semigroup associated to (11) is the ∞–Wasserstein distance. The validity of similar results for other distances (to our knowledge) is still an open problem.

Representation of the Wasserstein distance in $1d$.

In one space dimension, the Wasserstein metrics d_p, $p \in [1, +\infty]$, have a simple interpretation in terms of the pseudo-inverses of the primitive of the involved densities (see e. g. the book of Villani, Springer, 2003).

Given $u_1, u_2 \in L^1_+$ with compact support, let us denote

$$v_i(x) = \int_{-\infty}^{x} u_i(y)dy, \quad i = 1, 2$$

and define their pseudo–inverses $v_i^{-1} : [0, 1] \to \mathbb{R}$ as follows

$$v_i^{-1}(\xi) = \inf\{x : v_i(x) > \xi\}.$$

Then, for all $p \in [1, +\infty]$,

$$d_p(u_1, u_2) = \|v_1^{-1} - v_2^{-1}\|_{L^p([0,1])}.$$

Contraction result for scalar inviscid conservation laws

We introduce the functional space

$$\mathcal{B}_c = \left\{ u \in L^\infty(\mathbb{R}),\ u \geq 0,\ \begin{array}{l}\mathrm{supp}(u) \text{ compact} \\ \text{and connected}\end{array},\int_{\mathbb{R}} u(x)dx = 1 \right\}.$$

It is easily seen that the metric space $(\mathcal{B}_c, d_\infty)$ is *not dense* in $(\mathbb{P}_c(\mathbb{R}), d_\infty)$. We shall denote the closure of $\mathcal{B}_c$ in the d_∞ topology by $\overline{\mathcal{B}_c}$. We remark that $\overline{\mathcal{B}_c}$ contains the set of *compactly supported L^∞ probability densities with finite number of connected components* and the set of *bounded variation compactly supported L^∞ probability densities*.

Theorem 3. [Contraction w. r. t. d_∞ for (11)] *Let us consider solutions u and v to* (11) *with initial data $\bar{u}, \bar{v} \in \overline{\mathcal{B}_c}$ and assume the flux f in* (11) *is convex. Then, for all $t > 0$,*

$$d_\infty(u(t), v(t)) \leq d_\infty(\bar{u}, \bar{v}).$$

A representation formula for the pseudo–inverse

The proof of the previous theorem (cf. [CDL '07]) relies on the following lemma.

Lemma 3. *Let* f *be a uniformly convex function, let* $\bar{u} \in \mathcal{B}_c$ *and let* $u(t)$ *be the solution to (11) with initial datum* $\bar{u}$*. Let* $\bar{v}(x) = \int_{-\infty}^{x} \bar{u}(y)dy$*. Then, the function* $v(x,t) = \int_{-\infty}^{x} u(y,t)dy$ *is strictly increasing from* 0 *to* 1 *on a connected interval of* $\mathbb{R}$*. Moreover, for any* $\xi \in (0,1)$*,* $v^{-1}(\xi,t)$ *verifies*

$$v^{-1}(\xi,t) = \max_{0 \leq w \leq \xi} \left\{ tF\left(\frac{\xi - w}{t}\right) + \bar{v}^{-1}(w) \right\},$$

where F *is the inverse of* f^* *restricted to* $[0,+\infty)$*.*

Sketch of the proof of Lemma 3

$v(x,t)$ satisfies the Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} v_t + f(v_x) = 0 \\ v(x,0) = \bar{v}(x), \end{cases}$$

with $v(t) \in \mathrm{Lip}(\mathbb{R})$ for all $t > 0$. Hence, the Lax–Hopf formula yields

$$v(x,t) = \min_{y\in\mathbb{R}} \left\{ t f^* \left(\frac{x-y}{t} \right) + \bar{v}(y) \right\}, \tag{12}$$

where f^* is the Legendre transform of f. Formula (12) can be inverted to

$$v^{-1}(\xi,t) = \sup_{\{y\in\mathbb{R}:\ \bar{v}(y)\leq\xi\}} \left\{ x :\ t f^* \left(\frac{x-y}{t} \right) + \bar{v}(y) = \xi \right\}.$$

By computing x in the above formula we obtain the desired assertion.

Sketch of the proof of Theorem 3

$\mathrm{S{\scriptsize TEP}}\ 1$. Assume $\bar{u}_1, \bar{u}_2 \in \mathcal{B}_c$. Then, we can apply the result in Lemma. In particular, given v_1^{-1} and v_2^{-1} the pseudo–inverses of the primitives of the solutions $u_1(t)$ and $u_2(t)$ respectively, we can find $\bar{w}_1 \in (0,1)$ such that

$$\begin{aligned} v_1^{-1}(\xi,t) - v_2^{-1}(\xi,t) &\leq tF\left(\frac{\xi-\bar{w}_1}{t}\right) + \bar{v}_1^{-1}(\bar{w}_1) - tF\left(\frac{\xi-\bar{w}_1}{t}\right) + \bar{v}_2^{-1}(\bar{w}_1) \\ &= \bar{v}_1^{-1}(\bar{w}_1) - \bar{v}_2^{-1}(\bar{w}_1) \leq \sup_{0\leq w\leq\xi} |\bar{v}_1^{-1}(w) - \bar{v}_2^{-1}(w)|. \end{aligned}$$

Finally, interchanging the role of v_1 and v_2 we get

$$|v_1^{-1}(\xi,t) - v_2^{-1}(\xi,t)| \leq \sup_{0\leq w\leq\xi} |\bar{v}_1^{-1}(w) - \bar{v}_2^{-1}(w)|,$$

which reduces to the desired assertion taking the supremum over all ξ.
$\mathrm{S{\scriptsize TEP}}\ 2$. Via approximation procedure.

Estimate of the speed of propagation

As it is well known [Carrillo–Gualdani–Toscani 2003], the estimate proved in the previous theorem gives also a control of the speed of propagation of the supports of the two solutions $u(t)$ and $v(t)$:

Corollary 1. *Let us consider solutions* u *and* v *to* (11) *with initial data* $\bar{u}, \bar{v} \in \mathcal{B}_c$ *and assume the flux* f *in* (11) *is convex. Then*

$$\left|\inf\left[\mathrm{supp}(u(t))\right] - \inf\left[\mathrm{supp}(v(t))\right]\right| \leq d_\infty(\bar{u}, \bar{v}),$$

$$\left|\sup\left[\mathrm{supp}(u(t))\right] - \sup\left[\mathrm{supp}(v(t))\right]\right| \leq d_\infty(\bar{u}, \bar{v}).$$

Asymptotic behavior — previous results

When $f(u) = u^m$, it is described by the N–waves

$$N(x,t) = \begin{cases} (f')^{-1}\left(\frac{x}{t}\right) & 0 \le x \le b(t) \\ 0 & \text{otherwise} \end{cases}$$

where

$$b(t) = t(f^*)^{-1}\left(\frac{M}{t}\right),$$

where f^* is the Legendre transform of f (well defined iff $f''(u) > 0$ for $u > 0$). For $f(u) = u^m$, $m > 1$, Liu and Pierre (JDE, 1984) proved

$$\lim_{t \to +\infty} t^{(r-1)/mr} \|u(t) - N(t)\|_r = 0.$$

Asymptotic behavior — previous results

In Dolbeault–Escobedo (Asymptot. Anal. 2005), time dependent scaling

$$u(x,t) = (1+mt)^{-1/m} v\left((1+mt)^{-1/m}x, \frac{1}{m}\log(1+mt)\right),$$

yielding the rescaled equation

$$v_\tau = (yv - v^m)_y,$$

where the rescaled N–wave is a stationary solution, in the spirit (e.g.) of [Carrillo, Toscani - Indiana Univ. Math. J. 2000] for the PME.

Y. J. Kim (JDE 2003) improved the rate of convergence of Liu–Pierre (under further assumptions on the initial data) and extended the class of fluxes to those f's satisfying the property

$$\lim_{u \to 0} \frac{u f'(u)}{f(u)} = m > 1.$$

No previous results without any growth condition at the zero state for the convex flux f.

Remark: The N–waves solutions can be written in the self–similar form $N(x,t) = \alpha(t) U(\alpha(t) x)$ for a suitably (decaying for large times) function $\alpha(t)$ only if f is homogeneous.

Intermediate asymptotics for scalar conservation laws

In order to apply the abstract framework developed before, we need to work in the closed subspace $\overline{\mathcal{B}_c(\mathbb{R}^d)} \subset \mathbb{P}_c(\mathbb{R})$ endowed with the d_∞ distance. Therefore, our reference moment will be the maximal transport moment $\sigma[u(t)]$. Since the d_∞ is non expansive, we need to prove that $\sigma[u(t)]$ diverges to $+\infty$ for large times for all solutions $u(t)$ with initial data in $\mathcal{C}$. In order to make sure this happens, we require the additional assumption on f

$$\exists\ \alpha \in (0,1), \quad r \mapsto f(r)^{1-\alpha} \text{ is convex on } (0,+\infty). \tag{13}$$

The above assumption (cf. [Liu–Pierre, JDE '84]) ensures the L^∞ decay

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq f^{-1}\left(\frac{C(\alpha)}{t}\|u(0)\|_{L^1(\mathbb{R})}\right), \tag{14}$$

which implies $\sigma[u(t)] \geq \theta_2[u(t)]^{1/2} \to +\infty$ as in Lemma 2.

Intermediate asymptotics for scalar conservation laws

Then, we have the following theorem (cf. [CDL '07]):

Theorem 4. *Let* $f : [0, +\infty) \to [0, +\infty)$ *be a* C^1 *convex function such that* $f(0) = f'(0) = 0$ *and such that* (13) *is satisfied. Then, there exist a fixed* $t^* > 0$ *and a one parameter family of functions* $\{v_t^\infty\}_{t \geq t^*} \subset \mathcal{M}_\infty \cap \mathcal{B}_c$ *such that, for any* $u_0 \in \mathcal{M}_\infty \cap \overline{\mathcal{B}_c(\mathbb{R}^d)}$ *we have*

$$d_\infty(\mathcal{T}_t[u_0], v_t^\infty) \to 0, \qquad \textit{as} \;\; t \to +\infty,$$

where T_t *is the* ∞*–Toscani map for the scalar conservation law (11) and* $u(t)$ *is the unique entropy solution to (11) with* u_0 *as initial datum. Moreover, for any fixed* $t > t^*$, v_t^∞ *is characterized as the unique fixed point of the* ∞*–Toscani map.*

Remarks and open problems

- We recall that the fixed point family v_t^∞ is constant in time when $f(u) = u^m$ and it coincides with the N–wave $N_m(x, t_0)$ at the time t_0 when $\sigma[N(t_0)] = 1$.
- When $f(u) = u^m + h(u)$ with $h(u)u^{-m} \to 0$ as $u \to 0$, we prove that $v_t^\infty \to N_m(t_0)$ as $t \to +\infty$ with respect to the d_∞ metric.
- In [CDL '07] we apply the scaling technique to the case of increasing solutions, thus characterizing the self–similar rarefaction waves as the fixed points of the renormalized map.
- Characterization of fluxes f for which the families v_t^∞ are independent on time in the spirit of [Carrillo, Vazquez '07] is still open.
- The contraction result is optimal without further assumption on the initial data (e.g. by comparing a N–wave with a space translation of it). Can one obtain strict contractivity by fixing some initial parameter?

Results for viscous conservation laws

- Viscous Burgers:
$$u_t + \left(\frac{1}{2}u^2\right)_x = u_{xx} \tag{15}$$

- Viscous conservation laws with slow diffusion:
$$u_t + f(u)_x = g(u)_{xx}, \quad g'(u) \geq 0,\ g'(0) = 0 \tag{16}$$

Summary of results:

- Contraction in d_∞: (16) via operator splitting [Evje-Karlsen 1999]; (15) via direct calculation following [DF, Markowich – Contemp. Math. 2004].
- Intermediate asymptotics for (16): as before, via Toscani map.

Viscous Burgers' equation

Theorem 5. *Let* $p \in [1, +\infty]$*. Let* u_1 *and* u_2 *be solutions to* (15) *with compactly supported initial data* $\bar{u}_1$*,* $\bar{u}_2 \in L^1_+(\mathbb{R})$*, both with total masses equal to one. Then the Wasserstein distance* $d_p(u_1(t), u_2(t))$ *satisfies the estimates*

$$d_p(u_1(t), u_2(t)) \leq e^{1/2p} d_p(\bar{u}_1, \bar{u}_2),$$

for $p \in [1, +\infty)$,

$$d_\infty(u_1(t), u_2(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2),$$

for $p = +\infty$*.*

Proof:

- Use Hopf – Cole transformation to rewrite (15) as heat equation
- use contraction in $d_p' s$ for heat equation

Viscous conservation laws

Theorem 6. *Let $u_1(x,t)$ and $u_2(x,t)$ be the weak entropy solutions to (16) with nonnegative initial data $\bar{u}_1(x)$, $\bar{u}_2(x) \in \mathcal{P}_c \cap BV(\mathbb{R})$. Assume that f and g are locally C^2 functions, f is convex and $g'(0) = 0$. Then*

$$d_\infty(u_1(t), u_2(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2).$$

Proof. Fix $T > 0$ and a time step $\triangle t > 0$ such that $N\triangle t = T$ and define approx. solution u^n by induction:

- $n = 0$, choose as first term u^0 the initial datum $\bar{u}$
- if $u^n(x)$ is the approximate solution at a time $t_n = n\triangle t$, $n = 0, \dots N-1$, we construct the successive term $u^{n+1}(x)$ via operator splitting method

Operator splitting

Let $\mathcal{S}_t^1$ be the semigroup defining the unique weak entropic solution for the Cauchy problem associated with the nonlinear conservation law

$$u_t + f(u)_x = 0.$$

Define
$$u^{n+1/2}(x) = \mathcal{S}_{\triangle t}^1 u^n(x).$$

Let $\mathcal{S}_t^2$ be the semigroup defining the unique weak solution for the Cauchy problem associated with the nonlinear diffusion equation

$$u_t = g(u)_{xx}.$$

Define
$$u^{n+1}(x) = \mathcal{S}_{\triangle t}^2 u^{n+1/2}(x) = (\mathcal{S}_{\triangle t}^2 \circ \mathcal{S}_{\triangle t}^1) u^n(x)$$

Approx. solution:
$$u_n(x,t) = u^n(x),$$

for any $(x,t) \in \mathbb{R} \times (t_n, t_{n+1}]$ and $n = 0, \dots N-1$.

Properties of approx. solutions and contractivity

- L^∞ stability: $\|u_n(\cdot,t)\|_\infty \le \|\bar{u}\|_\infty$
- BV stability: $\|u_n(\cdot,t)\|_{BV(\mathbb{R})} \le \|\bar{u}\|_{BV(\mathbb{R})}$
- $u_n(\cdot,t) \to u(\cdot,t)$ in $L^1(\mathbb{R})$ and bounded almost everywhere
- $d_p(u_{1,n}(t), u_{2,n}(t)) \le d_\infty(u_{1,n}(t), u_{2,n}(t)) \le d_\infty(\bar{u}_1, \bar{u}_2)$ for any pair of solutions
- Lower semi–continuity of the d_p's:

$$d_p(u_1(t), u_2(t)) \le \liminf_{n\to+\infty} d_p(u_{1,n}(t), u_{2,n}(t)) \le d_\infty(\bar{u}_1, \bar{u}_2).$$

- Final result sending $p \nearrow +\infty$ above

An open problem: nonlocal interaction PDE's.

A drawback of this method: consider the following nonlocal equation

$$\partial_t \rho = \mathrm{div}(\rho \nabla I * \rho)$$

where $I : \mathbb{R}^d \to \mathbb{R}^d$ represents an *interaction kernel*. Let us take a very special case $I(x) = \frac{|x|^2}{2}$ (I is an attractive potential). The temperature of any measure valued solution $\mu(t)$ can be evaluated as follows (suppose that the center of mass of $\mu(t)$ is zero):

$$\frac{d}{dt} \int |x|^2 d\mu(t)(x) = -\int \int |x-y|^2 d\mu(t)(x) d\mu(t)(y) = -2 \int |x|^2 d\mu(t)(x),$$

which implies

$$\theta_2[\mu(t)] = e^{-2t} \theta_2[\mu(0)].$$

Now consider the special class of 'two particle' solutions (with zero center of mass)

$$\mu(t) = \alpha\delta_{x(t)} + (1-\alpha)\delta_{y(t)}, \qquad y(t) = -\frac{\alpha}{1-\alpha}x(t),$$

parameterized by $\alpha \in (0,1)$. For this class of solutions, the temperature reads

$$\theta_2[\mu(t)] = \frac{\alpha}{1-\alpha}|x(t)|^2.$$

The unit temperature initial condition then implies $|x(0)|^2 = \frac{1-\alpha}{\alpha}$. A rough computation of the 2–Wasserstein distance between two solutions in this class μ and ν, parameterized by α and β respectively, shows that $d_2(\mu(t),\nu(t)) = O(e^{-t})$. Therefore, the condition 3 of the abstract framework is not satisfied.

The above example suggests that when the model does not enjoy a little bit of smoothing effect, the present method does not work. To support this idea, we observe that a result of self–similar large time blow up for the friction equation of granular media in 1–d (where $I(x) = |x|^3/3$, cf. [Benedetto, Caglioti, Pulvirenti - 1997]) holds only if the initial datum is absolutely continuous with respect to the Lebesgue measure.

Further investigations on this topic are in progress.

Example of nonlocal repulsion where our approach works:

Consider the simple 1–d equation

$$\partial_t u = -\partial_x(u(\partial_x(\log|\cdot| * u))).$$

Let us evaluate the temperature of a solution $u(t)$ with unit mass:

$$\frac{d}{dt}\int |x|^2 u dx = 2\int xu(\partial_x(\log|\cdot| * u))dx$$
$$= \int\int (x-y)\frac{1}{(x-y)}u(x)u(y)dxdy = 1,$$

which implies

$$\theta_2[u(t)] = t + \theta_2[u(0)].$$

On the other hand, if we consider the equation for the pseudo–inverse variable $v(z,t)$, $z \in [0,1]$, we have

$$\partial_t v = \int_0^1 \frac{1}{v(z,t) - v(\xi,t)} d\xi.$$

Hence, for two solutions u_1 and u_2 with corresponding pseudo–inverse variables v_1 and v_2, we can compute the evolution of the 2–Wasserstein distance ad follows:

$$\frac{d}{dt} d_2^2(u_1(t), u_2(t)) = \frac{d}{dt} \int_0^1 (v_1(z,t) - v_2(z,t))^2 dz$$

$$= 2 \int_0^1 \int_0^1 \left(\frac{1}{v_1(z,t) - v_2(\xi,t)} - \frac{1}{v_2(z,t) - v_2(\xi,t)} \right) (v_1(z,t) - v_2(z,t)) d\xi dz.$$

By choosing the initial center of mass equal to zero, it is easy to check that the last quantity above is non positive, which implies that the 2–Wasserstein distance is non expansive. This proves that we are in the situation of the abstract framework developed above.

Thank you!